Noname manuscript No.
(will be inserted by the editor)

April, 2014

Claudio Borile · Paolo Dai Pra · Markus Fischer · Marco Formentin · Amos Maritan

Time to absorption for a heterogeneous neutral competition model

the date of receipt and acceptance should be inserted later

Abstract Neutral models aspire to explain biodiversity patterns in ecosystems where species difference can be neglected and perfect symmetry is assumed between species. Voter-like models capture the essential ingredients of the neutral hypothesis and represent a paradigm for other disciplines like social studies and chemical reactions. In a system where each individual can interact with all the other members of the community, the typical time to reach an absorbing state with a single species scales linearly with the community size. Here we show, by using a rigorous approach based on a large deviation principle and confirming previous approximate and numerical results, that in a mean-field heterogeneous voter model the typical time to reach an absorbing state scales exponentially with the system size.

Mathematics Subject Classification (2000) 60K35; 60F10; 60K37; 92D25

Keywords Voter Model with disorder, Neutral models of biodiversity, Large deviations, Stochastic dynamics with quenched disorder

1 Introduction

Models of interacting degrees of freedom are nowadays widely spread in different scientific disciplines—from Physics and Mathematics to Biology, Ecology, Finance and Social Sciences—, and more than

Claudio Borile
Dipartimento di Fisica "G. Galilei", Università di Padova, via Marzolo 8, I-35151 Padova, Italy
E-mail: borile@pd.infn.it

Paolo Dai Pra
Dipartimento di Matematica, Università di Padova, via Trieste 63, I-35121 Padova, Italy
E-mail: daipra@math.unipd.it

Markus Fischer
Dipartimento di Matematica, Università di Padova, via Trieste 63, I-35121 Padova, Italy
E-mail: fischer@math.unipd.it

Marco Formentin
Dipartimento di Fisica "G. Galilei", Università di Padova, via Marzolo 8, I-35151 Padova, Italy
E-mail: marco.formentin@rub.de

Amos Maritan
Dipartimento di Fisica "G. Galilei", Università di Padova, via Marzolo 8, I-35151 Padova, Italy
E-mail: maritan@pd.infn.it

ever in the last few years there has been a growing effort in connecting the phenomenology observed at a macroscopic level with a simplified "microscopic" modeling of very disparate complex systems. Clearly, this idea is extremely appealing to statistical physicists and can provide a good benchmark for developing new ideas and methods. A famous and particularly successful example of this approach, which reconciles interdisciplinarity and pure research in statistical physics, can be found in the ecological literature in the so-called *neutral theory of species diversity*, that aims at giving a first null individual-based modeling of the dynamic competition among individuals of different species in the same trophic level, i.e. at the same position in the food chain of an ecosystem [1,2,3,5,6]. The most basic ecological assumption of neutral theory is the complete equivalence of all the individuals in the community –independently of the species they belong to– regarding the basic feature governing the dynamics of the system, like the rates of birth, death, migration, diffusion, etc. This is of course a gross simplification of the real dynamics, but proved very effective in reproducing static and dynamic distributions of real ecosystems [2,3,4].
The neutral hypothesis finds its mathematical equivalent in the voter model (VM) [7] and its generalizations [8], which, in turn, in the mean-filed version considered in this paper, is equivalent to the well-known Moran model in genetics [9]. This model has been deeply studied and has gradually become a paradigmatic example of non-equilibrium lattice models. It is conceptually simple but nevertheless has a very rich phenomenology with applications in many different scientific areas [10, 11,12,14,15]. Despite the fact that the original formulation of the VM can be exactly solved in any spatial dimension [7]—fact that contributed greatly to its rise—, any slight modification made in order to improve the realism of the model complicates drastically its analysis.

Among the possible modifications of the original VM, there has been a recent interest in studying the asymptotic behavior of the VM in the presence of quenched random-field-like disorder, whose motivations span from ecology [16,17,18] to social modeling [19,20] through models of chemical reactants [21] and more fundamental research [22]. A particularly interesting problem is to assess the typical time needed by a finite-size system to reach one of the absorbing states of the model; depending on the particular interpretation of the model, that would mean the typical time for the extinction of a species in an ecosystem, or the reaching or not of a consensus on a particular topic in a society. In all these cases, it is known that heterogeneities, in the habitat of an ecosystem or in the ideologies of groups of people, play a major role in shaping the global dynamics of the complex system. The dynamics of an ecological system, are always influenced by the simultaneous action of neutral and niche processes. The success in invading a new patch of territory can be, say, highly resource dependent, creating an effective fitness over the considered area and locally differentiating otherwise equal species [23,24]. Mathematically this can be emulated by an external random disorder, that we choose quenched to emphasize the different time scales involved. By means of numerical simulations, it has been shown in [19] that a quenched (random-field-like) disorder creating an intrinsic preference of each individual for a particular state/opinion hinders the formation of consensus, hence favoring coexistence. In the context of neutral ecology this corresponds to a version of the VM in which at each location there is an intrinsic preference for one particular species, leading to mixed states lasting for times that grow exponentially with system size [16,19].

Here, we propose a rigorous mathematical development of a mean-field disordered VM intended as a general model of neutral competition in a heterogeneous environment. Supporting the previous findings [16,19] based on computational investigations or approximation arguments, we will show that a heterogenous environment indeed favors significantly the maintenance of the active state, and the typical time needed to reach an absorbing phase passes from a power-law dependence in the system's size, typical of the neutral theories, to an exponentially long time, signature of an asymptotic active phase, i.e., spin states 0 and 1 coexist in the stationary state in the limit, $N \to \infty$.

This will be achieved by setting up a large deviation principle for the considered model and will thus provide a first attempt of an extreme value theory for systems with multiple symmetric absorbing states.

The remainder of the paper is organized as follows. In Section 2 we define the model and study its macroscopic limit which is given in Theorem 1. Then, in the next section, we are ready to present our main result as Theorem 2. We prove that in a mean-field VM the presence of quenched disorder favoring one of the two species at each site, makes the coalescence time pass from linear to exponential in the system size N. In the last section we consider the normal fluctuations of the model around its thermodynamic limit. In particular, Theorem 3 shows that in the limiting dynamics fluctuations are not self-avering in the disorder.

2 Macroscopic limit

The state of the system is described by a vector of spins $\eta = (\eta_1, \eta_2, \ldots, \eta_N) \in \{0,1\}^N$. The random environment consists of N independent and identically distributed random variables $h_1, h_2, \ldots, h_N$, taking the values 0 and 1 with probability, respectively, $1-q$ and $q \in (0,1)$. Moreover, let $\rho \in [0,1]$ be a given parameter. While the random environment remains constant, the state η evolves in time according to the following rules:

– each site $1, 2, \ldots, N$ has its own independent random clock. A given site i after a waiting time with exponential distribution of mean 1 chooses at random, with uniform probability, a site $j \in \{1, 2, \ldots, N\}$.

– If $\eta_j = h_i$, then the site i updates its spin from η_i to η_j. If $\eta_j \neq h_i$, then the site i updates its spin from η_i to η_j with probability ρ, while it keeps its spin η_i with probability $1-\rho$.

Thus, the site i has a preference to agree with sites whose spins equal its local field h_i. For $\rho = 1$, this effect is removed, and we obtain the standard Voter model. Note that, by symmetry, there is no loss of generality in assuming $q \geq 1/2$, as we will from now on.

In more formal terms, for every realization of the random environment, the spins evolve as a continuous-time Markov chain with generator L_N acting on a function $f : \{0,1\}^N \to \mathbb{R}$ according to

$$\begin{aligned} L_N f(\eta) := \sum_{i=1}^{N} \frac{1}{N} \sum_{j=1}^{N} & \mathbb{I}_{h_i=\eta_j} \left(f(\eta^{j\to i}) - f(\eta) \right) \\ & + \rho \mathbb{I}_{h_i \neq \eta_j} \left(f(\eta^{j\to i}) - f(\eta) \right), \end{aligned} \tag{2.1}$$

where $\mathbb{I}_A$ denotes the indicator function of the set A and $\eta^{j\to i}$ is the configuration obtained from η by replacing the value of the spin at the site i with that of the spin at the site j. This Markov chain has two absorbing states, corresponding to all spin values equal to zero and all equal to one. We denote by T_N the random time needed to reach one of the two absorbing states.

It is useful to review the main properties of the model in the case $\rho = 1$. In this case, the dynamics are independent of q and the unique order parameter for the model is given by $K_N := \sum_{i=1}^{N} \eta_i$, i.e., the number of spins with value 1. It is easy to check, using the generator (2.1), that K_N evolves as a random walk on $\{0, 1, \ldots, N\}$: if $K_N = k$, then it moves to either $k+1$ or $k-1$ with the same rate $\frac{(N-k)k}{N}$. By standard arguments on birth and death processes (see e.g. [25]), one shows that $\langle T_N \rangle \sim N \ln(2)$ as $N \to +\infty$: the mean absorption time grows linearly in N.

Consider now the general case $\rho \leq 1$. Here the system is described in terms of two integer-valued order parameters, namely $\sum_{i=1}^{N} h_i \eta_i$ and $\sum_{i=1}^{N} (1-h_i)\eta_i$, that will be convenient to properly scale as follows:

$$m_N^+ := m_N^+(\eta) := \frac{1}{N}\sum_{i=1}^N h_i\eta_i$$
$$m_N^- := m_N^-(\eta) := \frac{1}{N}\sum_{i=1}^N (1-h_i)\eta_i$$

Note that the pair (m_N^+, m_N^-) belongs to the subset of the plane $\{(x,y) \in [0,1]^2 : x+y \leq 1\}$. Note, however, that when the limit as $N \to +\infty$ is considered, $m_N^+ \leq \frac{1}{N}\sum_{i=1}^N h_i \to q$, where this last convergence follows from the law of large numbers. Similarly, $m_N^- \leq \frac{1}{N}\sum_{i=1}^N (1-h_i) \to 1-q$. Thus, limit points of the sequence (m_N^+, m_N^-) belong to $[0,q] \times [0,1-q]$. Given an initial, possibly random, state $\eta(0)$ for the dynamics of N spins, we denote by $m_N^\pm(t)$ the (random) value at time t of the order parameters $m_N^\pm$. In what follows we also denote by μ_N the distribution of $\eta(0)$.

Theorem 1 *Assume there exists a non-random pair $(\bar{m}^+, \bar{m}^-) \in [0,q] \times [0,1-q]$ such that, for every $\epsilon > 0$,*

$$\lim_{N\to+\infty} \mu_N\left(\left|m_N^\pm(0) - \bar{m}^\pm\right| > \epsilon\right) = 0.$$

Then the stochastic process $(m^+(t), m^-(t))_{t\geq 0}$ converges in distribution to the unique solution of the following system of ODEs:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}m^+ = -\rho m^+(1-m^- - m^+) \\ \qquad\qquad +(q-m^+)(m^+ + m^-) \\ \dfrac{\mathrm{d}}{\mathrm{d}t}m^- = -m^-(1-m^- - m^+) \\ \qquad\qquad +\rho(1-q-m^-)(m^+ + m^-) \\ m^\pm(0) = \bar{m}^\pm \end{cases} \tag{2.2}$$

Proof: Denote by $\mathcal{G}$ the generator of the semigroup associated to the deterministic evolution (2.2), i.e.,

$$\mathcal{G}f(m^+, m^-) := V^+(m^+, m^-)\frac{\partial f}{\partial m^+} + V^-(m^+, m^-)\frac{\partial f}{\partial m^-},$$

with

$$\begin{aligned} V^+(m^+, m^-) &= -\rho m^+(1-m^- - m^+) \\ &\quad + (q-m^+)(m^+ + m^-) \\ V^-(m^+, m^-) &= -m^-(1-m^- - m^+) \\ &\quad + \rho(q-m^-)(m^+ + m^-) \end{aligned}$$

Let $f : [0,1]^2 \to \mathbb{R}$. By direct computation one finds that

$$L_N[f(m_N^+, m_N^-)](\eta)$$

depends on η only through m_N^+, m_N^-, which implies that the process $(m_N^+(t), m_N^-(t))_{t\geq 0}$ is a Markov process, whose associated semigroup has a generator $\mathcal{G}_N$ that can be identified by the identity

$$L_N[f(m_N^+, m_N^-)](\eta) = [\mathcal{G}_N f](m_N^+(\eta), m_N^-(\eta)),$$

which yields

$$\begin{aligned}
&\mathcal{G}_N f(x,y) \\
&:= N\Big((q-x)\,(x+y)\,\big(f(x+\tfrac{1}{N},y)-f(x,y)\big)\big) \\
&\quad + \rho x\,(1-(x+y))\,\big(f(x-\tfrac{1}{N},y)-f(x,y)\big) \\
&\quad + \rho\,(1-q-y)\,(x+y)\,\big(f(x,y+\tfrac{1}{N})-f(x,y)\big) \\
&\quad + y\,(1-(x+y))\,\big(f(x,y-\tfrac{1}{N})-f(x,y)\big)\Big).
\end{aligned} \tag{2.3}$$

Moreover, if f is smooth with bounded derivatives, one checks that

$$\lim_{N\to+\infty} \sup_{(m^+,m^-)\in[0,1]^2} \big|\mathcal{G}_N f(m^+,m^-) - \mathcal{G} f(m^+,m^-)\big| = 0.$$

The conclusion then follows by a standard result of convergence of Markov processes, cf. [26], Ch. 4, Corollary 8.7. □

This first theorem formalizes and extends a useful result for the infinite size system (hydrodynamic scaling), already obtained by means of different techniques in some previous works [19, 17]. It is a dynamic law of large numbers that quantifies the deterministic evolution of the order parameters as obtained from the limiting dynamics described by L_N neglecting fluctuations. The stability analysis of the fixed points of Eq.(2.2) provides some immediate results on the global dynamics of the model in the infinite size limit: For $\rho = 1$, equations (2.2) trivialize: the only relevant variable is $m = m^+ + m^-$, which satisfies $\frac{\mathrm{d}}{\mathrm{d}t}m = 0$. This is simply the macroscopic consequence of the fact that $K_N = N(m_N^+ + m_N^-)$ evolves as a symmetric random walk. The picture changes as $\rho < 1$. When $\rho < 1$, the system (2.2) has three equilibrium points:

1. $(m^+, m^-) = (q, 1-q)$, which represents a limiting behavior where all the spins equal 1;

2. $(m^+, m^-) = (0, 0)$, which is the case with all spins equal to 0;

3. $(m^+, m^-) = \Big(\frac{q(1+\rho)-\rho}{(1+\rho)(1-\rho)}, \rho\frac{q(1+\rho)-\rho}{(1+\rho)(1-\rho)}\Big)$.

It is easily checked that equilibrium 3 lies inside $[0, q] \times [0, 1-q]$, hence is admissible, if and only if the condition

$$\rho < \frac{1-q}{q} \tag{2.4}$$

holds (remember we are assuming $q \geq 1/2$). The stability analysis of the three equilibria is also easily done: for $\frac{1-q}{q} < \rho < 1$ equilibrium 1 is stable, and attracts all initial conditions except $(0, 0)$, which is an unstable equilibrium (see panel (a) in Fig.1), while for $\rho < \frac{1-q}{q}$ both $(q, 1-q)$ and $(0, 0)$ are unstable, and the stable equilibrium 3 emerges, attracting all initial conditions except the unstable equilibria (see panel (b) in Fig.1). Note that, for $q = 1/2$, only this second regime exists. Thus, in the case $q > 1/2$ and $\frac{1-q}{q} < \rho < 1$, the asymmetric disorder stabilizes the equilibrium $(q, 1-q)$ (see panel (c) in Fig.1); lower values of ρ increase the effects of the disorder, so that a new stable equilibrium appears .

3 Large deviations and time to absorption

In order to get information on the behavior of the system when the total number of "individuals" N is large but finite, we need to go beyond the law of large numbers in Theorem 1. In particular, our

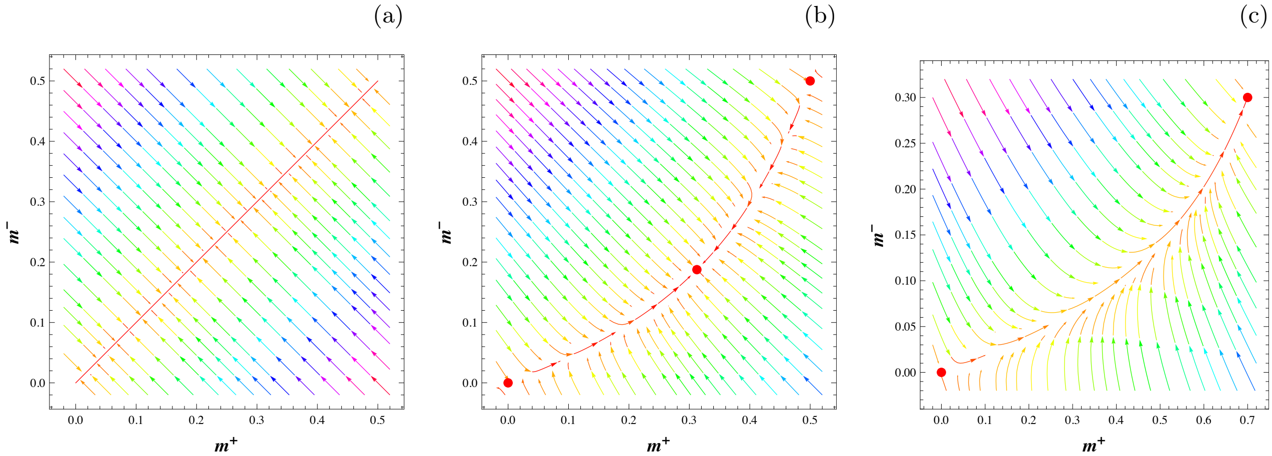

Fig. 1: The plots represent the streamlines of the system (2.2) for different values of ρ and q. In the pictures, red dots are the equilibria while blue/violet arrows indicate a strong vector field that becomes weaker as arrows turn to red. In (a), with $\rho = 1$ the disorder plays no role and any initial condition is attracted, along $m = m^+ + m^-$, to the fixed-point manifold $m^+ = m^-$. When the disorder is introduced (i.e. $\rho < 1$) the line $m^+ = m^-$ "breaks down" into multiple equilibria. In (b), for $\rho < \frac{1-q}{q}$, the equilibria $(0,0)$ and $(q, 1-q)$ are unstable and a third stable equilibria arises. This stable equilibrium is not necessarily admissible, as it may be outside $[0,q] \times [0,1-q]$ and disappears when $q > 1/2$ and $\frac{1-q}{q} < \rho < 1$ as shown in panel (c). In this case, $(q, 1-q)$ becomes stable and attracts all the initial condition but $(0,0)$.

next aim is to show that, whenever equilibrium 3 is present for the macroscopic dynamics (2.2), the absorption time for the microscopic system grows exponentially in N. To this end, we use the Freidlin and Wentzell theory for randomly perturbed dynamical systems (see [27]). This theory, based on finite time Large Deviations, yields asymptotic estimates characterizing the long-time behavior of the perturbed system (here the microscopic system described by (m_N^+, m_N^-)) as the noise intensity tends to zero (equivalent here to $N \to \infty$). See also [28,29] for an introduction to Large deviations. For the system with N finite but large, the escape for the neighborhood of the stable equilibrium 3, which leads to the absorption, has several analogies with *metastability* phenomena. We note that sharp estimates for the time to escape a metastable equilibrium have been recently obtained via a potential theoretic approach (see [30,31]). This approach, however, is well understood only in the case of reversible dynamics. For this reason, for our model which is non-reversible, we follow the more traditional Freidlin and Wentzell theory.
We also mention another line of research which is close in spirit, though much more general, to what we present in this paper. The absorption is a *finite size* phenomenon, it does not occur for the infinite system (2.2) whenever equilibrium 3 is present. The time scale at which absorption occurs corresponds to the *critical time scale* in the *finite system scheme* developed in [32] (see also [33]).

For simplicity, we assume $q = 1/2$, so that equilibrium 3 exists for every $\rho < 1$. For $\mathbf{x} = (x, y) \in [0, 1/2]^2$ set

$$\begin{aligned} l_1(\mathbf{x}) &= \rho x(1-x-y) \\ r_1(\mathbf{x}) &= (1/2 - x)(x+y) \\ l_2(\mathbf{x}) &= y(1-x-y) \\ r_2(\mathbf{x}) &= \rho(1/2 - y)(x+y) \end{aligned}$$

Notice that the vector field $b(\mathbf{x}) = (b_1(\mathbf{x}), b_2(\mathbf{x}))$ defined by $b_i(\mathbf{x}) := r_i(\mathbf{x}) - l_i(\mathbf{x})$ appears in (2.2), the equation of the macroscopic dynamics, which we interpret as the unperturbed dynamical system. Define the family of point measures, parametrized by $\mathbf{x} \in [0,1/2]^2$:

$$\mu_{\boldsymbol{x}} := r_1(\boldsymbol{x})\delta_{(1,0)} + l_1(\boldsymbol{x})\delta_{(-1,0)} + r_2(\boldsymbol{x})\delta_{(0,1)} + l_2(\boldsymbol{x})\delta_{(0,-1)},$$

where δ indicates Dirac measure. Then the generator $\mathcal{G}_N$ in (2.3) can be rewritten in a diffusion-like form as

$$\begin{aligned}\mathcal{G}_N(f)(\boldsymbol{x}) = N \int_{\mathbb{R}^2\setminus\{0\}} \big(f(\boldsymbol{x} + \tfrac{1}{N}\boldsymbol{\gamma}) - f(\boldsymbol{x})\big)\, \mu_{\boldsymbol{x}}(d\boldsymbol{\gamma})\\ &= \langle b(\boldsymbol{x}), \nabla f(\boldsymbol{x})\rangle \\ &+ N \int \left(f(\boldsymbol{x} + \tfrac{1}{N}\boldsymbol{\gamma}) - f(\boldsymbol{x}) - \frac{1}{N}\langle \boldsymbol{\gamma}, \nabla f(\boldsymbol{x})\rangle \right) \mu_{\boldsymbol{x}}(d\boldsymbol{\gamma}),\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^2$. Let $H\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be the Hamiltonian associated with the operators $\mathcal{G}_N$, $N \in \mathbb{N}$:

$$H(\boldsymbol{x}, \boldsymbol{\alpha}) := \langle b(\boldsymbol{x}), \boldsymbol{\alpha}\rangle + \int \left(\exp\left(\langle \boldsymbol{\gamma}, \boldsymbol{\alpha}\rangle\right) - 1 - \langle \boldsymbol{\gamma}, \boldsymbol{\alpha}\rangle\right) \mu_{\boldsymbol{x}}(d\boldsymbol{\gamma}).$$

It follows that

$$H(\boldsymbol{x}, \boldsymbol{\alpha}) = \sum_{i=1}^{2} \left[r_i(\boldsymbol{x}) \left(e^{\alpha_i} - 1\right) + l_i(\boldsymbol{x}) \left(e^{-\alpha_i} - 1\right) \right].$$

Let L be the Legendre transform of H, given by

$$L(\boldsymbol{x}, \boldsymbol{\beta}) := \sup_{\boldsymbol{\alpha} \in \mathbb{R}^2} \left\{ \langle \boldsymbol{\beta}, \boldsymbol{\alpha} \rangle - H(\boldsymbol{x}, \boldsymbol{\alpha}) \right\}.$$

It is easy to show that

$$L(\boldsymbol{x}, \boldsymbol{\beta}) = \tilde{L}\big(l_1(\boldsymbol{x}), r_1(\boldsymbol{x}); \beta_1\big) + \tilde{L}\big(l_2(\boldsymbol{x}), r_2(\boldsymbol{x}); \beta_2\big), \tag{3.1}$$

where $\tilde{L} : [0,\infty)^2 \times \mathbb{R} \to [0,\infty]$ is given by

$$\begin{aligned}\tilde{L}(l, r; \beta) &= \sup_{\alpha \in \mathbb{R}} \left\{ \beta \cdot \alpha - r \cdot (e^{\alpha} - 1) - l \cdot (e^{-\alpha} - 1) \right\} \\ &= \beta \log \left(\tfrac{\beta + \sqrt{\beta^2 + 4rl}}{2r} \right) - \sqrt{\beta^2 + 4rl} + l + r,\end{aligned}$$

taking appropriate limits for the boundary cases $l = 0$ or $r = 0$. In particular, $\tilde{L}(l, r; \beta) = \infty$ if and only if either $l = 0$ and $\beta < 0$ or $r = 0$ and $\beta > 0$. The *Lagrangian* L in (3.1) allows to define the *action functional*: for $T > 0$, $\varphi : [0,T] \to \mathbb{R}^2$, set

$$S_T(\varphi) := \int_0^T L\left(\varphi(t), \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t)\right) dt, \tag{3.2}$$

where $S_T(\varphi)$ is meant to be equal to $+\infty$ if φ is not absolutely continuous. The action functional controls the *quenched* Large Deviations of the stochastic process $(m_N^+(t), m_N^-(t))_{t\geq 0}$: if B_φ is a small neighborhood of a trajectory $\varphi : [0,T] \to \mathbb{R}^2$, $h = (h_1, h_2, \ldots, h_N)$ is a realization of the random environment, and P_h is the law of the Markov process generated by (2.1) for h *fixed*, then for almost every realization h

$$\frac{1}{N} \log P_h \left[(m_N^+(t), m_N^-(t))_{t\in[0,T]} \in B_\varphi \right] \simeq -S_T(\varphi) \tag{3.3}$$

for N large. This fact falls within the range of the Freidlin-Wentzell Large Deviations results (see [27]), although several modifications of the original proof are needed here, following [34]. For simplicity, we assume that the initial condition $(m_N^+(0), m_N^+(0))$ is deterministic conditional on the realization of the environment h and that $(m_N^+(0), m_N^+(0))$ converges to some $\mathbf{x} = \mathbf{x}_h \in [0, 1/2]^2$ as $N \to \infty$. The Large Deviations estimate (3.3) then holds provided $(m_N^+(0), m_N^+(0)) \to \varphi(0)$ as $N \to \infty$.

As shown in [27], the control of the Large Deviations provides control on the hitting times of subsets of the state space $[0, 1/2]^2$ of the process $(m_N^+(t), m_N^-(t))_{t\geq 0}$, in particular of the time T_N needed to reach the absorbing states. Denote by $\mathbf{z}$ the stable equilibrium for the macroscopic dynamics:

$$\mathbf{z} = \left(\frac{1}{2(1+\rho)}, \frac{\rho}{2(1+\rho)} \right).$$

For $\mathbf{x} \in [0, 1/2]^2$, define the *quasi-potential* by

$$V(\mathbf{x}) := \inf\{S_T(\varphi) : T > 0, \varphi(0) = \mathbf{z}, \varphi(T) = \mathbf{x}\}.$$

Calculating the quasi-potential is equivalent to solving a deterministic optimal control problem with calculus of variations dynamics and unbounded time horizon. An approximate solution to this global optimization problem could be obtained using numerical methods based on the principle of dynamic programming; see [35] and Chapter 15 in [36]. Let D be a domain in $[0, 1/2]^2$ containing $\mathbf{z}$ with smooth boundary ∂D such that $\partial D \subseteq (0, 1/2)^2$ and the vector field $b(\mathbf{x})$ is directed strictly inside D. Let τ_N denote the first time the process (m_N^+, m_N^-) hits the complement of D. By construction, $\tau_N \leq T_N$. Let us assume that, for almost every realization of the environment h, $(m_N^+(0), m_N^+(0))$ is deterministic and that $(m_N^+(0), m_N^+(0)) \to \mathbf{x}_h$ as $N \to \infty$ for some $\mathbf{x}_h \in D$.

Theorem 2 *For every $\mathbf{x} \neq \mathbf{z}$ we have $V(\mathbf{x}) > 0$. Moreover, for almost every realization of the environment h, every $\varepsilon > 0$,*

$$\lim_{N\to+\infty} P_h \left(e^{N(V_{\partial D}-\varepsilon)} \leq \tau_N \leq e^{N(V_{\partial D}+\varepsilon)} \right) = 1$$

where

$$V_{\partial D} := \min \{V(\mathbf{x}) : \mathbf{x} \in \partial D\} > 0.$$

Proof: In order to show that $V(\mathbf{x}) > 0$ for every $\mathbf{x} \neq \mathbf{z}$, it suffices to check that, for every $\delta_0 > 0$ small enough, $\inf_{\mathbf{x}\in\partial \mathbf{B}_{\delta_0}(\boldsymbol{z})} V(\mathbf{x}) > \mathbf{0}$.

Set $\mathbf{r}_* := (\frac{\rho}{4(1+\rho)}, \frac{\rho}{4(1+\rho)})$; thus $\mathbf{r}_* = (r_i(\mathbf{z}), l_i(\mathbf{z}))$, $i \in \{1, 2\}$. Let $l, r > 0$. Then $\tilde{L}(l, r; \beta)$ as a function of $\beta \in \mathbb{R}$ is smooth, non-negative, strictly convex with minimum value zero attained at $\beta = r - l$ and of super-linear growth. Second order Taylor expansion around $\beta = r - l$ yields

$$\tilde{L}(l, r; \beta) = \frac{1}{2(r+l)} (\beta - (r-l))^2 + \mathcal{O}\left((\beta - (r-l))^3 \right).$$

It follows that for every $\delta_* > 0$ small enough there are a constant $c > 0$ and a continuous function $\underline{L}\colon \overline{B_{\delta_*}(\mathbf{r}_*)} \times \mathbb{R} \to [0, \infty)$ such that $\tilde{L}(l, r; \beta) \geq \underline{L}(l, r; \beta)$, $\underline{L}(l, r; .)$ is strictly convex with super-linear growth and for every $(l, r) \in \overline{B_{\delta_*}(\boldsymbol{r}_*)}$,

$$\underline{L}(l, r; \beta) = c\, (\beta - (r-l))^2 \text{ if } \beta \in [-4\delta_*, 4\delta_*].$$

Choose such δ_*, c, $\underline{L}$. By continuity of the functions r_1, l_1, r_2, l_2, we can choose $\delta_0 > 0$ such that $(l_1(\boldsymbol{x}), r_1(\boldsymbol{x})), (l_2(\boldsymbol{x}), r_2(\boldsymbol{x})) \in \overline{B_{\delta_*}(\boldsymbol{r}_*)}$ for all $\boldsymbol{x} \in \overline{B_{\delta_0}(\boldsymbol{z})}$. Recall that $b_i = r_i - l_i$. It follows that

$$\inf_{\boldsymbol{x}\in\partial B_{\delta_0}(\boldsymbol{z})} \bar{V}(\boldsymbol{x}) \geq \inf \sum_{i=1}^{2} \int_0^T \underline{L}\left(l_i(\varphi(t)), r_i(\varphi(t)); \frac{\mathrm{d}}{\mathrm{d}t}\varphi_i(t) \right) dt,$$

where the infimum on the right-hand side is over all $\varphi \in \mathbf{C}_a([0,\infty), \overline{B_{\delta_*}(\boldsymbol{r}_*)})$, $T > 0$ such that $\varphi(0) = \boldsymbol{z}$, $\varphi(T) \in \partial B_{\delta_0}(\boldsymbol{z})$. Using a time transformation argument analogous to that of Lemma 4.3.1 in [27] and the convexity and super-linear growth of $\underline{L}(l, r, \beta)$ in β, one finds that the infimum can be restricted to $\varphi \in \mathbf{C}_a([0,\infty), \overline{B_{\delta_*}(\boldsymbol{r}_*)})$ such that $\left|\frac{\mathrm{d}}{\mathrm{d}t}\varphi_i(t)\right| \le 4\delta_*$ and $\left|\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t)\right| = |b(\varphi(t))|$ for almost all $t \in \mathbb{R}$. Thus

$$\inf_{\boldsymbol{x}\in\partial B_{\delta_0}(\boldsymbol{z})} \bar{V}(\boldsymbol{x}) \ge \inf \int_0^T c \cdot \left| b(\varphi(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) \right|^2 dt.$$

The Jacobian of b at $\boldsymbol{z}$ has two strictly negative eigenvalues. Choosing, if necessary, a smaller δ_* and corresponding $c > 0$, $\underline{L}$, it follows that

$$\inf_{\boldsymbol{x}\in\partial B_{\delta_0}(\boldsymbol{z})} \bar{V}(\boldsymbol{x}) \ge \inf \int_0^T c \cdot \left| Db(\boldsymbol{z})\varphi(t) - \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) \right|^2 dt > 0,$$

which establishes the strict positivity of V away from $\mathbf{z}$.

The second part of the assertion is established in a way analogous to the proofs of Theorems 4.4.1 and 4.4.2 in [27], see Section 5.4 therein. □

Theorem 2 implies in particular that the time to reach any small neighborhood of the absorbing states grows exponentially in N for any $\rho < 1$. This is a generalization of the Kramers's formula for the noise activated escape from a potential well [37]. This exponential behavior in N suggests the existence of an active phase where both spin states / species, 0 and 1, coexist in the stationary state in the infinite size limit, $N \to \infty$. It is a common wisdom supported by the work of Durrett (see for example [13]), that when the mean-field version of a model has an attracting fixed point, coexistence is expected in the spatially explicit model.

4 Normal fluctuation

As seen in the previous sections, on a time scale of order 1 the process $(m_N^+(t), m_N^-(t))_{t\ge 0}$ remains close to its thermodynamic limit: i.e., Eq.(2.2). In this section we consider the normal fluctuations around this limit. Suppose the assumptions of Theorem 1 are satisfied; moreover, for the sake of simplicity, we assume $q = 1/2$, and $(m^+, m^-) = \mathbf{z}$ with $\mathbf{z} = \left(\frac{1}{2(1+\rho)}, \frac{\rho}{2(1+\rho)}\right)$, so that the limiting dynamics starts in equilibrium. We define the fluctuation processes

$$\begin{aligned} x_N(t) &:= \sqrt{N}\left(m_N^+(t) - m^+\right) \\ y_N(t) &:= \sqrt{N}\left(m_N^-(t) - m^-\right). \end{aligned}$$

Theorem 3 *The stochastic process $(x_N(t), y_N(t))$ converges in distribution to a Gauss-Markov process (X, Y) which solves the stochastic differential equation*

$$\begin{cases} dX = \left(-\frac{1+\rho^2}{2(1+\rho)}X + \frac{\rho}{1+\rho}Y + \frac{1}{2}\mathcal{H}\right)dt \\ \qquad + \frac{1}{\sqrt{2}}\sqrt{\frac{\rho}{1+\rho}}dB_1 \\ dY = \left(-\frac{1+\rho^2}{2(1+\rho)}Y + \frac{\rho}{1+\rho}X - \rho\frac{1}{2}\mathcal{H}\right)dt \\ \qquad + \frac{1}{\sqrt{2}}\sqrt{\frac{\rho}{1+\rho}}dB_2 \end{cases} \tag{4.1}$$

Here, B_i, $i = 1,2$ are two independent standard Brownian motions and $\mathcal{H}$ is a zero average standard Gaussian random variable, independent of B_1, B_2.

The proof of Theorem 3 uses the method of convergence of generators as that of Theorem 1, and is omitted. Unlike in Theorem 1, the environment does not fully self-averages since $\mathcal{H}$ is not identically equal to zero. The quenched random variable $\mathcal{H}$ in Theorem 3 is due to the normal fluctuations of the environment $(h_1, h_2, \dots, h_N)$.

5 Discussion and conclusions

It is well known that habitat heterogeneity impacts on biodiversity and causes the introduction of niche-like effects in the system [38,23,24]. At large scale, e.g. at regional or larger level, geomorphological changes may induce genetic isolation whereas at smaller scales the complexity induced by, for example, vegetation, sediment types, moisture and temperature leads to the coexistence of several species and to the emergence of niches. To our knowledge, however, quantitative estimates of the relation between the degrees of heterogeneity and biodiversity and the time of coexistence of species have not been obtained. Here we have rigorously proved that even a small habitat disorder in a neutral competition-like model dramatically enhances the typical time biodiversity persists; more specifically, we have shown that the typical time to loss of biodiversity, τ_N, scales exponentially with the population size N, leading, for large size systems, to an unobservable long time scale beyond which extinction occurs. This is in contrast to what happens in absence of habitat heterogeneity, where the typical time to loss of biodiversity is typically small, growing as the system's size, N. We have also obtained the scaling exponent of τ_N in terms of a suitable *quasi-potential* $V(\mathbf{x})$, that encodes the minimum "cost" of a trajectory to reach a given point x of the phase-space. The consequences of these findings could be particularly relevant, for example, in conservation ecology: In a given area different species at the same trophic level compete for space and nutrients in a neutral fashion; for example, think of a tropical forest, where the neutral theory provides a very good null model [2].
Lastly, we have shown that the fluctuations around the metastable symmetric fixed point obey a Brownian motion dynamics with drift where the environmental disorder does not show self-averaging.

Acknowledgments: AM acknowledges Cariparo foundation for financial support. We thank Miguel Muñoz for useful discussions, comments and suggestions.

References

1. S.P. Hubbell. *The Unified Neutral Theory of Biodiversity and Biogeography.* Monographs in Population Biology. Princeton University Press, 2008.
2. I. Volkov, J.R. Banavar, S.P. Hubbell, and A. Maritan. Neutral theory and relative species abundance in ecology. *Nature*, 424(2):1035–1037, 2003.
3. S. Azaele, S. Pigolotti, J.R. Banavar, and A. Maritan. Dynamical evolution of ecosystems. *Nature*, 444:926–928, 2006.
4. J. Chave Neutral theory and community ecology. *Ecology Letters*, 7: 241-253, 2004.
5. M. Vallade and B. Houchmandzadeh. Analytical solution of a neutral model of biodiversity. *Physical Review E*, 68(6):061902, 2003.
6. D. Alonso, R.S. Etienne, and A.J. McKane. The merits of neutral theory. *Trends in Ecology & Evolution*, 21(8):451–457, 2006.
7. T.M. Liggett. *Interacting particle systems.* Springer, 2005.
8. O. Al Hammal, H. Chaté, I. Dornic, and M.A. Muñoz. Langevin description of critical phenomena with two symmetric absorbing states. *Phys. Rev. Lett.*, 94:230601, 2005.
9. M. Kimura and N. Takahata. *Population Genetics, Molecular Evolution, and the Neutral Theory: Selected Papers.* Evolutionary biology. University of Chicago Press, 1995.

10. M. Henkel, H. Hinrichsen, and S. Lübeck. *Non-equilibrium Phase Transitions. Volume I: Absorbing phase transitions*. Theoretical and Mathematical Physics. Springer / Canopus, 2008.
11. I. Dornic, H. Chaté, J. Chave, and H. Hinrichsen. Critical coarsening without surface tension: The universality class of the voter model. *Phys. Rev. Lett.*, 87:045701, 2001.
12. R. Durrett and S.A. Levin. Stochastic spatial models: a user's guide to ecological applications. *Philosophical Transactions of the Royal Society of London. Series B: Biological Sciences*, 343(1305):329–350, 1994.
13. R. Durrett. Coexistence in stochastic spatial models. *The Annals of Applied Probability* , 477-496, 2009.
14. C. Castellano, S. Fortunato, and V. Loreto. Statistical physics of social dynamics. *Rev. Mod. Phys.*, 81:591–646, 2009.
15. R.A. Blythe and A.J. McKane. Stochastic models of evolution in genetics, ecology and linguistics. *Journal of Statistical Mechanics: Theory and Experiment*, 2007(07):P07018, 2007.
16. S. Pigolotti and M. Cencini. Coexistence and invasibility in a two-species competition model with habitat-preference. *Journal of Theoretical Biology*, 265(4):609 – 617, 2010.
17. C. Borile, A. Maritan, and M.A. Muñoz. The effect of quenched disorder in neutral theories. *Journal of Statistical Mechanics: Theory and Experiment*, 2013(04):P04032, 2013.
18. D. Tilman, R.M. May, C.L. Lehman, and M.A. Nowak. Habitat destruction and the extinction debt. *Nature*, 371:65–66, 1994.
19. N. Masuda, N. Gibert, and S. Redner. Heterogeneous voter models. *Phys. Rev. E*, 82:010103, 2010.
20. N. Masuda and S. Redner. Can partisan voting lead to truth? *Journal of Statistical Mechanics: Theory and Experiment*, 2011(02):L02002, 2011.
21. P.L. Krapivsky, S. Redner, and E. Ben-Naim. *A Kinetic View of Statistical Physics*. Cambridge University Press, 2010.
22. G. Ódor. *Universality In Nonequilibrium Lattice Systems: Theoretical Foundations*. World Scientific, 2008.
23. D. Tilman. Niche tradeoffs, neutrality, and community structure: a stochastic theory of resource competition, invasion, and community assembly. *Proceedings of the National Academy of Sciences of the United States of America*, 101(30), 10854-10861, 2004.
24. P.B. Adler, J. HilleRisLambers, and J.M. Levine. A niche for neutrality. *Ecology Letters*, 10.2: 95-104, 2007.
25. A. Lambert. Population dynamics and random genealogies. *Stochastic Models*, 24(S1):45–163, 2008.
26. S.N. Ethier and T.G. Kurtz. *Markov processes: characterization and convergence*. Wiley, 2009.
27. M.I. Freidlin and A.D. Wentzell. *Random perturbations of dynamical systems*. Springer, 2012.
28. F. Den Hollander. *Large deviations*. American Mathematical Soc., 2008.
29. H. Touchette. The large deviation approach to statistical mechanics. *Physics Reports*, 478(1):1–69, 2009.
30. A. Bovier. Metastability: a potential theoretic approach. International Congress of Mathematicians. Vol. III, 499?518, Eur. Math. Soc., Zurich, 2006.
31. A. Bianchi, A. Bovier and D. Ioffe. Sharp asymptotics for metastability in the random field Curie-Weiss model. Electron. J. Probab. 14, no. 53, 1541?1603, 2009.
32. J. T. Cox and A. Greven. The finite systems scheme: an abstract theorem and a new example. *Measure-valued processes, stochastic partial differential equations, and interacting systems* (Montreal, PQ, 1992), 55?67, CRM Proc. Lecture Notes, 5, Amer. Math. Soc., Providence, RI, 1994.
33. A. Greven, V. Limic and A. Winter. Representation theorems for interacting Moran models, interacting Fisher-Wright diffusions and applications. Electron. J. Probab. 10, no. 39, 1286?1356, 2005.
34. A. Budhiraja, P. Dupuis, and V. Maroulas. Variational representations for continuous time processes. *Ann. Inst. Henri Poincaré Probab. Stat.*, 47(3):725–747, 2011.
35. M. Falcone. Numerical solution of dynamic programming equations. Appendix A in: M. Bardi and I. Capuzzo-Dolcetta. *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*. Birkhäuser, 1997.
36. H.J. Kushner and P. Dupuis. *Numerical Methods for Stochastic Control Problems in Continuous Time*. Springer, 2001.
37. P. Hanggi. Escape from a metastable state. *Journal of Statistical Physics*, 42(1-2):105–148, 1986.
38. C.R. McClain and J.P. Barry. Habitat heterogeneity, disturbance, and productivity work in concert to regulate biodiversity in deep submarine canyons. *Ecology*, 91(4):964–976, 2010.