LAPLACE BELTRAMI OPERATOR IN THE BARAN METRIC AND PLURIPOTENTIAL EQUILIBRIUM MEASURE: THE BALL, THE SIMPLEX AND THE SPHERE

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ABSTRACT. The Baran metric δ_E is a Finsler metric on the interior of $E \subset \mathbb{R}^n$ arising from Pluripotential Theory. We consider the few instances, namely E being the ball, the simplex, or the sphere, where δ_E is known to be Riemannian and we prove that the eigenfunctions of the associated Laplace Beltrami operator (with no boundary conditions) are the orthogonal polynomials with respect to the pluripotential equilibrium measure μ_E of E. We conjecture that this may hold in wider generality.

The considered differential operators have been already introduced in the framework of orthogonal polynomials and studied in connection with certain symmetry groups. In this work instead we highlight the relationships between orthogonal polynomials with respect to μ_E and the Riemannian structure naturally arising from Pluripotential Theory.

CONTENTS

1. INTRODUCTION

1.1. **Potential Theory and polynomials.** The study of Approximation Theory in the complex plane and on the real line (by polynomials and rational functions) is deeply

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related to Logarithmic Potential Theory (i.e., the study of subharmonic functions and the Laplace operator). The relations between Logarithmic Potential Theory and Approximation Theory are manifested in Markov, Bernstein and Nikolski type polynomial inequalities, the asymptotics of optimal polynomial interpolation arrays and Fekete points, *overconvergence phenomena* (i.e. uniformly convergent sequence of polynomials defining a holomorphic function in a larger open set) and its quantitative version, the Bernstein Walsh Theorem , and the asymptotics of orthogonal polynomials, random polynomials and random matrices. Moreover, most of such relations extend to the more general case of weighted polynomials and Logarithmic Potential Theory in presence of an external field. We refer to [52, 51, 54, 55, 49] and the references therein for extensive treatments of these subjects.

More recently a non linear potential theory in multi dimensional complex spaces has been introduced and many analogies with the linear case have been shown, provided there is a suitable "translation" of the quantities that come into the play. *Pluripotential Theory* (see for instance [35, 36]) is the study of plurisubharmonic functions (i.e., functions which are subharmonic along each complex line) and the complex Monge Ampere operator; [9].

Though the lack of linearity makes this new theory much more difficult and requires working with different tools, many connections with polynomial approximation have been extended to this multi dimensional framework; see [16, 18, 38]. Indeed, polynomial inequalities in $\mathbb{C}^n$ are usually obtained by means of Pluripotential Theory, see for instance [4, 2], the Bernstein Walsh Theorem has been extended by Siciak to $\mathbb{C}^n$ [53], to more general complex spaces by Zeriahi [63], and very recently to different polynomial spaces by Bos and Levenberg [21]. In his seminal work [56, 57, 58], Zaharjuta extended the equivalence between (a suitably re-defined version of) the Chebyshev Constant (i.e., the asymptotics of the uniform norms of monic polynomials) and the Transfinite Diameter (i.e., the asymptotics of the maximum of the Vandermonde determinant). Very recently, Berman Boucksom and Nystrom [11, 12] showed that Fekete points converge weak* to the *pluripotential equilibrium measure* of the considered set in $\mathbb{C}^n$ and in much more general settings. This is a deep extension of the one dimensional case which can be obtained only by means of the weighted theory. The work of Berman and Boucksom stimulated different lines of research such as L^2 theory and general orthogonal polynomials [15], the study of multi-variate random polynomials and holomorphic sections [59, 19, 48], the theory of sampling and interpolation arrays [40, 13, 41] and the study of Bernstein Markov measures [45, 20]. From the point of view of Approximation Theory, the widely used heuristic that *the equilibrium measure is the "best" measure for producing uniform polynomial approximations by L^2 projection* has been fully motivated and theoretically explained in [12] also in its multivariate setting.

For a wide class of compact subsets E in $\mathbb{R}^n \subset \mathbb{C}^n$, there is a natural Finsler metric δ_E associated to E called the Baran metric (see (9) below). In particular, for a convex body E (i.e., $E \subset \mathbb{R}^n$ is compact, convex and has non-empty interior) this metric, arising from pluripotential theory, has been well-studied [27, 25, 24, 23, 3, 2]. Baran metric is closely related to polynomial approximation and interpolation. Indeed the *Baran Inequality* (see [1, Th.1.1.4] and [3, 5, 6])

$$\frac{\left|\frac{d}{dt}p(x_0+tv)|_{t=0}\right|}{\sqrt{1-p^2(x_0)}} \leq (\deg p)\,\delta_E(x_0,v), \tag{1}$$
$$\forall x_0 \in \operatorname{int} E, v \in \mathbb{S}^{n-1}, p \in \mathscr{P}(\mathbb{C}^n), \|p\|_E \leq 1,$$

can be understood as a generalization of the classical Bernstein Inequality and has applications in polynomial sampling. For instance, if $E \subset \mathbb{C}^n$ is a compact and polynomial determining set and $N \subset E$ is such that, denoting by d_E the Finsler distance induced by

δ_E (see (10) below), we have

$$\sup_{x\in E}\min_{x_0\in N} d_E(x,x_0)\leq \frac{1}{ck}$$

for some $k\in\mathbb{N}$ and $c>1$, then N is a *norming set* for E and constant $c/(c-1)$ for the space of polynomials of degree not greater than k, i.e.,

$$\|p\|_E\leq \frac{c}{c-1}\|p\|_N,\quad \forall p\in\mathscr{P}^k(\mathbb{C}^n).$$

This essentially follows by the Baran Inequality (1). Even more importantly in connection with the present work, in [24] it is shown that, for E being the simplex or the ball or the sphere, Fekete points of degree k of E (arrays of points maximizing the modulus of the Vandermonde determinant and thus near optimal for polynomial interpolation) have spacing of order $1/k$ on E. These results may be used to construct good sampling sets for polynomials, namely admissible meshes, see [28, 37, 26, 39, 44, 46], that have applications in polynomial approximation and optimization [47].

In what follows we will focus only on the case when the Baran metric turns out to be Riemannian.

The present work attempts on the one hand to (partially) extend to the $\mathbb{C}^n$ case another *connection between polynomials and Potential Theory*, and on the other hand, to highlight how *polynomial L^2 approximation with respect to the equilibrium measure may be regarded as Fourier Analysis on a suitable Riemannian manifold.* These ideas rest upon the relation between the Laplace Beltrami operator relative to the Baran metric and the orthogonal polynomials with respect to the pluripotential equilibrium measure.

We would like to introduce such relations starting by some examples that treat the instances of the interval $[-1,1]$ and the unit sphere.

1.2. **Two motivational examples.**

1.2.1. *Chebyshev polynomials.* The Chebyshev polynomials $T_n(x):=\arccos(n\cos x)$ are the orthogonal polynomials with respect to $\frac{1}{\pi\sqrt{1-x^2}}dx$, the equilibrium measure of the interval $[-1,1]$ as a subset of $\mathbb{C}$, i.e., the unique minimizer of the logarithmic potential $-\int\log|z-w|d\mu(z)d\mu(w)$ among all Borel probability measures μ on the interval $[-1,1]$. Another classical characterization of Chebyshev polynomials is given by the eigenfunctions of the Sturm-Liouville eigenvalue problem

$$\begin{cases}\mathscr{S}[\varphi](x):=(1-x^2)\varphi''(x)-x\varphi'(x)=-\lambda\varphi(x), & x\in]-1,1[\\ \varphi'(x_0)=0, & x_0\in\{-1,1\}\end{cases}. \tag{2}$$

The set of eigenvalues turns out to be $\{n^2:n\in\mathbb{N}\}$ and $\mathscr{S}[T_n]=n^2T_n$.

Instead, we re-write this eigenvalue problem as

$$\frac{1}{\frac{1}{\sqrt{1-x^2}}}\frac{d}{dx}\left(\frac{1}{\sqrt{1-x^2}}(1-x^2)\varphi'(x)\right)=-n^2\varphi(x),\quad x\in]-1,1[. \tag{3}$$

This apparently useless manipulation actually illustrates another property of Chebyshev polynomials. To explain this property, we first recall that the Laplace Beltrami operator relative to a metric g can be written in local coordinates as

$$\Delta_{LB}f=\frac{1}{\sqrt{\det g}}\sum_{i=1}^n\partial_{x_i}\left(\sqrt{\det g}\sum_j g^{i,j}\partial_{x_j}f\right), \tag{4}$$

where $g^{i,j}$ are the components of the inverse of the matrix representing g.

Let us endow $]-1,1[$ with the Riemannian metric $g(x):=\frac{1}{1-x^2}$, we canonically obtain the Riemannian distance $d(x_0,x_1)=\int_{x_0}^{x_1}\frac{1}{\sqrt{1-x^2}}dx$. Note that, up to a re-normalization, the resulting volume form is precisely the equilibrium measure of $[-1,1]$. If we plug $g(x) := \frac{1}{1-x^2}$ in the expression (4) for the Laplace Beltrami operator, we obtain precisely the left hand side of (3). In other words, we observe that:

(†) *Chebyshev polynomials are eigenfunctions of the Laplace Beltrami operator with respect to the density of the equilibrium measure of the interval.*

It is relevant to notice that the density of the equilibrium measure on $[-1,1]$ at x is obtained as the normal (i.e., purely complex) derivative of the Green function of $\mathbb{C}\setminus[-1,1]$ with pole at infinity; see [52, Ch II.1]. This operation has a multidimensional counterpart (see [2]) that, under some assumptions, leads to the so called *Baran metric* ([17, 3]), see equation (9) below.

Remark 1. The observation (†) above can also be understood in a more general framework. To this aim, let us recall that a weighted Riemannian manifold is a triple (M, g, ρ) where (M, g) is a Riemannian manifold and ρ is a positive smooth function on M. In such a setting one defines the *weighted Laplace Beltrami operator* Δ_ρ acting on smooth functions by setting

$$\Delta_\rho u := \frac{1}{\rho\sqrt{\det g}} \sum_{i=1}^n \partial_{x_i} \left(\rho\sqrt{\det g} \sum_j g^{i,j} \partial_{x_j} u \right).$$

It turns out that the classical orthogonal polynomials on the interval $[-1,1]$, e.g., Legendre and Gegenbauer orthogonal polynomials, are indeed eigenfunctions of Δ_ρ on $([-1,1], \frac{1}{1-x^2}, (1-x^2)^\beta)$ with an appropriate choice of β, this has already been shown in [29].

1.2.2. *Spherical harmonics.* We mention another relevant example of this relation between eigenfunctions of the Laplace Beltrami operator with respect to the metric defined by (pluri-)potential theory and the (pluripotential) equilibrium measure. In contrast to case of Chebyshev polynomials, now we work in a multi dimensional setting and the flat euclidean space $\mathbb{C}^n$ is replaced by a complex manifold. A more detailed account of this example requires some preliminary notions in addition to the ones of Subsection 2.1, and so we decided to give the explicit computations in Appendix A, together with the needed facts from pluripotential theory on algebraic varieties. At this stage we only sketch the results to underline the analogy with the case of Chebyshev polynomials.

Let us consider the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ endowed with the *round metric* g induced by the flat metric on $\mathbb{R}^n$ and denote by Δ the Laplace Beltrami operator on $\mathbb{S}^{n-1}$. It is well known that *spherical harmonics* are a dense orthogonal system of $L^2(\mathbb{S}^{n-1})$ which consists of polynomials that are eigenfunctions of Δ.

Let us look at $\mathbb{S}^{n-1}$ as a compact subset of the *complexified sphere* $\mathcal{S}^{n-1} := \{z \in \mathbb{C}^n : \sum z_i^2 = 1\}$. By a fundamental result due to Sadullaev [50], since $\mathcal{S}^{n-1}$ is a irreducible algebraic variety, one can relate (see Appendix A) the traces of polynomials on $\mathcal{S}^{n-1}$ to *pluripotential theory on the complex manifold* of $\mathcal{S}^{n-1}$. On the other hand, due to Lemma 3 below, we can define a smooth Riemannian metric $g_{\mathbb{S}^{n-1}}$ on $\mathbb{S}^{n-1}$ suitably modifying the construction (see eq. (9)) of the Baran metric of convex real bodies. In particular such a definition is given by the generalization of the case of the real interval $[-1,1]$. Indeed, it turns out that $g_{\mathbb{S}^{n-1}} = g$ and its volume form is, up to a constant scaling factor (chosen to make it a probability measure), the pluripotential equilibrium measure of $\mathbb{S}^{n-1}$, as a compact subset of $\mathcal{S}^{n-1}$. In other words

(‡) *the eigenfunctions of the Laplace Beltrami operator of* $(\mathbb{S}^{n-1}, g)$ *are the orthogonal polynomials with respect to the pluripotential equilibrium measure of* $\mathbb{S}^{n-1}$, *seen as a compact subset of* $\mathcal{S}^{n-1}$; see Corollary 1.

1.3. **Our results and conjecture.** The aim of the present paper is to present a conjecture on the extension to the $\mathbb{C}^n$ case of the relation between potential theory and certain Riemannian structure that holds in the examples above. We support it by full proofs of all the few known instances, see Theorems 1 and 2 below, fulfilling the required hypothesis, i.e., their Baran metrics are Riemannian.

Conjecture 1. *Let $\mathcal{C}$ denote either $\mathbb{C}^n$ or any irreducible algebraic sub-variety of it. Let $E \subset \mathcal{C}$ be a fat*[1] *real compact set. Assume that the* Baran metric δ_E *of* E *is a Riemannian metric on* $\operatorname{int}_{\mathbb{R}^n\cap\mathcal{C}} E$, *then the orthonormal polynomials with respect to the pluripotential equilibrium measure* $\mu_{E,\mathcal{C}}$ *of* E *in* $\mathcal{C}$ *are eigenfunctions of the Laplace Beltrami operator relative to the metric* δ_E.

Remark 2. We stress that the orthogonal bases used in our proofs as well as most of their properties are already known in the framework of orthogonal polynomials (see [33], [30], [31] and the references therein). Moreover, our differential operators (i.e., Laplace Beltrami operators with respect to the metrics arising from Pluripotential Theory) turn out to be already studied in relation to certain symmetry groups [33, Ch. 8], but they have not been related to any potential theoretic aspects before. More precisely, the Laplace Beltrami operator on the ball endowed with its Baran metric turns out to be the operator $\mathscr{D}_\mu$ in [33, pg. 142] with the parameter choice $\mu = 0$. Instead, in the simplex case, Δ is precisely the operator defined in [33, eq. 5.3.4] (see equation 29 and Theorem 4 below) if we set (in the authors notation) $\kappa = (0, \dots, 0) \in \mathbb{R}^{n+1}$.

Our goal is precisely to relate such families of functions and their properties to the Riemannian structure that comes from Pluripotential Theory.

Remark 3. Note that we will deal with the *open* sets S^n and B^n but we will refer, by a slight abuse of notation and nomenclature that aims to an easier notation, to *their* Baran metrics δ_{S^n} and δ_{B^n} instead of $\delta_{\overline{S^n}}$ and $\delta_{\overline{B^n}}$.

Theorem 1 (Laplace Beltrami on the Baran Ball)**.** *Let us denote by Δ the Laplace Beltrami operator of the Riemannian manifold (B^n, δ_{B^n}) acting on*

$$\mathscr{C}^2_b(B^n) := \left\{ u \in \mathscr{C}^2(B^n) : \max_{|\alpha|\le 2} \sup_{x\in B^n} |\partial^\alpha u(x)| < \infty \right\},$$

where $B^n := \{x \in \mathbb{R}^n : |x| < 1\}$ and the Baran Metric $\delta_{B^n}(x)$ of the ball, see (12), is represented by the matrix

$$G_{B^n}(x) := \begin{bmatrix} 1 + \frac{x_1^2}{1-\sum_{i=1}^n x_i^2} & \frac{x_1x_2}{1-\sum_{i=1}^n x_i^2} & \cdots & \cdots & \frac{x_1x_n}{1-\sum_{i=1}^n x_i^2} \\ \frac{x_1x_2}{1-\sum_{i=1}^n x_i^2} & 1 + \frac{x_2^2}{1-\sum_{i=1}^n x_i^2} & \cdots & \cdots & \frac{x_2x_n}{1-\sum_{i=1}^n x_i^2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{x_nx_1}{1-\sum_{i=1}^n x_i^2} & \frac{x_nx_2}{1-\sum_{i=1}^n x_i^2} & \cdots & \cdots & 1 + \frac{x_n^2}{1-\sum_{i=1}^n x_i^2} \end{bmatrix}.$$

The operator Δ is symmetric and unbounded, it has discrete spectrum

$$\sigma(\Delta) = \{\lambda_s := s(s+n-1) : s \in \mathbb{N}\}$$

and the eigen-space of λ_s is $\operatorname{span}\{\varphi_\alpha, |\alpha| = s\}$, *where φ_α (see Proposition 4) are orthonormal polynomials with respect to the pluripotential equilibrium measure*

$$\mu_{B^n} := \frac{1}{\sqrt{1-|x|^2}} \chi_{B^n} \operatorname{Vol}_{\mathbb{R}^n} = \operatorname{Vol}_{\delta_{B^n}}.$$

Moreover, Δ can be closed to a self-adjoint operator $\mathcal{D}(\Delta) \to L^2(B^n, \delta_{B^n})$ (having the same spectrum), where

$$\mathcal{D}(\Delta) := \left\{ u \in L^2(B^n, \delta_{B^n}) : \sum_{s=0}^{\infty} \lambda_s^2 \sum_{|\alpha|=s} |\hat{u}_\alpha|^2 < \infty \right\} \subset H^1(B^n, \delta_{B^n})$$

[1]This should be intended as the closure in $\mathcal{C}$ of the interior in $\mathbb{R}^n \cap \mathcal{C}$ of E equals to E itself.

and $\hat{u}_\alpha$ *is the Fourier coefficient* $\int_{B^n} u \frac{\varphi_\alpha}{\|\varphi_\alpha\|^2_{L^2(\mu_{B^n})}} d\mu_{B^n}$.

The operator $\Delta^{1/2}$ *has domain*

$$\mathcal{D}(\Delta^{1/2}) := \left\{ u \in L^2(B^n, \delta_{B^n}) : \sum_{s=0}^{\infty} \lambda_s \sum_{|\alpha|=s} |\hat{u}_\alpha|^2 < \infty \right\} = H^1(B^n, \delta_{B^n}).$$

Here and from now on we denote by α a integer multi-index and by $|\alpha|$ its length. For a precise definition of the Sobolev space $H^1(B^n, \delta_{B^n})$ see Subsection 2.2.2 below.

Theorem 2 (Laplace Beltrami on the Baran Simplex)**.** *Let us denote by* Δ *the Laplace Beltrami operator on the Riemannian manifold* (S^n, δ_{S^n})*, acting on*

$$\mathscr{C}^2_b(S^n) := \left\{ u \in \mathscr{C}^2(S^n) : \max_{|\alpha|\leq 2} \sup_{x \in S^n} |\partial^\alpha u(x)| < \infty \right\},$$

where $S^n := \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i < 1, x_i > 0 \,\forall i = 1, 2, \dots, n\}$ *and the Baran metric* $\delta_{S^n}(x)$ *of the simplex, see Equation* (13)*, is represented by the matrix*

$$G_{S^n}(x) := \begin{bmatrix} x_1^{-1} & 0 & \dots & 0 \\ 0 & x_2^{-1} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & x_n^{-1} \end{bmatrix} + \frac{1}{1 - \sum_{i=1}^n x_i} \begin{bmatrix} 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & \dots & \dots & 1 \end{bmatrix}.$$

The operator Δ *is symmetric and unbounded, it has discrete spectrum*

$$\sigma(\Delta) = \left\{ \lambda_s := s\left(s + \frac{n-1}{2}\right) : s \in \mathbb{N} \right\}$$

and the eigen-space of λ_s *is* $\operatorname{span}\{\psi_\alpha, |\alpha| = s\}$*, where* ψ_α *(see Proposition 5) are orthonormal polynomials with respect to the pluripotential equilibrium measure of the simplex*

$$\mu_{S^n} := \frac{1}{\sqrt{(1 - \sum_{i=1}^n x_i) \prod_{i=1}^n x_i}} \chi_{S^n}(x) \operatorname{Vol}_{\mathbb{R}^n} = \operatorname{Vol}_{\delta_{S^n}}.$$

Moreover, Δ *can be closed to a self-adjoint operator (still denoted by* Δ*)* $\mathcal{D}(\Delta) \to L^2(S^n, \delta_{S^n})$ *(having the same spectrum) where*

$$\mathcal{D}(\Delta) := \left\{ u \in L^2(S^n, \delta_{S^n}) : \sum_{s=0}^{\infty} \lambda_s^2 \sum_{|\alpha|=s} |\hat{u}_\alpha|^2 < \infty \right\} \subset H^1(S^n, \delta_{S^n})$$

and $\hat{u}_\alpha$ *is the Fourier coefficient* $\int_{B^n} u \frac{\psi_\alpha}{\|\psi_\alpha^2|_{L^2(\mu_{S^n})}} d\mu_{S^n}$.

The operator $\Delta^{1/2}$ *has domain*

$$\mathcal{D}(\Delta^{1/2}) := \left\{ u \in L^2(S^n, \delta_{S^n}) : \sum_{s=0}^{\infty} \lambda_s \sum_{|\alpha|=s} |\hat{u}_\alpha|^2 < \infty \right\} = H^1(S^n, \delta_{S^n}).$$

Remark 4. In view of Remark 1 above one may ask if Theorem 1 and Theorem 2 have their counterparts if the considered Riemannian manifold is endowed with a weight ρ. It is possible to prove that this is indeed the case, provided that ρ satisfies $\rho = \eta d \operatorname{Vol}_{\mathbb{R}^n}$, where η is a positive power of the density of μ_{B^n} (or μ_{S^n}) with respect to $\operatorname{Vol}_{\mathbb{R}^n}$ (or Vol_{S^n}, respectively).

Remark 5. In order to better understand how the Baran metrics of the ball and the simplex look like, it is worth recalling their special relation with a certain portion of the sphere.

Let us denote by $(\mathbb{H}^n_+, g_{\mathbb{H}^n_+})$ the *upper unit hemisphere*, i.e., the Riemannian manifold which can be obtained by intersecting the unit sphere $\mathbb{S}^n$ (thought as a sub-manifold of $\mathbb{R}^{n+1}$ endowed with the euclidean metric) with the positive half space $\{\xi \in \mathbb{R}^{n+1} : \xi_{n+1} > 0\}$. The map $\pi : \mathbb{H}^n_+ \to B^n$, $\pi(\xi) := (\xi_1, \dots, \xi_n)$ clearly is a one-to-one $\mathscr{C}^\infty$ map of manifolds. Therefore we can define a metric g on B^n by means of the pull-back operator with respect to $F := \pi^{-1}$:

$$g(v, w) := F^* g_{\mathbb{H}^n_+}(v, w) = g_{\mathbb{H}^n_+}(dFv, dFw), \forall v, w \in TB^n.$$

One can verify by direct computations that indeed $g \equiv \delta_{B^n}$.

Similarly, we can define the map Sqrt : $S^n = \{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i \leq 1\} \to B^n \cap \{x \in \mathbb{R}^n : x_i > 0, \forall i = 1, \dots, n\}$, $\text{Sqrt}(x) := (\sqrt{x_1}, \sqrt{x_2}, \dots, \sqrt{x_n})$, and pull back by Sqrt on S^n the Baran metric of the ball, see (12) and (13) below. Again this new metric indeed coincides with the Baran metric of the simplex; [23].

Question 1. Since the the property of a compact set of having a Riemannian Baran metric is stable under taking the pre-image of such a set via certain polynomial maps, we expect that it is possible to prove the equivalence of Theorem 1 and Theorem 2.

Note that, since the manifolds (B^n, δ_{B^n}) and (S^n, δ_{S^n}) are isometric to certain portions of $\mathbb{S}^n$, the local differential and metric properties of this manifolds are the same as those of $\mathbb{S}^n$. We recall that a Riemannian manifold (M, g) is termed Einstein when its metric tensor is a solution of the *Einstein vacuum field equation*

$$\mathbf{Ric} = kg. \tag{5}$$

Here

$$\mathbf{Ric}_{i,j} := \sum_{l=1}^n (\partial_l \Gamma^l_{j,i} - \partial_j \Gamma^l_{l,i}) + \sum_{l,k=1}^n (\Gamma^l_{l,k}\Gamma^k_{j,i} - \Gamma^l_{j,k}\Gamma^k_{l,i})$$

is the Ricci tensor (written by means of the Christoffel symbols $\Gamma^i_{j,k}$) and $k > 0$. Since it is a well known fact that $(\mathbb{S}^n, g_{\mathbb{S}^n})$ is Einstein, we get the following proposition as a consequence of Remark 5.

Proposition 1. *The unit ball and the unit simplex, endowed with their Baran metric respectively, are* Einstein Manifolds*.*

Since for all cases where the Baran metric is known to be Riemannian it happens that it solves Equation (5), the following question naturally arises.

Question 2. Assume that E is a Baran body in the sense of Definition 1 below. Is it necessary for its Baran metric tensor to solve the Einstein vacuum field equation (5)?

Remark 6. Recently, Zelditch [61] studied the spectral theory of the Laplace Beltrami operator on a real analytic Riemannian manifold M in connection with the Pluripotential Theory of the so called Bruhat-Whitney complexification $M_\mathbb{C}$ of M. In particular, working under the assumption of ergodicity of the geodesic flow, [62, 60] present asymptotic results on the zero distribution of the eigenfunctions and series of functions with random Fourier coefficients. These results closely resemble the relation between the behaviour of zeros of orthogonal polynomials (or random polynomials) and the pluripotential equilibrium measure.

Even though our study is far from being as general as the context of the above references, in the author's opinion our result may be cast within this framework and offer concrete examples where explicit computations are performed. Indeed our Appendix A exactly fits in the framework of [61].

The paper is structured as follows. In Section 2 we furnish all the required definitions from Pluripotential Theory, Operator Theory and Differential Geometry. In Section 3 we prove Theorems 1 and 2, giving a precise spectral characterization of the involved Sobolev spaces. Finally, in Appendix A it is shown how to define the Baran metric on the sphere and its equivalence with the standard round metric.

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2. Preliminaries and tools

2.1. **The Pluripotential Theory framework.** Pluripotential Theory is the study of *plurisubharmonic* functions, i.e., any uppersemicontinuous function $u : \Omega \to [-\infty, +\infty[$ being subharmonic along each one complex dimension affine variety in $\Omega \subseteq_{open} \mathbb{C}^n$. We use the operators $d := \partial + \bar\partial$ and $d^c := i(-\partial + \bar\partial)$, where

$$\partial := \sum_{j=1}^n \frac{\partial}{\partial z_j} \cdot dz_j, \quad \bar\partial := \sum_{j=1}^n \frac{\partial}{\partial \bar z_j} \cdot d\bar z_j.$$

The operator $\mathrm{dd^c}$ is sometimes referred as the complex Laplacian and correspond with the usual Laplacian (up to a scaling factor) when $n = 1$.

Since $\mathrm{dd^c}$ is a linear operator, one can consider $\mathrm{dd^c}\, u$ for a L^1_{loc} function in the sense of currents (distributions on the space of differential forms) and it turns out that, for an uppersemicontinuous function u, $\mathrm{dd^c}\, u \geq 0$ if and only iff u is plurisubharmonic.

The *complex Monge Ampere operator* $(\mathrm{dd^c}\,)^n$ is defined for $\mathscr{C}^2$ functions as

$$(\mathrm{dd^c}\, u)^n := \mathrm{dd^c}\, u \wedge \mathrm{dd^c}\, u \wedge \cdots \wedge \mathrm{dd^c}\, u = c_n \det(\mathrm{dd^c}\, u) d\,\mathrm{Vol}_{\mathbb{C}^n}\,. \tag{6}$$

Clearly trying to define wedge products of factors of the type $\mathrm{dd^c}\, u$ for any plurisubharmonic function u leads to serious difficulties due to the lack of linearity. Bedford and Taylor [9] showed that the definition of equation (6) can be extended to any locally bounded plurisubharmonic function, with $(\mathrm{dd^c}\, u)^n$ being a positive Borel measure.

One may think to plurisubharmonic functions in $\mathbb{C}^n$ as "the correct counterpart" (see [35, Preface]) of subharmonic functions on $\mathbb{C}$, while harmonic functions should be replaced in this multi dimensional setting by *maximal plurisubharmonic functions*, i.e., functions u dominating on any subdomain Ω' any plurisubharmonic function v such that $u \geq v$ on $\partial\Omega'$. Locally bounded maximal plurisubharmonic functions satisfy $(\mathrm{dd^c}\, u)^n = 0$.

The multi dimensional counterpart of the Green function for the unbounded component of the complement of a compact set E is the *pluricomplex Green function* (also known as Siciak-Zaharjuta extremal function) V_E^*. Let $E \subset \mathbb{C}^n$ be a compact set, then we set

$$\begin{aligned} V_E(\zeta) &:= \sup\{u(\zeta), u \in \mathcal{L}(\mathbb{C}^n), u|_E \leq 0\}, \\ V_E^*(z) &:= \limsup_{\zeta \to z} V_E(\zeta). \end{aligned} \tag{7}$$

Here $\mathcal{L}(\mathbb{C}^n)$ is the Lelong class of plurisubharmonic functions on $\mathbb{C}^n$ of logarithmic growth, i.e., $u(z) - \log|z|$ is bounded above at infinity.

It is worth recalling that, as in the one dimensional case, due to [53] (see also [35]) we can express V_E^* by means of polynomials $\mathscr{P}(\mathbb{C}^n)$. That is

$$V_E(\zeta) = \sup\left\{\frac{1}{\deg p}\log^+|p(\zeta)|, p \in \mathscr{P}(\mathbb{C}^n), \|p\|_E \leq 1\right\}.$$

Here $\log^+(x) := \max\{\log x, 0\}$.

The function V_E^* is either identically $+\infty$ or a locally bounded plurisubharmonic function on $\mathbb{C}^n$, maximal on $\mathbb{C}^n \setminus E$ (i.e., $(\mathrm{dd}^c V_E^*)^n$ is a positive Borel measure with support in E) having logarithmic growth at ∞; if the latter case occurs we say that E is *non pluripolar*. In principle V_E^* is only a uppersemicontinuous function. When V_E^* is continuous the compact set E is said to be *regular*. It is worth recalling that it turns out that V_E^* is continuous if and only if V_E^* identically vanishes on E. We will treat only such a case in what follows.

For any non pluripolar compact set $E \subset \mathbb{C}^n$ the *pluripotential equilibrium measure* of E is defined as

$$\mu_E := (\mathrm{dd}^c V_E^*)^n, \tag{8}$$

this is a Borel *probability* measure supported on E. We stress that, since $\mu_E(E) = 1$ for any non pluripolar set [9], the total mass of the measures (and volume forms) that we are going to deal with is not important. We avoid introducing normalizing constants in the metrics to keep the notation simple.

Let E be a real convex body, Baran showed that in this case

$$\delta_E(x, v) := \limsup_{t \to 0^+} \frac{V_E^*(x + itv)}{t} \tag{9}$$

exists for any $x \in \operatorname{int} E$, $v \in \mathbb{R}^n$. We refer to $\delta_E(x, v)$ as the *Baran metric* of E. It is worth mentioning that in many cases $\limsup_{t \to 0^+}$ can be replaced by $\lim_{t \to 0^+}$ in the above definition, [5, 10]. We refer the reader to [25] for a study on the connections among this metric, polynomial inequalities and polynomial sampling. The Baran metric defines in general a Finsler distance on E

$$d_E(x, y) := \inf \left\{ \int_0^1 \delta_E(\gamma(s), \gamma'(s))ds, \gamma \in Lip([0, 1], E), \gamma(0) = x, \gamma(1) = y \right\}, \tag{10}$$

however it may happen that $\delta_E(x, v)$ is indeed Riemannian, i.e.,

$$\delta_E(x, v) = \sqrt{v^t G_E(x) v}$$

for a positive definite matrix $G_E(x)$. Note that $G_E(x)$ is then well defined by the *parallelogram law*. More precisely we have

$$u^T G_E(x) v = \frac{\delta_E^2(x, u + v) - \delta_E^2(x, u - v)}{4}.$$

We believe that the following definition is worth being introduced.

Definition 1 (Baran body)**.** Let $\mathcal{C}$ denote either $\mathbb{C}^n$ or a irreducible algebraic variety of pure dimension n, and let $\mathcal{C}_{\mathbb{R}}$ denote the real points of $\mathcal{C}$. Let $E \subset \mathcal{C}_{\mathbb{R}}$ a compact fat[2] non pluripolar set. If the Baran metric of E is a Riemannian metric for the real interior of E, then we term E a *Baran body*.

In [25], the Baran metrics of the real ball, real simplex are computed (see Theorem 1 and Theorem 2 above), showing in particular that they are Baran bodies. For the sake of completeness, we recall how δ_{B^n} and δ_{S^n} can be computed. The pluricomplex Green function of the real unit ball is given by the Lundin Formula (see for instance [35, Th. 5.4.6])

$$V_{B^n}(x) = \frac{1}{2} \log \left[h \left(\sum_{i=1}^n |x_i|^2 + \left| \sum_{i=1}^n x_i^2 - 1 \right| \right) \right], \quad \forall x \in \mathbb{C}^n, \tag{11}$$

where $h : \mathbb{C} \setminus [-1, 1] \to \mathbb{C} \setminus \{z \in \mathbb{C} : |z| \leq 1\}$ is the inverse Joukowsky map $z \mapsto z + \sqrt{z^2 - 1}$. The function $V_{B^n}(x)$ is differentiable at any $x \notin B^n$ so that one can compute the Baran

[2]This mean that the closure in $\mathcal{C}_{\mathbb{R}}$ of the interior of E in $\mathcal{C}_{\mathbb{R}}$ coincides with E.

metric taking the limit

$$\delta_{B^n}(x,v) := \limsup_{t\to 0^+} \frac{V_{B^n}(x+itv)}{t} = \lim_{t\to 0^+} \partial_{i\cdot v} V^*_{B^n}(x+itv).$$

A direct computation leads to

$$\delta_{B^n}(x,v) = \sqrt{v^t G_{B^n}(x) v},$$

where

$$G_{B^n}(x) := \begin{bmatrix} 1+\frac{x_1^2}{1-\sum_{i=1}^n x_i^2} & \frac{x_1x_2}{1-\sum_{i=1}^n x_i^2} & \dots & \dots & \frac{x_1x_n}{1-\sum_{i=1}^n x_i^2} \\ \frac{x_1x_2}{1-\sum_{i=1}^n x_i^2} & 1+\frac{x_2^2}{1-\sum_{i=1}^n x_i^2} & \dots & \dots & \frac{x_2x_n}{1-\sum_{i=1}^n x_i^2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{x_nx_1}{1-\sum_{i=1}^n x_i^2} & \frac{x_nx_2}{1-\sum_{i=1}^n x_i^2} & \dots & \dots & 1+\frac{x_n^2}{1-\sum_{i=1}^n x_i^2} \end{bmatrix}. \tag{12}$$

The square map $(x_1, x_2, \dots, x_n) \mapsto (x_1^2, x_2^2, \dots, x_n^2)$ is a polynomial map from the unit ball to the simplex satisfying the so called *Klimek condition*, see [35, pg. 196], therefore applying [35, Thm. 5.3.1] one has

$$V_{S^n}(z_1, z_2, \dots, z_n) = 2V_{B^n}(\sqrt{z_1}, \sqrt{z_2}, \dots, \sqrt{z_n}).$$

The chain rule leads to

$$G_{S^n}(x) = diag \begin{pmatrix} \sqrt{x_1} \\ \sqrt{x_2} \\ \vdots \\ \sqrt{x_n} \end{pmatrix}^T G_{B^n}(\sqrt{x_1}, \sqrt{x_2}, \dots, \sqrt{x_n})\, diag \begin{pmatrix} \sqrt{x_1} \\ \sqrt{x_2} \\ \vdots \\ \sqrt{x_n} \end{pmatrix}$$

$$= \begin{bmatrix} x_1^{-1} & 0 & \dots & 0 \\ 0 & x_2^{-1} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & x_n^{-1} \end{bmatrix} + \frac{1}{1-\sum_{i=1}^n x_i} \begin{bmatrix} 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & \dots & \dots & 1 \\ 1 & 1 & \dots & \dots & 1 \end{bmatrix}. \tag{13}$$

Remark 7. To the best of the author's knowledge, these are all the examples of Baran compact sets in $\mathbb{C}^n$ known in the literature. Indeed, such a property seems to be very rare: for instance in [25] some counterexamples are given as the regular hexagon and the cordinate square $[-1, 1]$. We offer a further instance of a Baran manifold in Appendix A: the real sphere as subset of the complexified sphere.

Remark 8. In [7] Barthleme intruduces for the first time a natural Laplace operator on Finsler manifolds. It is an interesting question to investigate if it is possible to prove a statement equivalent to Theorem 1 and Theorem 2 at least for particular cases of Baran manifolds arising from pluripotential theory. A natural starting point for this should be the case of the standard coordinate square $K = [-1, 1]^n$ endowed with its Baran metric, see [14] and references therein,

$$\delta_K(x,v) = \max_{i\in\{1,2,\dots,n\}} \frac{|v_i|}{\sqrt{1-x_i^2}}.$$

We postpone this study for a future work.

2.2. **Differential operators and Sobolev spaces on a Riemannian manifold.**

2.2.1. *Differential operators.* We recall that a linear connnection on a vector bundle $\pi : E \to M$ (built on the differentiable manifold M) is a mapping (here $\mathcal{E}(M)$ is the space of smooth sections of the vector bundle E and $\mathcal{T}(M)$ is the tangent bundle)

$$\begin{aligned} \nabla : \mathcal{T}(M) \times \mathcal{E}(M) &\longrightarrow \mathcal{E}(M) \\ (X, V) &\longrightarrow \nabla_X V \end{aligned}$$

such that it is $\mathscr{C}^\infty$-linear in X, $\mathbb{R}$-linear in V, and for which holds the Liebnitz Rule $\nabla_X(fV) = VX(f) + f\nabla_X(V)$ holds for any $f \in \mathscr{C}^\infty(M)$. In particular we have $\nabla_X f = X(f)$.

Let (M, g) be a (possibly non compact) Riemannian manifold. It is well known that there exists a unique torsion-free linear connection on $\mathcal{T}(M)$ that is compatible with the metric g; namely the *Levi-Civita connection.* Since we will deal only with such a connection we will still denote it by ∇.

Note that, for a given $u \in \mathscr{C}^\infty(M)$, ∇u is a $(1, 0)$ tensor field (i.e., point-wise it is a linear form) having the property that $(X, \nabla u)_g = \nabla_X u = X(u)$ and thus it can be written in local coordinates

$$\nabla u = \sum_j g^{ij} \frac{\partial u}{\partial x_j} dx_j.$$

Here $(\cdot, \cdot)_g$ is the canonical duality induced by g and g^{ij} are the components of the matrix representing g^{-1}. Hence it is convenient to define the tangent vector

$$(\operatorname{grad} u)_i := \left(\sum_j g^{ij} \frac{\partial u}{\partial x_j} \right)_i,$$

namely the *covariant gradient* of u, having the property that $(X, \nabla u)_g = \langle X, \operatorname{grad} u\rangle_g$.

The *divergence* operator acting on $X \in \mathcal{T}(M)$ is defined by

$$\operatorname{div} X := \nabla \cdot X = \frac{1}{\sqrt{\det g}} \sum_i \partial_i(\sqrt{\det g} X^i).$$

Finally we can recall the definition of the *Laplace Beltrami operator* Δ.

$$\Delta u := \operatorname{div}(\operatorname{grad} u) = \frac{1}{\sqrt{\det g}} \sum_i \partial_i(\sqrt{\det g} (\operatorname{grad} u)_i). \tag{14}$$

2.2.2. *Sobolev Spaces.* Let (M, g) be a Riemannian manifold. Let us introduce on $\mathscr{C}^\infty(M)$ the norm

$$\|u\|_{1,2} := \left(\int |u|^2 d\operatorname{Vol}_g \right)^{1/2} + \left(\int |\operatorname{grad} u|^2 d\operatorname{Vol}_g \right)^{1/2},$$

where $|\operatorname{grad} u|^2 = \langle \operatorname{grad} u, \operatorname{grad} u\rangle_g$. Let us denote by $\mathscr{C}^\infty_{1,2}(M)$ the space $\{u \in \mathscr{C}^\infty(M), \|u\|_{1,2} < \infty\}$.

The *Sobolev space* $H^1(M, g)$ is defined as the closure of $\mathscr{C}^\infty_{1,2}(M)$ with respect to $\|\cdot\|_{1,2}$ in the space of square integrable functions, also we introduce the space $H^1_0(M, g)$ defined as the closure of $\mathscr{C}^\infty_c(M)$ in the same norm. Note that in general $H^1_0(M, g) \subseteq H^1(M, g)$.

An important fact about Sobolev spaces and manifold is that the above two spaces may coincide, that is

$$H^1_0(M, g) \equiv H^1(M, g) \tag{15}$$

Our interest in this phenomena is mainly due to the fact that the Laplace operator does not need to be complemented with boundary conditions in such a case.

Indeed, $H^1_0(M, g) \equiv H^1(M, g)$ for any complete Riemannian manifold M; see [34, Th. 3.1]. We recall for the reader's convenience that a Riemannian manifold (M, g) is said to be complete if the metric space (M, d_g) is complete, where

$$d_g(x, y) := \inf \left\{ \int_0^1 \sqrt{\langle \gamma'(s), \gamma'(s)\rangle_{g(\gamma(s))}} ds, \gamma \in Lip([0, 1], M), \gamma(0) = x, \gamma(1) = y \right\}.$$

The Hopf-Rinow Theorem asserts that the completeness of (M,g) is equivalent to the fact that any relatively closed bounded subset of M is compact.

We denote by $\mathscr{C}_b^\infty(M)$ the set uniformly bounded functions that have uniformly bounded partial derivatives of any order. Since for a complete manifold $\mathscr{C}_c^\infty(M) \subseteq \mathscr{C}_b^\infty(M) \subset H^1(M,g)$, it follows that for any complete manifold (M,g), $\mathscr{C}_b^\infty(M)$ is dense in $H^1(M,g)$.

Unfortunately, both $(\operatorname{int} B^n, \delta_{B^n})$ and $(\operatorname{int} S^n, \delta_{S^n})$ fail to be complete: it is very easy to construct a Cauchy sequence in B^n not converging in B^n. For instance take $\{x_k\} := \cos(2^{-k})u$ for any unit vector $u \in \mathbb{R}^n$. Since $d(x_k, x_l) \leq 2^{-\min(k,l)}$, this is a Cauchy sequence, however $x_k \to u \notin B^n$. Nevertheless, one may wonder whether equation (15) holds in these instances. This fact indeed depends on finer properties of the manifolds than completeness. Namely, Masamune [43, 42] showed that equality (15) holds if and only if the metric completion of M lies in the category of manifolds with almost polar boundary.

We recall that the Riemannian manifold $(M \cup \Gamma, g)$ with boundary Γ is said to have *almost polar boundary* if the outer capacity $\operatorname{cap}(\Gamma)$ of Γ vanishes. Here we use the notation $\operatorname{cap}(A)$ for the Sobolev (outer) capacity of the Borel subset A of $M \cup \Gamma$, where for any open subset O of $M \cup \Gamma$ we set

$$\operatorname{cap}(O) := \inf\{\|u\|_{1,2}, u \in \mathscr{C}_c^\infty(M \cup \Gamma), 0 \leq u \leq 1, u|_O \equiv 1\}$$

and for for any Borel subset S we set

$$\operatorname{cap}(A) := \inf\{\operatorname{cap}(O), A \subset O\}.$$

It is clear that one can replace $\mathscr{C}_c^\infty(M \cup \Gamma)$ by $H_0^1(M \cup \Gamma, g)$ in the definition of $\operatorname{cap}(O)$ obtaining an equivalent definition.

At this stage we can observe that ∂B^n fails the *sufficient* condition (see [43, Th. 7]) to be polar

$$\liminf_{\varepsilon \to 0^+} \frac{\log \operatorname{Vol}(\{x \in B^n : d(x, \partial B^n) < \varepsilon\})}{\log \varepsilon} \geq 2. \tag{16}$$

Here the equality case is considered since ∂B^n itself is a manifold (see [43, Th. 7]).

Let us denote by N_ε the set $\{x \in B^n : d(x, \partial B^n) < \varepsilon\}$, we have $N_\varepsilon = B^n \setminus (\cos \varepsilon) \cdot B^n$, moreover

$$\operatorname{Vol}(N_\varepsilon) = \pi \beta(1/2, n/2, 1 - (\cos \varepsilon)^2).$$

Here $\beta(a,b,z)$ denotes the Incomplete Beta Function $\int_0^z t^{a-1}(1-t)^{b-1} dt$. Hence

$$\frac{\operatorname{Vol} N_\varepsilon}{\varepsilon^2} \sim \frac{\operatorname{Vol} N_\varepsilon}{1 - (\cos \varepsilon)^2} \frac{1 - (\cos \varepsilon)^2}{\varepsilon^2} \sim 2 \frac{\operatorname{Vol} N_\varepsilon}{1 - (\cos \varepsilon)^2}, \quad \text{as } \varepsilon \to 0^+.$$

Note that

$$\liminf_{\varepsilon \to 0^+} \frac{\operatorname{Vol} N_\varepsilon}{1 - (\cos \varepsilon)^2} = \lim_{z \to 0^+} \frac{\beta(1/2, n/2, z)}{z} = \lim_{z \to 0^+} z^{-1/2}(1-z)^{n/2-1} = +\infty.$$

Thus we have $\liminf_{\varepsilon \to 0^+} \frac{\operatorname{Vol} N_\varepsilon}{\varepsilon^2} = +\infty$ that in particular implies $\frac{\log \operatorname{Vol} N_\varepsilon}{\log \varepsilon} < 2$ for any $\varepsilon < \varepsilon_0$.

Since the condition (16) is not fulfilled by ∂B^n nor ∂S^n we wonder if the ball and the simplex, endowed with their Baran metrics, are not manifolds with almost polar boundary. Indeed this is the case, as stated in the following proposition. We stress that, since these conclusions are obtained as a consequence of Theorem 1 and Theorem 2 respectively, we cannot use them in the proof of these theorems.

Proposition 2. *The manifolds (B^n, δ_{B^n}) and (S^n, δ_{S^n}) are not manifolds with almost polar boundary and*

$$H^1(B^n, \delta_{B^n}) \neq H_0^1(B^n, \delta_{B^n}), \quad H^1(S^n, \delta_{S^n}) \neq H_0^1(S^n, \delta_{S^n}). \tag{17}$$

Remark 9. We warn the reader that $H^1(B^n, \delta_{B^n}) \neq H^1_0(B^n, \delta_{B^n})$ does not imply in general that the eigenvalue problem $\Delta u = \lambda u$ is not well posed when we do not impose any boundary condition. The motivation depends on the following proposition which allows us to write the weak formulations (25) and (31) of the Laplace Beltrami operator used in the proofs of Theorem 1 and 2 which is based on $\mathscr{C}^\infty_b$ functions (for which the boundary terms appearing in the integration by parts formulas we use vanish).

Proposition 3. *Let (M, g) be (B^n, δ_{B^n}) or (S^n, g_{S^n}). The space $\mathscr{C}^\infty_b(M)$ is dense in $\mathscr{C}^\infty_{1,2}(M)$ with respect to the norm $\|\cdot\|_{1,2}$. Thus $\mathscr{C}^\infty_b(M)$ is dense in $H^1(M, g)$.*

Before proving Proposition 3 we need the following technical Lemmas whose proofs are omitted since it is sufficient to check the statements by easy direct computations.

Lemma 1 (The inverse Baran metric of the ball)**.** *Let us denote by $G^{-1}_{B^n}$ the inverse of the matrix G_{B^n} which represents the Baran metric of the n-dimensional ball. Then we have*

$$G^{-1}_{B^n}(x) := \begin{bmatrix} 1-x_1^2 & -x_1x_2 & \dots & \dots & -x_1x_n \\ -x_2x_1 & 1-x_2^2 & \dots & \dots & -x_2x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -x_nx_1 & \dots & \dots & -x_nx_{n-1} & 1-x_n^2 \end{bmatrix}. \tag{18}$$

The matrix $G^{-1}_{B^n}(x)$ has eigenvalues $\{1, 1-|x|^2\}$, where the eigen-space of 1 is the tangent space at x to the sphere of radius $|x|$ and centred at zero, while the eigen-space of $1-|x|^2$ is the Euclidean normal to this sphere at x.

Lemma 2 (The inverse Baran metric of the simplex)**.** *Let us denote by $G^{-1}_{S^n}$ the inverse of the matrix G_{S^n} which represents the Baran metric of the n-dimensional simplex. Then we have*

$$G^{-1}_{S^n}(x) := \begin{bmatrix} (1-x_1)x_1 & -x_1x_2 & \dots & \dots & -x_1x_n \\ -x_2x_1 & (1-x_2)x_2 & \dots & \dots & -x_2x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -x_nx_1 & \dots & \dots & -x_nx_{n-1} & (1-x_n)x_n \end{bmatrix}. \tag{19}$$

Moreover we have

$$G^{-1}_{S^n}(x) = diag \begin{pmatrix} \sqrt{x_1} \\ \vdots \\ \vdots \\ \sqrt{x_n} \end{pmatrix} G^{-1}_{B^n} diag \begin{pmatrix} \sqrt{x_1} \\ \vdots \\ \vdots \\ \sqrt{x_n} \end{pmatrix}. \tag{20}$$

Proof of Proposition 3. Let us start by considering the case $M = B^n \subset \mathbb{R}^n$. We denote by $\mathbb{S}^n$ the n dimensional unit real sphere endowed with the standard *round metric* $g_{\mathbb{S}^n}$ and we introduce the embedding map

$$E : \mathscr{C}^\infty_{1,2}(M) \to H^1_{\text{even}}(\mathbb{S}^n, g_{\mathbb{S}^n}),$$

where

$$E[f]\left(x_1, \dots, x_n, \pm\sqrt{1 - \sum_{i=1}^n x_i^2}\right) := \frac{1}{\sqrt{2}} \begin{cases} f(x_1, \dots, x_n) & , \sum_{i=1}^n x_i^2 \neq 1 \\ \limsup_{B \ni \xi \to x} f(\xi_1, \dots, \xi_n) & , \sum_{i=1}^n x_i^2 = 1 \end{cases}$$

and

$$\begin{aligned} &H^1_{\text{even}}(\mathbb{S}^n, g_{\mathbb{S}^n}) \\ :=& \left\{ g \in H^1(\mathbb{S}^n, g_{\mathbb{S}^n}),\ g(x_1, \dots, x_{n+1}) = g(x_1, \dots, -x_{n+1}) \operatorname{Vol}_{\mathbb{S}^n} -a.e. \right\}. \end{aligned}$$

We claim that E is an *isometry* of Hilbert spaces.

Before proving this claim we stress that this would conclude the proof for the case of the ball. For, by standard mollifications we can construct a sequence $\{\tilde f_k\}$ of function in $\mathscr{C}^\infty(\mathbb{S}^n)$ converging to $E[f]$ in $H^1(\mathbb{S}^n, g_{\mathbb{S}^n})$. To ensure that $\tilde f_k \in H^1_{\text{even}}(\mathbb{S}^n, g_{\mathbb{S}^n})$ we replace $\tilde f_k$ by $(\hat f_k(x_1, \dots, x_{n+1}) + \hat f_k(x_1, \dots, -x_{n+1}))/2$. Finally define $\{f_k\} := \{E^{-1}[\tilde f_k]\}$ and note that the claim above implies that $f_k \to f$ in $H^1(B^n, \delta_{B^n})$.

We stress that, while the injectivity of E is trivial, one needs to notice that the global boundedness of $\tilde f_k$ together with its derivatives ensure that $E^{-1}[\tilde f_k]$ is a well defined element of $\mathscr{C}^\infty_{1,2}(M)$ which in particular is in $\mathscr{C}^\infty_b(M)$.

Let us go back to prove that E is an isometric embedding. For simplicity we work in the easy case of $n = 2$, the general case can be proved in a completely equivalent way. Consider spherical coordinates

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ \sin\theta \end{pmatrix}.$$

We recall that the round metric represented in these coordinates is

$$g_{\mathbb{S}^2}(\theta, \varphi) := \begin{bmatrix} 1 & 0 \\ 0 & \cos^2\theta \end{bmatrix}$$

and the corresponding volume form can be written $d\operatorname{Vol}_{\mathbb{S}^2} = \cos\theta d\theta d\varphi$. It follows that, for any $h \in H^1(\mathbb{S}^2, g_{\mathbb{S}^2})$ we have

$$\|h\|^2_{H^1(\mathbb{S}^2)} = \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \left(|h|^2 + |\partial_\theta h|^2 + \frac{|\partial_\varphi h|}{\cos^2\theta} \right) \cos\theta d\theta d\varphi.$$

To compute $\|E[f]\|_{H^1(\mathbb{S}^2)}$ we perform the change of variables suggested by the first two components of the spherical coordinates, i.e.,

$$(x_1, x_2) \mapsto (\cos\theta\cos\varphi, \cos\theta\sin\varphi).$$

Then

$$\begin{aligned} &\|E[f]\|^2_{H^1(\mathbb{S}^2)} \\ =& \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \left(|E[f]|^2 + |\partial_\theta E[f]|^2 + \frac{|\partial_\varphi E[f]|}{\cos^2\theta} \right) \cos\theta d\theta d\varphi \\ =& 2\int_0^{2\pi} \int_0^{\pi/2} \left(|E[f]|^2 + |\partial_\theta E[f]|^2 + \frac{|\partial_\varphi E[f]|}{\cos^2\theta} \right) \cos\theta d\theta d\varphi \\ =& 2\int_{B^2} \left(\frac{|f|^2}{2} + (1 - x_1^2 - x_2^2)\frac{|\partial_n f|^2}{2} + \frac{|\partial_t f|^2}{2} \right) \frac{1}{\sqrt{1 - x_1^2 - x_2^2}} dx_1 dx_2 \\ =& \int_{B^2} \left(|f|^2 + |\operatorname{grad} f|^2_{g_{B^2}} \right) d\operatorname{Vol}_{g_{B^2}} \\ =& \|f\|^2_{H^1(B, g_{B^2})}. \end{aligned}$$

Let us now consider the case $M = S^n$. We introduce the embedding map

$$F : \mathscr{C}^\infty_{1,2}(S^n, \delta_{S^n}) \to V,$$

where

$$\begin{aligned} V := \{ &f \in \mathscr{C}^\infty_{1,2}(B^n, \delta_{B^n}), \\ &f(\xi_1, \dots, \xi_j, \dots, \xi_n) = f(\xi_1, \dots, -\xi_j, \dots, \xi_n), \forall j \in \{1, \dots, d\} \} \end{aligned}$$

and

$$F[h]\,(\xi_1,\ldots,\xi_n) := \frac{1}{2}h(\xi_1^2,\ldots,\xi_n^2).$$

Again if the closure of F to $H^1(S^n,\delta_{S^n})$ is an isometric embedding we are done, since, for any given target function $h \in H^1(S^n,\delta_{S^n})$ we can pull back to $\mathscr{C}_b^\infty(S^n)$ any sequence of $\mathscr{C}_b^\infty(B^n)$ approximations to $F[h]$.

To this aim, we introduce the partition $Q_1,\ldots,Q_{2^n}$ of $[-1,1]^n$ given by the coordinate hyperplanes, we denote by $T: S^n \to B^n$ the map $(\xi_1,\ldots,\xi_n) \mapsto (\xi_1^2,\ldots,\xi_n^2) = x$ and we notice that, for any $f \in \mathscr{C}^\infty(S^n)$, we have

$$\int_{B^n\cap Q_j} f\circ T d\,\mathrm{Vol}_{B^n} = \frac{1}{2^n}\int_{S^n} f d\,\mathrm{Vol}_{S^n}\,.$$

Finally we compute

$$\begin{aligned}
&\|F[h]\|_{H^1(B^n,\delta_{B^n})}\\
=&\sum_{j=1}^{2^n}\int_{B^n\cap Q_j}\left(|F[h](\xi)|^2 + |\operatorname{grad} F[h](\xi)|^2_{\delta_{B^n}}\right)d\,\mathrm{Vol}_{B^n}(\xi)\\
=&\frac{1}{4}\sum_{j=1}^{2^n}\int_{B^n\cap Q_j}\left(|h\circ T(\xi)|^2 + Dh^t\circ TJT^t\delta_{B^n}^{-1}JTDh\circ T(\xi)\right)d\,\mathrm{Vol}_{B^n}(\xi)\\
=&\frac{1}{4\cdot 2^n}\sum_{j=1}^{2^n}\int_{T^{-1}(B^n\cap Q_j)}\left(|h(x)|^2 + Dh^t(JT^t\delta_{B^n}^{-1}JT)\circ T^{-1}Dh(x)\right)d\,\mathrm{Vol}_{S^n}(x)\\
=&\frac{1}{4\cdot 2^n}2^n\int_{S^n}\left(|h(x)|^2 + Dh^t(JT^t\delta_{B^n}^{-1}JT)\circ T^{-1}Dh(x)\right)d\,\mathrm{Vol}_{S^n}(x).
\end{aligned}$$

Since, due to equation (20),

$$(JT^t\delta_{B^n}^{-1}JT)\circ T^{-1} = 4diag(\xi)\delta_{B^n}^{-1}diag(\xi)\Big|_{\xi=\sqrt{x}} = 4\delta_{S^n}(x)$$

we conclude that $\|F[h]\|_{H^1(B^n,\delta_{B^n})} = \|h\|_{H^1(S^n,\delta_{S^n})}$. In view of the above reasoning this concludes the proof. □

2.3. **Unbounded linear operators on Hilbert spaces, some tools.** We need to recall some concepts from Operator Theory that allow a more precise and compact formulation of our results. A *linear operator* on a Banach space $\mathscr{B}$ is a couple $(\mathcal{D}_{\mathscr{B}}(T),T)$, where $\mathcal{D}_{\mathscr{B}}(T)$ is a dense linear subspace of $\mathscr{B}$ and T is a linear map $\mathcal{D}_{\mathscr{B}}(T) \to \mathscr{B}$.

Let $(\mathcal{D}_{\mathscr{B}}(T),T)$ be a linear operator. If for any sequence $\{f_n\}$ in $\mathcal{D}_{\mathscr{B}}(T)$ such that

- $\|f_n \to f\|_{\mathscr{B}} \to 0$ for some $f \in \mathscr{B}$,
- there exists $g \in \mathscr{B}$ with $\|Tf_n - g\|_{\mathscr{B}} \to 0$

it follows that $f \in \mathcal{D}_{\mathscr{B}}(T)$ and $Tf = g$, then the operator T is said to be *closed*. If $\mathscr{B}$ is not finite dimensional, the notion of spectrum and set of eigenvalues are not the same. More precisely, we denote by $\sigma(T)$ the *spectrum* of T

$$\sigma(T) := \{z \in \mathbb{C} : T - z\mathbb{I} \text{ is not invertible}\}.$$

Instead, λ is an eigenvalue of T if there exists an element $f \in \mathscr{B}$ such that $Tf = \lambda f$.

If an operator T is not closed we may try to find an extension of it, i.e., $(\tilde{T},\mathcal{D}_{\mathscr{B}}(\tilde{T}))$ such that $\mathcal{D}_{\mathscr{B}}(\tilde{T}) \supset \mathcal{D}_{\mathscr{B}}(T)$ and $\tilde{T}f = Tf$ for any $f \in \mathcal{D}_{\mathscr{B}}(T)$. If we can find such an extension in the category of closed operators, then T is said to be *closable* and its minimal closed extension $\overline{T}$ is termed *the closure* of T.

Now we replace the Banach space $\mathscr{B}$ by an Hilbert space $\mathscr{H}$, clearly the above terminologies are still well defined, since any Hilbert space is in particular Banach.

If for any $f,g \in \mathcal{D}_{\mathscr{H}}(T)$ we have $\langle Tf,g\rangle_{\mathscr{H}} = \langle f,Tg\rangle_{\mathscr{H}}$, then the operator T is said to be *symmetric*. It is a very useful fact that *any symmetric operator is closable to a* *symmetric operator*. Again, if $\mathscr{H}$ is infinite dimensional, one must pay attention to the difference between symmetric and self-adjoint operators.

The adjoint T^* of the operator T is defined by the relation

$$\langle Tf, g\rangle_{\mathscr{H}} = \langle f, T^* g\rangle_{\mathscr{H}}, \forall f \in \mathcal{D}_{\mathscr{H}}(T), g \in \mathcal{D}_{\mathscr{H}}(T^*),$$

where

$$\mathcal{D}_{\mathscr{H}}(T^*) := \{g \in \mathscr{H} : \exists h \in \mathscr{H} \text{ such that } \langle Tf, g\rangle_{\mathscr{H}} = \langle f, h\rangle_{\mathscr{H}}, \forall f \in \mathcal{D}_{\mathscr{H}}(T)\}.$$

Clearly, we term T self-adjoint when the two domains indeed coincide.

The proofs of our results, besides the explicit computations, rely on the following theorem which collects some classical results of Operator Theory; see for instance [32, Ch. 1 and Ch. 4].

Theorem 3. *Let T be a linear non negative unbounded operator on the separable Hilbert space $(\mathscr{H}, \|\cdot\|)$ with domain $\mathcal{D}(T)$. Assume that*

a) T is symmetric,

b) It has discrete real spectrum $\sigma(T) = \{\lambda_j\}_{j\in\mathbb{N}}$ diverging to $+\infty$.

Then

i) the closure $\bar{T}$ of T is a self-adjoint unbounded operator (i.e., T is essentially self-adjoint),

ii) $\sigma(\bar{T}) = \sigma(T)$,

iii) the domain of $\bar{T}$ is

$$\mathcal{D}(\bar{T}) = \{u \in \mathscr{H} : \sum_{j=1}^{\infty} \lambda_j^2 |\hat{u}_j|^2 < \infty\} \tag{21}$$

iv) the quadratic form

$$Q(u) := \langle T^{1/2}u, T^{1/2}u\rangle_{\mathscr{H}}$$

has domain

$$\mathcal{D}(Q) = \{u \in \mathscr{H} : \sum_{j=1}^{\infty} \lambda_j |\hat{u}_j|^2 < \infty\} \tag{22}$$

which is complete in the norm

$$|||u||| := \sqrt{Q(u)} + \|u\|_{\mathscr{H}}.$$

Here and throughout the paper $\hat{u}_j$ denotes the $j-$th Fourier coefficients of the function u.

3. Proofs

The strategy of the proofs of Theorems 1 and 2 is to show that the conditions $a)$ and $b)$ of Theorem 3 holds for T being the Laplace Beltrami operator (with respect to the considered metric), then to conclude applying Theorem 3. This will be done by considering the weak formulation of the Laplace Beltrami operator and performing explicit computations on a suitable orthogonal system.

3.1. Orthogonal polynomials in $L^2_{\mu_{B^n}}$.

The following family of orthogonal functions on the unit ball has been first introduced in the Approximation Theory framework, indeed the formula we will use is a special case of orthogonal polynomials for certain radial weight functions; see [33, Ch. 5].

Proposition 4 ([33])**.** *Let us set for any $\alpha \in \mathbb{N}^n$*

$$\varphi_\alpha := T_{\alpha_n}\left(\frac{x_n}{\sqrt{1-\sum_{k=1}^{n-1} x_k^2}}\right)\prod_{j=1}^{n-1}(1-\sum_{k=1}^{j-1} x_k^2)^{\alpha_j/2} C_{\alpha_j}^{\gamma_j}\left(\frac{x_j}{\sqrt{1-\sum_{k=1}^{j-1} x_k^2}}\right), \tag{23}$$

where T_k is the Chebyshev polynomial of degree k, $\gamma_j := \frac{n-j}{2} + \sum_{k=j+1}^n \alpha_k$ and C_t^s denote the monic Gegenbauer polynomials of degree t (i.e., $C_t^s := J_t^{s-1/2,s-1/2}$ and $J_t^{\alpha,\beta}$ is the monic Jacobi polynomial orthogonal on $[-1,1]$ with respect to the weight $(1-x)^\alpha(1+x)^\beta$).

The set $\{\varphi_\alpha : \alpha \in \mathbb{N}^n\}$ is a dense orthogonal system in $L^2(B^n, \delta_{B^n})$ and

$$\|\varphi_\alpha\|^2_{L^2(B^n,\delta_{B^n})} = \|T_{\alpha_n}\|^2_{-1/2,-1/2} \prod_{j=1}^{n-1} \|C_{\alpha_j}^{\gamma_j}\|^2_{\alpha_j-1/2,\alpha_j-1/2}, \tag{24}$$

where $\|f\|_{a,b} := \left(\int_{-1}^1 |f(t)|^2(1-t)^a(1+t)^b dt\right)^{1/2}$.

Note that the density of the linear subspace $\operatorname{span}\{\varphi_\alpha : \alpha \in \mathbb{N}^n\}$ in $H^1(B^n, \delta_{B^n})$ follows from Proposition 3.

3.2. **Proof of Theorem 1.** We **warn the reader** that we will denote throughout this section by Df the *Euclidean gradient of f*.

Proof of Theorem 1. We start showing that Δ acting on $\mathscr{C}_b^2(B^n)$ is a symmetric operator. Namely, for any $u, v \in \mathscr{C}_b^2(B^n)$, we have

$$\int_{B^n} u\Delta v d\operatorname{Vol}_{B^n} = -\int_{B^n} \langle \operatorname{grad} u, \operatorname{grad} v\rangle_{\delta_{B^n}} d\operatorname{Vol}_{B^n} = \int_{B^n} v\Delta u d\operatorname{Vol}_{B^n}. \tag{25}$$

In order to prove this formula we perform two integrations by parts on $B_r^n := \{x : |x| \le r\}$ letting $r \to 1^-$.

$$\begin{aligned}
&-\int_{B^n} v\Delta u d\operatorname{Vol}_{B^n} = -\int_{B^n} \operatorname{div}(\sqrt{\det \delta_{B^n}} G_{B^n}^{-1} Du) v dx \\
&= \lim_{r\to 1} -\int_{{B^n}_r} \operatorname{div}(\sqrt{\det \delta_{B^n}} G_{B^n}^{-1} Du) v dx \\
&= \lim_{r\to 1} \left(\int_{{B^n}_r} Dv^T G_{B^n}^{-1} Du \sqrt{\det \delta_{B^n}} dx - \int_{\partial {B^n}_r} \nu^T G_{B^n}^{-1} Du \sqrt{\det \delta_{B^n}} d\sigma \right) \\
&= \int_{B^n} \langle \operatorname{grad} u, \operatorname{grad} v\rangle_{\delta_{B^n}} d\operatorname{Vol}_{B^n} - \lim_{r\to 1} \int_{\partial {B^n}_r} \nu^T G_{B^n}^{-1} Du \sqrt{\det \delta_{B^n}} d\sigma \\
&= -\int_{B^n} \operatorname{div}(\sqrt{\det \delta_{B^n}} G_{B^n}^{-1} Dv) u dx + \\
&\qquad \lim_{r\to 1} \int_{\partial {B^n}_r} u\nu^T G_{B^n}^{-1} Dv \sqrt{\det \delta_{B^n}} d\sigma - \lim_{r\to 1} \int_{\partial {B^n}_r} v\nu^T G_{B^n}^{-1} Du \sqrt{\det \delta_{B^n}} d\sigma \\
&= -\int_{B^n} u\Delta v d\operatorname{Vol}_{B^n} + \lim_{r\to 1} \int_{\partial {B^n}_r} u\nu^T G_{B^n}^{-1} Dv \sqrt{\det \delta_{B^n}} d\sigma \\
&\qquad - \lim_{r\to 1} \int_{\partial {B^n}_r} v\nu^T G_{B^n}^{-1} Du \sqrt{\det \delta_{B^n}} d\sigma
\end{aligned}$$

Here ν is the (euclidean) unit outward normal to $\partial {B^n}_r := \{x \in \mathbb{R}^n : |x| = r\}$.

The proof of (25) is concluded if we show that

$$\lim_{r\to 1} \int_{\partial {B^n}_r} v\nu^T G_{B^n}^{-1} Du \sqrt{\det \delta_{B^n}} d\sigma = 0$$

for any $u, v \in \mathscr{C}_b^2(B^n)$. To see this, simply observe (see Lemma 1) that ν is an eigenvector of $G_{B^n}^{-1}$ for the eigenvalue $(\det g)^{-1}|_{|x|=r} = (1-r^2)$, thus we have

$$\begin{aligned}
&\lim_{r\to 1} \int_{\partial B_r} v\nu^T G_B^{-1} Du \sqrt{\det g} d\sigma = \lim_{r\to 1} \sqrt{1-r^2} \int_{\partial B_r} v\partial_\nu u d\sigma \\
&\le \lim_{r\to 1} \sqrt{1-r^2} \|u\|_{\mathscr{C}_b^2(B^n)} \|v\|_{\mathscr{C}_b^2(B^n)} = 0.
\end{aligned}$$

This shows that condition *a*) of Theorem 3 holds for Δ. To conclude the proof we need to show that *b*) holds as well, i.e., there exists a $L^2(B^n, \delta_{B^n})$-orthogonal system in $\mathscr{C}_b^2(B^n)$

dense in $L^2(B^n, \delta_{B^n})$ made of eigenfunctions of Δ such that the corresponding eigenvalues are a positive diverging sequence. We claim that such an orthogonal system is, indeed $\{\varphi_\alpha, \alpha \in \mathbb{N}^n\}$, see Proposition 4.

For the sake of readability we present here the case $n = 2$, which leads to slightly easier notation and computations with respect to the general case. However, all the elements of the proof of the general case are presented in such a simplified exposition. To simplify the notation we denote B^2 by B.

The orthogonal basis of Proposition 4 reads as

$$\varphi_{s,k}(x, y) := (1 - x^2)^{k/2} J_{s-k}^{k,k}(x) T_k\left(\frac{y}{\sqrt{1 - x^2}}\right), 0 \leq k \leq s \in \mathbb{N},$$

where we denoted by $J_m^{\alpha,\beta}$ the m-th Jacobi orthogonal polynomial with respect to $(1 - x)^\alpha (1 + x)^\beta$. We need to verify that

$$\langle -\Delta\varphi_{s,k}, \varphi_{m,l}\rangle_{L^2(B,\delta_B)} = \lambda_{s,k}\delta_{s,m}\delta_{k,l} = s(s + 1)\delta_{s,m}\delta_{k,l}.$$

Since $\varphi_{s,k}$ are elements of $\mathscr{C}_b^\infty(B)$ we can use the above weak formulation (25) to get

$$\langle -\Delta\varphi_{s,k}, \varphi_{m,l}\rangle_{L^2(B,\delta_B)} = \int_B D\varphi_{s,k}^T G_B^{-1} D\varphi_{m,l}\sqrt{\det \delta_B}dxdy.$$

Let us introduce a change of variables

$$(x, z) \mapsto \Psi(x, z) := (x, z\sqrt{1 - x^2}) = (x, y).$$

We denote by $J\psi$ the Jacobian matrix of Ψ so we get

$$\begin{aligned}
&\int_B Df_1^T G_B^{-1} Df_2 \sqrt{\det g}dxdy = \\
&= \int_{-1}^1\int_{-1}^1 D(f_1 \circ \Psi)^T J\Psi^{-T} G_B^{-1} J\Psi^{-1} D(f_2 \circ \Psi)dx\frac{dz}{\sqrt{1 - z^2}} \\
&= \int_{-1}^1\int_{-1}^1 D(f_1 \circ \Psi)^T \begin{bmatrix} 1 - x^2 & 0 \\ 0 & \frac{1-z^2}{1-x^2} \end{bmatrix} D(f_2 \circ \Psi)dx\frac{dz}{\sqrt{1 - z^2}}.
\end{aligned}$$

Note that not only Ψ is a change of variables that diagonalizes G_B^{-1}, it also has the property of giving to the basis functions $\varphi_{s,k}$ a tensor product structure. Indeed we have $\varphi_{s,k} \circ \Psi(x, z) = (1 - x^2)^{k/2} J_{s-k}^{k,k}(x) T_k(z)$, thus

$$\begin{aligned}
&\int_{-1}^1\int_{-1}^1 D(\varphi s, k \circ \Psi)^T \begin{bmatrix} 1 - x^2 & 0 \\ 0 & \frac{1-z^2}{1-x^2} \end{bmatrix} D(\varphi_{m,l} \circ \Psi)dx\frac{dz}{\sqrt{1 - z^2}} \\
&= \int_{-1}^1 \partial_x[(1 - x^2)^{k/2} J_{s-k}^{k,k}(x)]\partial_x[(1 - x^2)^{l/2} J_{m-l}^{l,l}(x)](1 - x^2)dx \times \\
&\quad \int_{-1}^1 T_k(z)T_l(z)\frac{dz}{\sqrt{1 - z^2}} + \\
&\quad \int_{-1}^1 (1 - x^2)^{(k+l)/2-1} J_{s-k}^{k,k}(x) J_{m-l}^{l,l}(x)]dx \times \\
&\quad \int_{-1}^1 \partial_z T_k(z)\partial_z T_l(z)\sqrt{1 - z^2}dz.
\end{aligned}$$

It is well known that

$$\int_{-1}^1 T_k(z)T_l(z)\frac{dz}{\sqrt{1 - z^2}} = 2^{\delta_k}\pi/2\delta_{l,k}.$$

Here we denoted by δ_k the Kroneker delta at 0, i.e., $\delta_k = 1$ if $k = 0$ and vanishes elsewhere.

Also one has $T_k' = kU_{k-1}$, where U_k are the orthogonal Chebyshev polynomials of the second kind, i.e.,

$$\int_{-1}^{1} U_k(z)U_l(z)\sqrt{1-z^2}dz = \pi/2\delta_{l,k}.$$

Using such orthogonality and differentiation relations in the above computation we get

$$\begin{aligned}
&\int_{-1}^{1} \partial_x[(1-x^2)^{k/2} J_{s-k}^{k,k}(x)]\partial_x[(1-x^2)^{l/2} J_{m-l}^{l,l}(x)](1-x^2)dx \cdot \\
&\qquad \int_{-1}^{1} T_k(z)T_l(z)\frac{dz}{\sqrt{1-z^2}} + \\
&\qquad \int_{-1}^{1} (1-x^2)^{(k+l)/2-1} J_{s-k}^{k,k}(x) J_{m-l}^{l,l}(x)]dx \cdot \\
&\qquad \int_{-1}^{1} \partial_z T_k(z)\partial_z T_l(z)\sqrt{1-z^2}dz \\
=&\frac{\pi}{2}\delta_{l,k}\Big(\int_{-1}^{1} \partial_x[(1-x^2)^{k/2} J_{s-k}^{k,k}(x)]\partial_x[(1-x^2)^{k/2} J_{m-k}^{k,k}(x)](1-x^2)dx \ \cdot 2^{\delta_k} \\
&\quad + k^2 \int_{-1}^{1} (1-x^2)^{k-1} J_{s-k}^{k,k}(x) J_{m-k}^{k,k}(x)dx\Big).
\end{aligned} \tag{26}$$

Now, letting $k = l$, we note that

$$\begin{aligned}
&\int_{-1}^{1} \partial_x[(1-x^2)^{k/2} J_{s-k}^{k,k}(x)]\partial_x[(1-x^2)^{k/2} J_{m-k}^{k,k}(x)](1-x^2)dx \\
=&\int_{-1}^{1} \partial_x[J_{s-k}^{k,k}(x)]\partial_x[J_{m-k}^{k,k}(x)](1-x^2)^{k+1}dx \\
&+ \int_{-1}^{1} -kx\partial_x[J_{s-k}^{k,k}(x) J_{m-k}^{k,k}(x)](1-x^2)^k dx \\
&+ k^2 \int_{-1}^{1} x^2 J_{s-k}^{k,k}(x) J_{m-k}^{k,k}(x)(1-x^2)^{k-1}dx.
\end{aligned}$$

Integration by parts in the second term leads to

$$\begin{aligned}
&\int_{-1}^{1} \partial_x[(1-x^2)^{k/2} J_{s-k}^{k,k}(x)]\partial_x[(1-x^2)^{k/2} J_{m-k}^{k,k}(x)](1-x^2)dx \\
=&\int_{-1}^{1} \partial_x[J_{s-k}^{k,k}(x)]\partial_x[J_{m-k}^{k,k}(x)](1-x^2)^{k+1}dx \\
&- 2k^2 \int_{-1}^{1} x^2 J_{s-k}^{k,k}(x) J_{m-k}^{k,k}(x)(1-x^2)^{k-1}dx \\
&+ k \int_{-1}^{1} J_{s-k}^{k,k}(x) J_{m-k}^{k,k}(x)(1-x^2)^k dx \\
&+ k^2 \int_{-1}^{1} x^2 J_{s-k}^{k,k}(x) J_{m-k}^{k,k}(x)(1-x^2)^{k-1}dx.
\end{aligned}$$

We plug this last identity to (26) with $k = l$ to get

$$\begin{aligned}
&\langle -\Delta_B \varphi_{s,k}, \varphi_{m,l}\rangle_{L^2(B,\delta_{B^n})}\\
=&\frac{\pi}{2}\delta_{l,k}2^{\delta_k}\Big(\int_{-1}^1 \partial_x[J^{k,k}_{s-k}(x)]\partial_x[J^{k,k}_{m-k}(x)](1-x^2)^{k+1}dx\\
&+k^2\int_{-1}^1(1-x^2)J^{k,k}_{s-k}(x)J^{k,k}_{m-k}(x)(1-x^2)^{k-1}\\
&+k\int_{-1}^1 J^{k,k}_{s-k}(x)J^{k,k}_{m-k}(x)(1-x^2)^k dx\Big)\\
=&\frac{\pi}{2}\delta_{l,k}2^{\delta_k}\Big(\int_{-1}^1 \partial_x[J^{k,k}_{s-k}(x)]\partial_x[J^{k,k}_{m-k}(x)](1-x^2)^{k+1}dx\\
&+k(k+1)\int_{-1}^1 J^{k,k}_{s-k}(x)J^{k,k}_{m-k}(x)(1-x^2)^k dx\Big).
\end{aligned}$$

The last term in the sum vanishes for any $m \neq s$, this follows from the orthogonality of the Jacobi polynomials. When instead $m = s$ we have (see for instance [33])

$$k(k+1)\int_{-1}^1 (J^{k,k}_{s-k}(x)(x))^2(1-x^2)^k dx = \frac{k(k+1)2^{2k+1}(s!)^2}{(2s+1)(s+k)!(s-k)!}.$$

For the first term, we recall that $\frac{d}{dx}J^{k,k}_{s-k} = \frac{s+k+1}{2}J^{k+1,k+1}_{s-k-1}$, Hence, using again the orthogonality, we get

$$\begin{aligned}
&\int_{-1}^1 \partial_x[J^{k,k}_{s-k}(x)]\partial_x[J^{k,k}_{m-k}(x)](1-x^2)^{k+1}dx\\
=&\left(\frac{s+k+1}{2}\right)^2\int_{-1}^1 (J^{k+1,k+1}_{s-k-1})^2(1-x^2)^{k+1}dx\\
=&(s+k+1)\frac{2^{2k+1}(s!)^2}{(2s+1)(s+k)!(s-k-1)!}.
\end{aligned}$$

We finally compute

$$\begin{aligned}
&\langle -\Delta_B \varphi_{s,k}, \varphi_{m,l}\rangle_{L^2(B,\delta_{B^n})}\\
=&\frac{\pi}{2}\delta_{l,k}2^{\delta_k}\delta_{s,m}\frac{2^{2k+1}(s!)^2}{(2s+1)(s+k)!(s-k)!}\Big(k(k+1)+(s+k+1)(s-k)\Big)\\
=&s(s+1)\frac{\pi}{2}\delta_{l,k}\delta_{s,m}2^{\delta_k}\frac{2^{2k+1}(s!)^2}{(2s+1)(s+k)!(s-k)!}\\
=&s(s+1)\|\varphi_{s,k}\|^2_{L^2(B,)}\delta_{l,k}\delta_{s,m}.
\end{aligned}$$

Here the last line is due to Proposition 4. □

3.3. Orthogonal polynomials in $L^2_{\mu_{S^n}}$.

Proposition 5 ([33]). *Let us set for any $\alpha \in \mathbb{N}^n$ and $x \in S^n$*

$$\psi_\alpha(x) := \prod_{j=1}^n \left(1-\sum_{k=1}^{j-1}x_k\right)^{\alpha_j} J^{a_j,-1/2}_{\alpha_j}\left(\frac{2x_j}{1-\sum_{k=1}^{j-1}x_k}-1\right), \tag{27}$$

where $J^{a,b}_m$ is the m-th Jacobi polynomial of parameters a, b and

$$a_j := 2\sum_{k=1}^{\min(n,j+1)}\alpha_k + \frac{n-j-1}{2}.$$

The set $\{\psi_\alpha : \alpha \in \mathbb{N}^n\}$ is a dense orthogonal system in $L^2(S^n, \delta_{S^n})$.

This result (see Th. 8.2.2 in [33]) plays a key role in our proof.

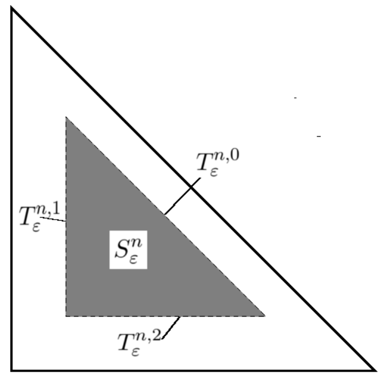

FIGURE 1. Some notations used in the proof of Theorem 2.

Theorem 4 ([33]). *Let us introduce the differential operator*

$$\mathcal{D}f := \sum_{i=1}^{n} x_i \partial_{i,i}^2 f - 2 \sum_{1\le i<j\le n} x_i x_j \partial_{i,j}^2 f + \frac{1}{2}\sum_{i=1}^{n}(1-(n+1)x_i)\partial_i f. \tag{28}$$

Then we have

$$\mathcal{D}\psi_\alpha = |\alpha|\left(|\alpha| + \frac{n+1}{2}\right)\psi_\alpha. \tag{29}$$

It will turn out in the proof of Theorem 2 that $\mathcal{D}$ agrees on smooth functions with the aforementioned Laplace Beltrami operator with respect to the Baran metric of the simplex.

3.4. **Proof of Theorem 2.**

Proof of Theorem 2. Since G_{S^n} blows up at the boundary of the simplex, our strategy is to carry out integration by parts on certain exhausting subsets of S^n and then we take the limit approaching the boundary. To this aim it is convenient to introduce, see Figure 1, the following notations for $\varepsilon > 0$

$$S_\varepsilon^n := \left\{ x \in S^n : x_i > \varepsilon, (1 - \sum_{k=1}^{n} x_i) > \varepsilon \right\},$$
$$T_\varepsilon^{n,0} := \left\{ x \in \partial S_\varepsilon^n : (1 - \sum_{k=1}^{n} x_i) = \varepsilon \right\},$$
$$T_\varepsilon^{n,i} := \{x \in \partial S_\varepsilon^n : x_i = \varepsilon\},\ i = 1, \dots, n.$$

Also let ν_i be the Euclidean unit normal to $T_\varepsilon^{n,i}$ (for any $\varepsilon > 0$). We note that $\partial S_\varepsilon^n = \cup_{j=0}^{n} T_\varepsilon^{n,i}$.

Following the first part of the proof of Theorem 1, we show that Δ is a symmetric operator on the space $\mathscr{C}_b^\infty(S^n)$ which is dense (see Proposition 3) in $H^1(S^n, \delta_{S^n})$.

To this aim we perform integration by parts twice. Let $u, v \in \mathscr{C}_b^\infty(S^n)$, then

$$
\begin{aligned}
&-\int_{S^n} v\Delta u d\,\mathrm{Vol}_{S^n} = -\int_{S^n} \mathrm{div}(\sqrt{\det \delta_{S^n}} G_{S^n}^{-1} Du) v dx \\
&= \lim_{\varepsilon\to 0^+} -\int_{S^n_\varepsilon} \mathrm{div}(\sqrt{\det \delta_{S^n}} G_{S^n}^{-1} Du) v dx \\
&= \lim_{\varepsilon\to 0^+} \left(\int_{S^n_\varepsilon} Dv^T G_{S^n}^{-1} Du \sqrt{\det \delta_{S^n}} dx - \sum_{i=0}^n \int_{T_\varepsilon^{n,i}} v\nu_i^T G_{S^n}^{-1} Du \sqrt{\det \delta_{S^n}} d\sigma \right) \\
&= \int_{S^n} \langle \mathrm{grad}\, u, \mathrm{grad}\, v\rangle_{\delta_{S^n}} d\,\mathrm{Vol}_{S^n} - \sum_{i=0}^n \lim_{\varepsilon\to 0^+} \int_{T_\varepsilon^{n,i}} \nu_i^T G_{S^n}^{-1} Du \sqrt{\det \delta_{S^n}} d\sigma \\
&= -\int_{S^n} v\Delta u d\,\mathrm{Vol}_{S^n} + \\
&\qquad \lim_{\varepsilon\to 0^+} \int_{T_\varepsilon^{n,i}} \left(u\nu_i^T G_{S^n}^{-1} Dv - v\nu_i^T G_{S^n}^{-1} Du \right) \sqrt{\det \delta_{S^n}} d\sigma.
\end{aligned}
$$

Thus we need to prove that for any $u, v \in \mathscr{C}_b^\infty(S^n)$ and any $i \in \{0, 1, \dots, n\}$ we have

$$
\lim_{\varepsilon\to 0^+} \int_{T_\varepsilon^{n,i}} u\nu_i^T G_{S^n}^{-1} Dv \sqrt{\det \delta_{S^n}} d\sigma = 0. \tag{30}
$$

To see this, it is sufficient to notice (using Lemma 2) that for any $x \in T_\varepsilon^{n,0}$

$$
\nu_0^T G_{S^n}^{-1} \sqrt{\det \delta_{S^n}} = \sqrt{\frac{\varepsilon}{\prod_{k=1}^n x_k}} (x_1, x_2, \dots, x_n)^T
$$

and for any $x \in T_\varepsilon^{n,i}$, $i = 1, 2, \dots, n$

$$
\begin{aligned}
&\nu_i^T G_{S^n}^{-1} \sqrt{\det \delta_{S^n}} = \\
&\sqrt{\frac{\varepsilon}{(1-\varepsilon-\sum_{k=1,k\neq i}^n x_k)\prod_{k=1,k\neq i}^n x_k}} (x_1, x_2, \dots, x_{i-1}, 1-\varepsilon, x_{i+1}, \dots, x_n)^T.
\end{aligned}
$$

Therefore we have

$$
\left| \int_{T_\varepsilon^{n,0}} u\nu_i^T G_{S^n}^{-1} Dv \sqrt{\det \delta_{S^n}} d\sigma \right| \leq \sqrt{\varepsilon} n \max_{S^n}(|Dv|_\infty |u|) \left\| \prod_{k=1}^n \sqrt{x_k} \right\|_{L^1(T_\varepsilon^{n,0})} \to 0
$$

and, for any $i = 1, 2, \dots, n$

$$
\begin{aligned}
&\left| \int_{T_\varepsilon^{n,i}} u\nu_i^T G_{S^n}^{-1} Dv \sqrt{\det \delta_{S^n}} d\sigma \right| \\
&\leq \sqrt{\varepsilon} n \max_{S^n}(|Dv|_\infty |u|) \left\| \left((1-\varepsilon - \sum_{k=1,k\neq i}^n x_k) \prod_{k=1,k\neq i}^n x_k \right)^{-1/2} \right\|_{L^1(T_\varepsilon^{n,i})} \to 0
\end{aligned}
$$

and thus (30) holds. This shows that Δ is a symmetric operator on $\mathscr{C}_b^\infty(S^n)$, i.e., for any such u and v

$$
\int_{S^n} u\Delta v d\,\mathrm{Vol}_{S^n} = -\int_{S^n} \langle \mathrm{grad}\, u, \mathrm{grad}\, v\rangle_{\delta_{S^n}} d\,\mathrm{Vol}_{S^n} = \int_{S^n} v\Delta u d\,\mathrm{Vol}_{S^n}. \tag{31}
$$

Now we want to show that Δ has discrete spectrum $\sigma(\Delta_S) = \{\lambda_s := s(s+\frac{n-1}{2}) : s \in \mathbb{N}\}$ and the eigen-space of λ_s is $\mathrm{span}\{\psi_\alpha, |\alpha| = s\}$ (see Proposition 5).

Instead of proving this directly, we rely on the known properties of the basis $\{\psi_\alpha\}$, namely (29), and we simply show that for smooth functions

$$
\Delta f = \mathcal{D} f, \tag{32}
$$

this allows us to characterize $\sigma(\Delta)$ due to Theorem 4. Then we apply Theorem 3 and the result follows.

We introduce the notation $h(x) := (1 - \sum_{k=1}^n x_k) \prod_{k=1}^n x_k$. It is worth noting that

$$\sqrt{h(x)}\partial_i \frac{x_i}{\sqrt{h(x)}} = \frac{1 - \sum_{k\neq i} x_k}{2(1 - \sum_{k=1}^n x_k)} = \frac{1}{2}\left(1 + \frac{x_i}{1 - \sum_{k=1}^n x_k}\right).$$

For any smooth f we have

$$\begin{aligned}
&\Delta_{S^n} f \\
=&\sqrt{h(x)} \sum_{i=1}^n \partial_i \left(\frac{x_i}{\sqrt{h(x)}} (\partial_i f - \sum_{j=1}^n x_j \partial_j f) \right) \\
=&\sum_{i=1}^n \left\{ \sqrt{h(x)}\partial_i \frac{x_i}{\sqrt{h(x)}} (\partial_i f - \sum_{j=1}^n x_j\partial_j f) + x_i \partial_i (\partial_i f - \sum_{j=1}^n x_j \partial_j f) \right\} \\
=&\sum_{i=1}^n \left\{ \frac{1}{2}\left(1 + \frac{x_i}{1 - \sum_{k=1}^n x_k}\right) (\partial_i f - \sum_{j=1}^n x_j \partial_j f) + x_i \partial_i (\partial_i f - \sum_{j=1}^n x_j \partial_j f) \right\} \\
=&-\frac{1}{2} \sum_{j=1}^n x_j \partial_j f \cdot \sum_{i=1}^n \left(1 + \frac{x_i}{1 - \sum_{k=1}^n x_k}\right) + \\
&\frac{1}{2}\sum_{i=1}^n \partial_i f + \frac{1}{2(1 - \sum_{k=1}^n x_k)} \sum_{i=1}^n x_i \partial_i f \;+ \\
&\sum_{i=1}^n \left\{ x_i \partial_i^2 f - x_i \sum_{j\neq i} x_j \partial_{i,j}^2 f - x_i^2 \partial_i^2 f - x_i \partial_i f \right\}
\end{aligned}$$

$$\begin{aligned}
=&\sum_{i=1}^n x_i(1 - x_i)\partial_i^2 f - 2 \sum_{1\leq j<i\leq n} x_i x_j \partial_{i,j}^2 f + \frac{1}{2}\sum_{i=1}^n \partial_i f \\
&+ \left(\sum_{i=1}^n x_i \partial_i f\right) \cdot \left\{ -\sum_{i=1}^n \left(\frac{1}{2} + \frac{x_i}{2\left(1 - \sum_{k=1}^n x_k\right)} \right) + \frac{1}{2\left(1 - \sum_{k=1}^n x_k\right)} - 1 \right\} \\
=&\sum_{i=1}^n x_i(1 - x_i)\partial_i^2 f - 2 \sum_{1\leq j<i\leq n} x_i x_j \partial_{i,j}^2 f + \frac{1}{2}\sum_{i=1}^n \partial_i f \\
&+ \left(\sum_{i=1}^n x_i \partial_i f\right) \cdot \left\{ -\frac{n+2}{2} + \frac{-\sum_{i=1}^n x_i + 1}{2(1 - \sum_{k=1}^n x_k)} \right\} \\
=&\sum_{i=1}^n x_i(1 - x_i)\partial_i^2 f - 2 \sum_{1\leq j<i\leq n} x_i x_j \partial_{i,j}^2 f + \frac{1}{2}\sum_{i=1}^n \partial_i f \\
=&\sum_{i=1}^n x_i \partial_{i,i}^2 f - 2 \sum_{1\leq i<j\leq n} x_i x_j \partial_{i,j}^2 f + \frac{1}{2}\sum_{i=1}^n (1 - (n+1)x_i)\partial_i f \\
=&\mathcal{D} f.
\end{aligned}$$

□

3.5. **Proof of Proposition 2.** Let us first recall a result of Masamune [42, Th. 3] which the proof of Proposition 2 relies. Assume (M, g) to be a compact Riemannian manifold and let Σ be a submanifold of M, let us define Δ_M as the standard Laplace Beltrami operator acting on $\mathscr{C}_c^\infty(M \setminus \Sigma)$. Then

(33) Δ_M is essentially self-adjoint if and only if $\dim(M) - \dim(\Sigma) > 3$.

Proof of Proposition 2. Let $M := S^n \subset \mathbb{R}^{n+1}$ and $\Sigma := \{x \in M : x_{n+1} = 0\}$. Also introduce the notation $(x_1, x_2, \dots, x_n, x_{n+1}) = (\xi, x_{n+1})$.

Let us assume for a contradiction that $\mathscr{C}_c^\infty(B^n)$ is dense in $H^1(B^n, \delta_{B^n})$. In view of the proof of Proposition 3 we have

$$\begin{aligned} \mathscr{H} &:= (\mathscr{C}_c^\infty(B^n), \|\cdot\|_{1,2,\delta_{B^n}}) \leftrightarrows_{\text{isometry}} \left(\mathscr{C}_{c,\text{even}}^\infty(M \setminus \Sigma), \|\cdot\|_{1,2,g_M}\right) =: \mathscr{E}_1. \\ &(\mathscr{C}_c^\infty(B^n), \|\cdot\|_{1,2,\delta_{B^n}}) \leftrightarrows_{\text{isometry}} \left(\mathscr{C}_{c,\text{odd}}^\infty(M \setminus \Sigma), \|\cdot\|_{1,2,g_M}\right) =: \mathscr{E}_2. \end{aligned} \tag{34}$$

Here $\mathscr{C}_{c,\text{odd}}^\infty(M \setminus \Sigma)$ denotes the subspace

$$\{u \in \mathscr{C}_c^\infty(M \setminus \Sigma), g_{\mathbb{S}^n}),\ u(\xi, x_{n+1}) = -u(\xi, -x_{n+1}) \,\forall (\xi, x_{n+1}) \in M \setminus \Sigma\}$$

and $\mathscr{C}_{c,\text{even}}^\infty(M \setminus \Sigma)$ is defined similarly. Note that, given $u \in \mathscr{C}_c^\infty(M \setminus \Sigma)$ we can define $u_{\text{even}} := 1/2(u(\xi, x_{n+1}) + u(\xi, -x_{n+1})) \in \mathscr{E}_1$ and $u_{\text{odd}} := 1/2(u(\xi, x_{n+1}) - u(\xi, -x_{n+1})) \in \mathscr{E}_2$ such that $u = u_{\text{even}} + u_{\text{odd}}$.

The assumption that $\mathscr{C}_c^\infty(B^n)$ is dense in $H^1(B^n, \delta_{B^n})$ together with Theorem 1 and the isometry property of the map E in the proof of Proposition 3 implies that the Laplace Beltrami operator Δ_1 acting on $\mathscr{E}_1$ and Δ_2 acting on $\mathscr{E}_2$ are essentially self-adjoint. Moreover, since $\Delta_M u = \Delta_1 u_{\text{even}} + \Delta_2 u_{\text{odd}}$ for any $u \in \mathscr{C}_c^\infty(M \setminus \Sigma)$, it follows that Δ_M itself is essentially self-adjoint.

On the other hand, $\dim \Sigma = n-1$ and $\dim M = n$, this is in contrast with Masamune's result (33) and thus $\mathscr{C}_c^\infty(B^n)$ can not be dense in $H^1(B^n, \delta_{B^n})$ and thus $H^1(B^n, \delta_{B^n}) \neq H_0^1(B^n, \delta_{B^n})$. Note that, in view of [43, Th. 1], this is equivalent to the fact that B^n is not a manifold with almost polar boundary.

The proof for the simplex can be done in a equivalent way but using the map F defined in the proof of Proposition 3 instead of the map E. □

Appendix A. Pluripotential Theory on the complexified sphere and spherical harmonics

In this section we consider $\mathbb{S}^{n-1}$ as a compact subset of the complexified sphere $\mathcal{S}^{n-1} := \{z \in \mathbb{C}^n : \sum z_i^2 = 1\}$. We can consider the space $\text{psh}(\mathcal{S}^{n-1})$ of plurisubharmonic functions on the complex manifold $\mathcal{S}^{n-1}$ and form the usual upper envelope

$$V_{\mathbb{S}^{n-1}}^*(z, \mathcal{S}^{n-1}) := \limsup_{\mathcal{S}^{n-1} \ni \zeta \to z} \sup \left\{u(\zeta) : u \in \mathcal{L}(\mathcal{S}^{n-1}), u|_{\mathbb{S}^{n-1}} \leq 0\right\},$$

where $\mathcal{L}(\mathcal{S}^{n-1})$ denotes the space of plurisubharmonic functions u on $\mathcal{S}^{n-1}$ such that $u - \frac{1}{2} \log \sum_{i=1}^{n-1} |z_i|^2$ is bounded above as $\sum_{i=1}^{n-1} |z_i| \to \infty$ along $\mathcal{S}^{n-1}$, *defining* the *extremal plurisubharmonic function*; compare this definition with equation (7). This is a locally bounded plurisubharmonic function which is maximal on $\mathcal{S}^{n-1} \setminus \mathbb{S}^{n-1}$; [63, 8].

On the other hand, it is clear that $\mathcal{S}^{n-1}$ is a irreducible algebraic sub-variety of $\mathbb{C}^n$ of pure dimension $n-1$,, hence we can use the result of Sadullaev [50] to get

$$V_{\mathbb{S}^{n-1}}^*(z, \mathcal{S}^{n-1}) = \limsup_{\mathcal{S}^{n-1} \ni \zeta \to z} \sup \left\{\frac{1}{\deg p} \log^+ |p(\zeta)| : p \in \mathscr{P}(\mathbb{C}^n), \|p\|_{\mathbb{S}^{n-1}} \leq 1\right\}.$$

Here $\mathscr{P}$ denotes the space of algebraic polynomials with complex coefficients. It is worth stressing that here deg denotes the degree of a polynomial on $\mathbb{C}^n$, not the degree over the coordinates ring of $\mathcal{S}^{n-1}$.

In [22, Prop. 4.1] the authors prove the formula

$$V_{\mathbb{S}^{n-1}}^*(z, \mathcal{S}^{n-1}) = \frac{1}{2} \log \left(|z|^2 + \sqrt{|z|^4 - 1}\right), \forall z \in \mathcal{S}^{n-1}, \tag{35}$$

we note that this function can be used to define the Baran metric on the sphere, due to the following differentiability property.

Lemma 3. *Let $x \in \mathbb{S}^{n-1}$, the function $V_{\mathbb{S}^{n-1}}(\cdot, \mathcal{S}^{n-1})$ has right tangent directional derivative at x in any direction $i \cdot v$, for any $v \in T_x\mathbb{S}^{n-1}$, that is*

$$\partial^+_{i\cdot v} V_{\mathbb{S}^{n-1}}(x, \mathcal{S}^{n-1}) := \frac{d}{dt} V_{\mathbb{S}^{n-1}}(\gamma(t), \mathcal{S}^{n-1})|_{t=0} \in \mathbb{R},$$

where $\gamma : [0,1] \mapsto \mathcal{S}^{n-1}$ is any differentiable arc with $\gamma(0) = x$, $\gamma'(0^+) = i \cdot v$.

Moreover we have $\partial_{i\cdot v} V_{\mathbb{S}^{n-1}}(x, \mathcal{S}^{n-1}) = |v|$.

Proof. The problem is clearly rotation independent. We can thus assume $x = (1, 0, \ldots, 0) = e_1$ and $v = |v|(0, 1, 0, \ldots, 0) = |v|e_2$ without loss of generality.

Let us introduce the arc

$$z(t) := \sqrt{1 + |v|^2 \log^2(1+t)}e_1 + |v| \log(1+t)e_2, t \in [0, +\infty[.$$

It is easy to verify that z enjoys the properties

$$\begin{aligned} &z(t) \in \mathcal{S}^{n-1}, \forall t \in [0, +\infty[,\\ &z(0) = x,\\ &\frac{d}{dt} z(0^+) = i \cdot v. \end{aligned}$$

Thus we are left to show that, setting $u(t) := V^*_{\mathbb{S}^{n-1}}(z(t), \mathcal{S}^{n-1})$, we have $\frac{d}{dt}u(0^+) = |v|$.

Let us note first that $|z(t)|^2 = 1 + 2|v|^2 \log^2(1+t)$, then we can compute

$$\begin{aligned} u(t) =&\frac{1}{2} \log \left[1 + 2|v|^2 \log^2(1+t) + \sqrt{4|v|^2 \log^2(1+t)(1 + |v|^2 \log^2(1+t))}\right]\\ =&\frac{1}{2} \log \left[1 + 2|v|^2 \log^2(1+t) + 2|v| \log(1+t)\sqrt{1 + |v|^2 \log^2(1+t)}\right]\\ \sim&\frac{1}{2} \log \left[1 + 2|v|^2 t^2 + 2|v|t\sqrt{1 + |v|^2 t^2}\right]\\ \sim&\frac{1}{2} \log(1 + 2|v|t) \sim |v|t, \quad \text{as } t \to 0^+. \end{aligned}$$

Therefore

$$u'(0^+) = \lim_{t\to 0^+} \frac{u(t) - u(0)}{t} = \lim_{t\to 0^+} \frac{u(t)}{t} = |v|.$$

□

Due to Lemma 3 we can define the *Baran metric* on the real unit sphere by setting

$$\delta_{\mathbb{S}^{n-1}}(x, v) := \partial_{i\cdot v} V^*_{\mathbb{S}^{n-1}}(x, \mathcal{S}^{n-1}) = |v|,$$

note the analogy with the partial derivative taken in the Lemma with the definition of the Baran metric in the standard "flat" case.

Using the Parallelogram Identity we can define for any $x \in \mathbb{S}^{n-1}$ and any $u, v \in T_x\mathbb{S}^{n-1}$ the scalar product related to the Baran metric as

$$\begin{aligned} \langle u, v\rangle_{g_{\mathbb{S}^{n-1}}(x)} :=&\frac{\delta^2_{\mathbb{S}^{n-1}}(x, u+v) - \delta^2_{\mathbb{S}^{n-1}}(x, u-v)}{4}\\ =&\frac{|u+v|^2 - |u-v|^2}{4} = \langle u, v\rangle_{\mathbb{R}^n}, \end{aligned}$$

that turns out to coincide with the standard (round) metric.

It is very well known that the Laplace Beltrami operator on the real unit sphere (endowed with the round metric) has a discrete diverging set of eigenvalues and its eigenfunctions are polynomials: the *spherical harmonics.*

These observations lead automatically to the desired conclusion that we state as a corollary.

Corollary 1. *The eigenfunctions of the Laplace Beltrami operator with respect to the Baran metric on the real unit sphere are the orthogonal polynomials with respect to the pluripotential equilibrium measure* $\mu_{\mathbb{S}^{n-1},\mathcal{S}^{n-1}}$ *of the real unit sphere* $\mathbb{S}^{n-1}$ *in the complexified sphere* $\mathcal{S}^{n-1}$.

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