Pluripotential Numerics

Computational Methods and Function Theory 2017
Lublin, July 10th - 15th, 2017

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What is this talk about?

Log. Pot. Th. and Approximation Th.

Many tasks in *univariate* Approximation Theory lead to the study of Logarithmic potential theory and subharmonic functions

- approximation numbers (for analytic functions) asymptotic and overconvergence
- quest for *good* polynomial interpolation arrays distribution
- polynomial inequalities
- uniform convergence of (discrete) least squares
- asymptotic behaviour of orthogonal polynomials
- ......

Back in the 60's people were looking for a multidimensional counterpart...

What Pluripotential Theory is?

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Morally speaking

Pluripotential Theory is the natural *non linear* extension of Logarithmic Potential Theory to multidimensional complex spaces.

- It reduces to log pot. th. if complex dimension is 1
- Many analytical and geometric analogies between the two theories (*but pay attention. . . !!*)

Motivations:

- Approximation Theory in $\mathbb{C}^n$, $n > 1$
- Complex (Kahler) Geometry
- Random polynomials, arrays and matrices

The players in Pluripotential Theory I

- *subharmonic func. shm*$(\mathbb{C}) \rightsquigarrow \mathrm{psh}(\mathbb{C}^n)$ plurisubharmonic func.

$$\mathrm{psh}(\mathbb{C}^n) := \{u : \mathbb{C}^n \to [-\infty, \infty[, \text{u.s.c. and shm along any complex line}\}.$$

- Laplacian $\Delta \rightsquigarrow (dd^c \cdot)^n$ complex Monge-Ampere operator

For $u \in C^2(\mathbb{C}^n)$

$$dd^c u = 2i \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial_i \bar{\partial}_j} dz_i \wedge d\bar{z}_j,$$

$$(dd^c u)^n = dd^c u \wedge dd^c u \wedge \cdots \wedge dd^c u = c_n \det[\partial_i \bar{\partial}_j u] d\,\mathrm{vol}_{\mathbb{C}^n},$$

For $u \in \mathrm{psh}(\mathbb{C}^n) \cap L^\infty_{\mathrm{loc}}(\mathbb{C}^n)$ the *product of positive currents* $(dd^c u)^n$ can be defined as positive measure

- Green function $g_{\mathbb{C}\setminus E}(z,\infty) \rightsquigarrow V_E^*(z)$ Pluricomplex Green function

$$V_E(\zeta) := \sup\{u(\zeta) \in \mathcal{L}(\mathbb{C}^n), u|_E \leq 0\},$$
$$V_E^*(z) := \overline{\lim}_{\zeta\to z} V_E(\zeta).$$

Here $\mathcal{L}(\mathbb{C}^n)$ is the Lelong class of psh functions of log growth at ∞.

$$\begin{cases} \left(dd^c V_E^*\right)^n = 0 & \text{in } \mathbb{C}^n \setminus E \\ V_E^* = 0 & \text{q.e. on} E \\ V_E^* \in \mathcal{L}(\mathbb{C}^n) & \end{cases}.$$

- eq. meas. $\mu_E := \Delta g_{\mathbb{C}\setminus E}(z,\infty) \rightsquigarrow \left(dd^c V_E^*(z)\right)^n =: \mu_E$ pluripotential equilibrium measure, $\mu_E(E) = 1, \mu_E(\mathbb{C}^n \setminus E) = 0$

- Fekete points *are almost the same*, i.e., maximizers of the Vandermonde determinant ($N_k = \dim \mathscr{P}^k(\mathbb{C}^n)$)

$$|\mathrm{VDM}(z_1, z_2, \ldots, z_{N_k})| = \max_{\zeta \in E^{N_k}} |\mathrm{VDM}(\zeta_1, \zeta_2, \ldots, \zeta_{N_k})|$$

- The transfinite diameter $\delta(E)$ is the asymptotic

$$\delta(E) := \lim_k \left(\max_{\zeta \in E^{N_k}} |\mathrm{VDM}(\zeta_1, \zeta_2, \ldots, \zeta_{N_k})|\right)^{\frac{n+1}{nkN_k}}$$

Aim and tools

Aim:

Start the study of numerical algorithms for the approximation of the pluripotential quantities for a given compact *"nice"* $E \subset \mathbb{C}^n$.

- Check conjectures or formulate new ones
- Get new heuristics in polynomial approximation
- Random sampling

Tools:

- L^2 theory in Pluripotential theory: deep results by Berman and Boucksom
- Admissible polynomial meshes based computations

People: N. Levenberg, M. Vianello (CAA group and friends).

Tools

Bernstein Markov measures

Definition

Let $E \subset\subset \mathbb{C}^n$ and μ be a positive Borel measure with $\operatorname{supp} \mu \subseteq E$. The couple (E, μ) has the Bernstein Markov property if for any sequence of polynomials $\{p_k\}$, $\deg p_k \leq k$ we have

$$\overline{\lim}_k \left(\frac{\|p_k\|_E}{\|p_k\|_{L^2_\mu}} \right)^{1/k} \leq 1.$$

Bergman function $B_k^\mu(z) := \sum_{j=1}^{N_k} |q_j(z, \mu)|^2$, q_j's o.n.b. $\mathscr{P}^k(\mathbb{C}^n)$

Equivalently

$$\overline{\lim}_k \|B_k\|_E^{1/2k} \leq 1.$$

Remarks & example

- Very close to Stahl and Totik's **Reg** class (extended by Bloom)
- There are plenty of generalizations
- Almost sharp sufficient condition is well studied
- *If and only if* condition is open problem

Example

$E := \{z \in \mathbb{C} : |z| = 1\}$, μ arc-length measure. Then $q_j(z,\mu) = z^j$ and $B_k^\mu(z) = \sum_{j=1}^{k+1} |z|^{2j} = k+1$ on E.

Counterexample

Can't fit in here. Any non trivial non BM measure looks extremely ugly and needs a rather long technical construction.

L^2 Theory in PLoripotential Th.

Assume E is pluriregular compact set and (E, μ) is a Bernstein Markov couple, then

- $\lim_k \frac{1}{2k} \log B_k^\mu(z) = V_E^*(z)$ uniformly
- $\lim_k (\det G_k(\mu, \mathcal{M}_k))^{\frac{n+1}{2nkN_k}} = \delta(E)$
- $\lim_k N_k^{-1} B_k^\mu(z)\mu = \mu_E$ weakly$*$

Recall: $B_k^\mu(z) := \sum_{j=1}^{N_k} |q_j(z, \mu)|^2$ is the Bergman function and

$$G_k(\mu, \mathcal{M}_k)_{i,j} := \int_E z_1^{i_1} \cdot \ldots z_n^{i_n} \cdot \bar{z}_1^{j_1} \cdot \ldots \bar{z}_n^{j_n} d\mu$$

is the Gram matrix in the monomial basis.

Admissible meshes

Definition

Let $\{A_k\}$ a sequence of fine subsets of the compact polynomial determining set $E \subset \mathbb{C}^n$, then $\{A_k\}$ is an admissible mesh for E if

- $\mathrm{Card}\, A_k = \mathcal{O}(k^s)$
- $\sup_{p \in \mathscr{P}^k \setminus 0} \frac{\|p\|_E}{\|p\|_{A_k}} \leq C$, for any $k \in \mathbb{N}$.

- originally introduced to show uniform convergence of DLS (Calvi and Levenberg 2008)
- fulfil many nice properties. . .
- construction available (algorithm!) on many "nice" geometries
- stable numerical computations available in the WAM-package (CAA group Padova Verona)

Key Idea is easy!

Heuristically speaking

Sequence of uniform probability measures on admissible polynomial meshes are *good discrete models* for Bernstein Markov measures.

Let $\{A_k\}$ be admissible for E and μ_k uniform probability on A_k, then

$$\|p\|_E \leq C\|p\|_{A_k} \leq C\sqrt{\operatorname{Card} A_k}\|p\|_{L^2_{\mu_k}}, \forall p \in \mathscr{P}^k$$

$$\boxed{B_k^{\mu_k}(z) \leq C\sqrt{\operatorname{Card} A_k}, \forall z \in E.}$$

Theoretical Results

Theorem 1 [P. 2016]

Let $E \subset \mathbb{C}^n$ be a compact $\mathcal{L}$-regular set and $\{A_k\}$ a (weakly-) admissible mesh for E, then, uniformly in $\mathbb{C}^n$, we have

$$\lim_k v_k := \lim_k \frac{1}{2k} \log B_k^{\mu_k} = V_E^*,$$
$$\lim_k u_k := \lim_k \frac{1}{k} \log \int_E |K_k^{\mu_k}(\cdot,\zeta)| d\mu_k(\zeta) = V_E^*,$$
$$\lim_k v_k := \lim_k \frac{1}{2k} \log B_k^{\nu_k} = V_E^*,$$
$$\lim_k u_k := \lim_k \frac{1}{k} \log \int_E |K_k^{\nu_k}(\cdot,\zeta)| d\nu_k(\zeta) = V_E^*.$$

Here $\nu_k := B_k^{\mu_k} N_k^{-1} \mu_k$, $K_k^{\mu_k}(z,\zeta) := \sum_{j=1}^{N_k} q_j(z,\mu_k)\bar{q}_j(\zeta,\mu_k)$.

Approximating $\delta(E)$

Theorem 2 [P. 2016]

Let $E \subset \mathbb{C}^n$ be a compact $\mathcal{L}$-regular set and $\{A_k\}$ a (weakly) admissible mesh for E then, denoting by μ_k the uniform probability measure on A_k and by ν_k the Bergman re-weighted measure, we have

$$\lim_k \left(\det G_k(\mu_k, \mathcal{M}_k)\right)^{\frac{n+1}{2nkN_k}} = \delta(E)$$

$$\lim_k \left(\det G_k(\nu_k, \mathcal{M}_k)\right)^{\frac{n+1}{2nkN_k}} = \delta(E).$$

Approximating μ_E

Th. [Bos, Calvi, Levenberg, Sommariva and Vianello, 2011]

Let E be a compact non pluripolar set, $\{A_k\}$ a (weakly-) admissible mesh for E. The approximate Fekete points arrays sequence $\{F_k\}$ extracted starting by $\{A_k\}$ by the AFP algorithm converges weak star to μ_E.

Theorem 3 [P. 2016]

Let $E \subset \mathbb{C}^n$ a non pluripolar compact set. Let $\{A_k\}$ be a weakly admissible mesh for E and let μ_k be the uniform probability measure supported on A_k.

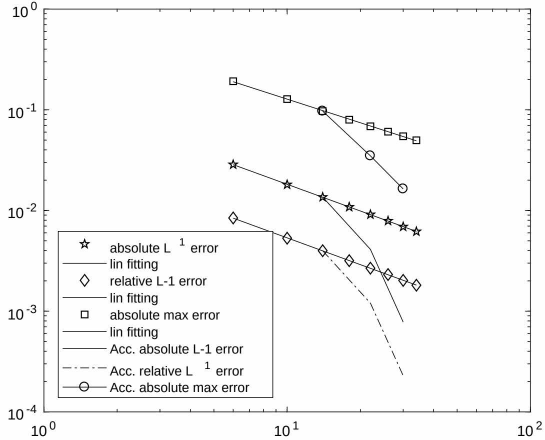
$$\lim_k \frac{B_k^{\mu_k}}{N_k}\mu_k = \mu_E, \text{ in the weak* topology.}$$

Few words on implementation

- Real algorithms for a complex theory, so far
- Key ingredient 1: *cope with ill conditioning* by heuristics + double orthogonalization (QR&backslash) implemented in the WAM-package
- Key ingredient 2: nice admissible meshes are the one for which $B_k^{\mu_k}$ is moderate oscillating on E.
- Straightforward implementation may fail! Example: $\delta(E)$ is computed in a stable basis and determinant of the change of basis is estimated by its known asymptotic.
- Convergence is monotone but *very slow*: extrapolation at infinity works very effectively (ρ-algorithm)

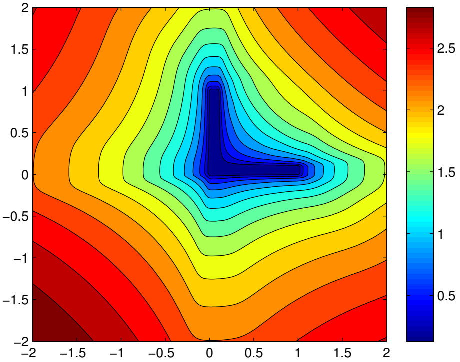
Numerical tests

Extremal function of a regular hexagon

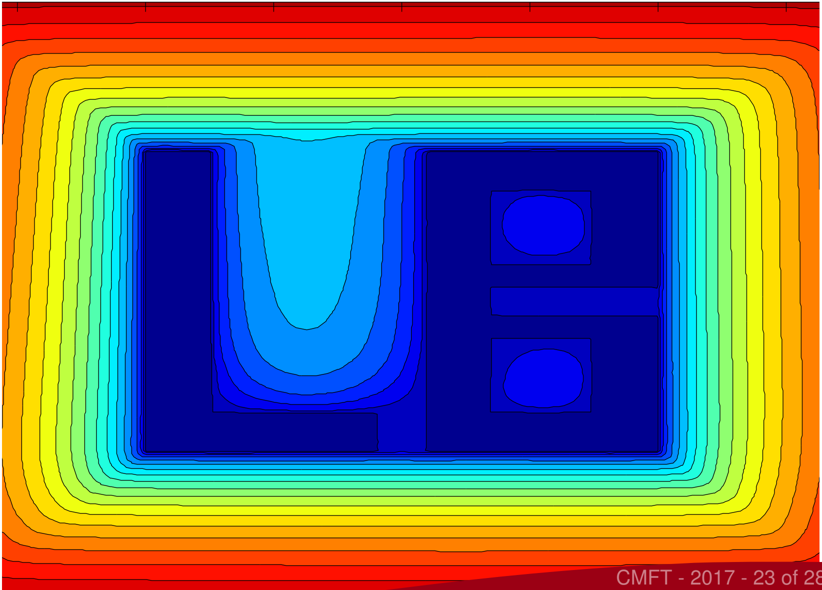


Target function computed by the Baran formula.

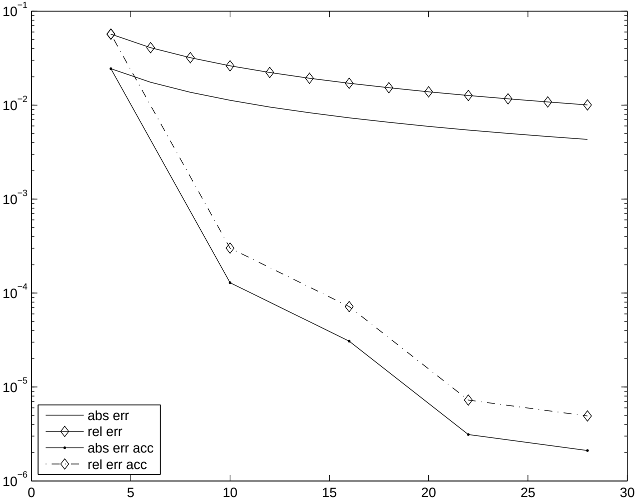
Len Bos asked about the L-set…



but after a year of hard work...



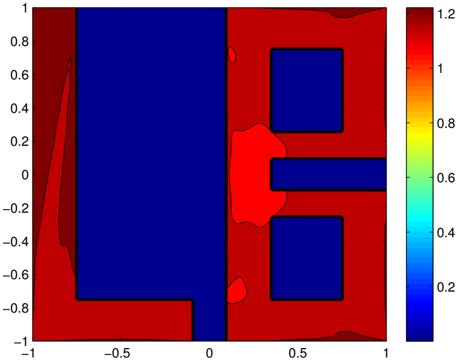
Transfinite diameter of the unit disk



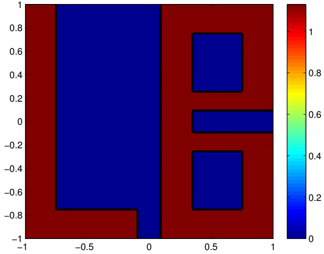
It has been analytically computed by Bos and Levenberg.

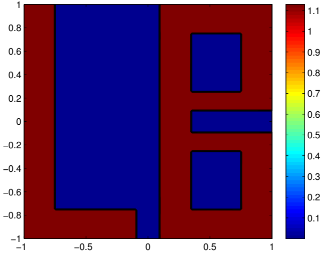
Morally:

Over-and-oversampling isn't enough, we need a *nice* oversampling.



Holding $B_k^{\mu_k}$ oscillations II





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Thank You!