Harmonic Admissible Meshes

Federico Piazzon
Dip. Matematica Univ Padova fpiazzon@math.unipd.it

Università degli Studi di Padova

$\mathcal{H}-$(W)AM

We use standard identification $\mathbb{C} \cong \mathbb{R}^2$ and let

$$\mathcal{H}^n := \{p \in \mathscr{P}^{2n}_{\mathbb{C}}(\mathbb{R}^2) : \Delta p \equiv 0\} \cong \left\{c + \sum_{i=1}^n a_j z^j + \sum_{i=1}^n \bar{a}_j \bar{z}^j, c, a_j \in \mathbb{C}\right\}$$

$$\mathcal{H} := \cup_n \mathcal{H}^n \quad \dim \mathcal{H}^n = 2n+1.$$

We introduce **Harmonic (Weakly) Admissible Meshes** in the spirit of [1].

Definition 1 ($\mathcal{H}-$AM)**.** *Let $K \subset \mathbb{C}$ be a $\mathcal{H}$-determining compact set. Let $\{A_n\}_{\mathbb{N}}$ be a sequence of finite subsets of K such that there exists $C < \infty$ such that*

(A) $\operatorname{Card} A_n$ grows polynomially w.r.t. n and

(B) $\|p\|_K \leq C\|p\|_{A_n}$ for any $p \in \mathcal{H}^n$.

Then A_n is said to be an Harmonic Admissible Mesh ($\mathcal{H}-$AM). If instead (B) is substituted by

(B') $\|p\|_K \leq C_n\|p\|_{A_n}$ for any $p \in \mathcal{H}^n$,

and C_n is growing at most polynomially w.r.t. n, then A_n is said to be an Harmonic Weakly Admissible Mesh ($\mathcal{H}-$WAM).

Basic Properties

(1) Possibly located on the exterior boundary (Maximum Principle).
(2) Stable under affine mapping.
(3) Well behaving under polynomial maps.
(4) Good $\mathcal{H}$ interpolation sets are $\mathcal{H}-$WAMs.

Fekete and $\mathcal{H}-$Fekete Points

An array of points of K maximizing the modulus of the **Vandermonde Determinant**

$$\operatorname*{VDM}_n(z_0, z_2 \dots z_n) := [z_i^j]_{i,j=0,1,\dots n}$$

is said a **Fekete** array of degree n for K. Similarly a $\mathcal{H}-$**Fekete** set of points is maximizing

$$\mathcal{H}\text{-}\operatorname*{VDM}_n(z_0, z_2 \dots z_{2n}) := \det \left[\begin{array}{c|c} [z_i^j] \;\; \begin{matrix} 0 \leq i \leq 2n \\ 0 \leq j \leq n \end{matrix} & [\bar{z}_i^j] \;\; \begin{matrix} 0 \leq i \leq 2n \\ 1 \leq j \leq n \end{matrix} \end{array} \right]$$

The **transfinite diameter** $\delta(K)$ of K is defined by taking any sequence F_n of Fekete sets and setting

$$\delta(K) := \lim_n \operatorname*{VDM}_n(F_n)^{\frac{1}{\alpha_n}}, \quad \text{where } \alpha_n := \binom{n+1}{2}.$$

Siciak [4] proved that in general one has the estimate for the **harmonic transfinite diameter** $0 \leq h(K) := \limsup_n \mathcal{H}\text{-VDM}_n(H_n)^{\frac{1}{2\alpha_n}} \leq \delta(K)$, but for $\mathcal{H}-$regular K (in particular for the closure of any finite union of bounded simply connected domain [2]) equality holds.

Logarithmic Energy

If K is a regular compact set of positive capacity (i.e. $\delta(K) < \infty$) the classical minimization problem in the space $\mathscr{M}(K)$ of probability measures supported on K w.r.t log energy functional $I[\mu] := \int\int \log \frac{1}{w-z} d\mu(z) d\mu(w)$ has an unique solution μ_K, the **Equilibrium Measure**.
It is well known that $\mu_K = \frac{1}{2\pi}\Delta V_K$ in a distributional sense, here V_K is the Green function for (Δ, K) with pole at ∞ having logarithmic growth. μ_K is the asymptotic of Fekete points (see prop below).
By standard techniques we can prove that

Proposition 1. *if $B_n \subset K$ is a sequence of finite sets having the property $\limsup_n \mathcal{H}\text{-VDM}(B_n)^{\frac{1}{2\alpha_n}} = \delta(K)$, then the uniform probability measures μ_n relative to B_n satisfies $\mu_n \rightharpoonup^* \mu_K$.*

In particular any Fekete array do.

Key Features

By mimiking and slightly modifying the case of WAMs we can prove some **theorems**:

(i) **Existence 1.** If K preserves a Markov Inequality for algebraic polynomials of 2 real variables with exponent r (i.e. there exists M such that $\|\nabla p(x,y)\|_K \leq M n^r \|p(x,y)\|_K$ for any $p \in \mathscr{P}^n(\mathbb{R}^2)$) and ∂K is a compact connected Lipschitz manifold, then there exists an $\mathcal{H}-$AM for K with $\mathcal{O}(n^r)$ points.

(ii) **Existence 2.** If K is the closure of a bounded domain, ∂K is a compact connected Lipschitz manifold and uniform (or even or $L^1(\partial K)$) interior ball condition holds true, then there exists an $\mathcal{H}-$AM for K with $\mathcal{O}(n)$ points. Namely **Optimal** $\mathcal{H}$**-AM**.

(iii) **Holomorphic Mapping.** If A_n is an $\mathcal{H}-$WAM for $Q \subset \Omega$ of cardinality $c(n)$, $f \in hol(\Omega, \mathbb{C})$ is proper and $K := f(Q)$ enjoys a Markov Inequality, then there exists a sequence $j(n) = \mathcal{O}(\log(n))$ such that $\tilde{A}_n := f(A_{n \cdot j(n)})$ is an $\mathcal{H}-$WAM for K (of cardinality $c(nj(n))$).

(iv) **Uniform Approximation by DLS.** Let K be compact and $\mathcal{H}-$determining let A_n be a $\mathcal{H}-$WAM of cardinality $c(n)$ and constant C_n, let us denote by Λ_{A_n} the discrete least squares projection onto $\mathcal{H}^n$ and set $d_K^h(f, \mathcal{H}^n) := \inf_{p \in \mathcal{H}^n} \|f - p\|_K$. Then for any $f \in \mathscr{C}^0(K)$ we have

$$\|f - \Lambda_n f\|_K \leq \left(1 + C_n(1 + \sqrt{c(n)})\right) d_K^h(f, \mathcal{H}^n).$$

(v) **Uniform Approximation of Derivatives.** In particular if $\Delta f = 0$ in the open set $\Omega \supset\supset K$ and K is convex with respect to $\mathcal{H}$ and $\mathcal{H}-$regular then $\|f - \Lambda_n f\|_K \to 0$ and $\|\partial_i f - \partial_i \Lambda_n f\|_K \to 0$.

Applications: Pseudo-spectral Methods

- Laplace $\begin{cases} \Delta u = 0 & \text{in int} K \\ u|_{\partial K} = g. \end{cases} \approx \begin{cases} u^{(n)} \in \mathcal{H}^n \\ u^{(n)}|_{\partial K} = \Lambda_n[g]. \end{cases}$

- Poisson $\begin{cases} \Delta u = f & \text{in int} K \\ u|_{\partial K} = g. \end{cases} \approx \begin{cases} u^{(n)} := u_0^{(n)} + u_1^{(n)} \\ u_0^{(n)} \in \mathcal{H}^n \ , \ u_1^{(n)} \in \mathscr{P}^{2n} \\ u^{(n)}|_{\partial K} = \Lambda_n[g] \ , \ \Delta u_1^{(n)} = \Pi_{2n} f. \end{cases}$

  where Π_{2n} is the DLS projection onto $\mathscr{I}^n \cong \mathscr{P}^{2n} / \ker \Delta$ w.r.t. $l^2(\tilde{A}_n)$ with $\tilde{A}_n \supset A_n$ any WAM for K.

- Constant coefficient divergence form second order elliptic operators are diagonalized by an affine invertible change of coordinates φ

  $\begin{cases} \operatorname{div}(A \nabla u) = 0 & \text{in int} K \\ u|_{\partial K} = g. \end{cases} \leftrightarrow \begin{cases} \Delta(u \circ \varphi) =: \Delta \tilde{u} = 0 & \text{in } \varphi^{-1}(\text{int} K) \\ \tilde{u}|_{\varphi^{-1}(\partial K)} = g \circ \varphi. \\ J\varphi^{-t} A J\varphi = \mathbb{I}. \end{cases}$

Approximate $\mathcal{H}-$Fekete Points algorithm

Finding *true* $\mathcal{H}-$Fekete points is an extremely hard task, however an approximate solution is provided by the (adapted version of the) greedy algorithm AFP [3], namely the $\mathcal{H}-$**AFP** ***that selects*** (QR factorization with partial pivoting) the ***maximum volume square rank*** $2n+1$ ***sub-matrix from the matrix*** $\mathcal{H}\text{-VDM}(z_0, z_2 \dots z_{2c_n})$.

Theorem 1. *Let $K \subset \mathbb{C}$ be a compact $\mathcal{H}$-regular and -convex set and let A_n be an $\mathcal{H}-$WAM of degree n of constant C_n for K let H_n be extracted from A_n by $\mathcal{H}-$AFP, then*

(i) H_n are unisolvent and have Lebesgue Constant $\Lambda_{H_n} \leq (2n+1)C_n$.

(ii) H_n are asymptotically Fekete (i.e.$\limsup_n \mathcal{H}\text{-VDM}_n(H_n)^{\frac{1}{2\alpha_n}} = \delta(K)$)

(iii) Let μ_n be the uniform probability measure canonically identified by H_n, we have $\mu_n \rightharpoonup^* \mu_K$.

References

[1] J.P. Calvi and N. Levenberg. Uniform approximation by discrete least squares polynomials. *JAT*, **152**:82–100, 2008.

[2] J. H. Curtiss. The transfinite diameter and extremal points for harmonic polynomial interpolation. *Journal d' Analyse Matematique*, **17**[66]:75–92, 1966.

[3] L.Bos, S. De Marchi, A.Sommariva, and M.Vianello. Weakly admissible meshes and discrete extremal sets. *Numer. Math. Theory Methods Appl.*, **4(1)**:1–12, 2011.

[4] J. Siciak. Asymptotic behavior of harmonic polynomials bounded on a compact set. *Annales Polonici Mathematici*, **20**:267–278, 1968.