The Bernstein Markov Property and Applications to Pluripotential Theory

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Setting and Notation

- K compact set in $\mathbb{C}$ or $\mathbb{C}^n$
- $\|f\|_K := \max_K |f|$.
- μ positive finite Borel measure, $\operatorname{supp}\mu \subseteq K$, $\mu \in \mathcal{M}_+(K)$.
- $\mathscr{P}^m(K)$ space of complex polynomials of deg $\leq m$ on K.
- $N_m := \dim \mathscr{P}^m = \binom{n+m}{m}$.
- $\mathscr{P}^m_\mu(K)$ Hilbert space $\left(\mathscr{P}^m, \langle \cdot, \cdot \rangle_{L^2_\mu}\right)$.

Since $\mathscr{P}^m$ is a finite dimensional TVS all norms are comparable. In particular there exists $0 < C(\mu, K, m) < \infty$ such that

$$\frac{1}{\sqrt{\mu(K)}}\|p\|_{L^2_\mu} \leq \|p\|_K \leq C(\mu, K, m)\|p\|_{L^2_\mu} \quad \forall p \in \mathscr{P}^m(K).$$

Bernstein Markov Property (BMP)

Let $K \subset \mathbb{C}^n$ be compact and $\mu \in \mathcal{M}_+(K)$, then the (K, μ) is said to enjoy the **Bernstein Markov Property** if there exists a sequence $\{C_m\}_{m \in \mathbb{N}}$ such that

$$\begin{aligned} \|p\|_K \leq C_m \|p\|_{L^2_\mu} \quad \forall p \in \mathscr{P}^m(K), \\ \limsup_m C_m^{1/m} \leq 1. \end{aligned} \tag{1}$$

Definitions . . . continued

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Several variants have been introduced

- **Weighted BMP** Given a weight function $w : K \to [0, +\infty[$ one looks at $\|pw^m\|_K$ and $\|pw^m\|_{L^2_\mu}$ for $p \in \mathscr{P}^m$.
- **Strong BMP** If for any $w \in \mathcal{C}(K)$ (K, μ, w) has the WBMP.
- **Rational BMP** For a given compact set $P, K \cap P = \emptyset$, we set

$$\mathscr{R}^m(P) := \{p_m/q_m \ : \ p_m, q_m \in \mathscr{P}^m(K),\ Z(q_m) \subset P\},$$

then we compare $\|r\|_K$ and $\|r\|_{L^2_\mu}$ for $r \in \mathscr{R}^m(P)$.

Weighted Rational and Strong Rational BMP etc.

Related studies

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- The first steps are made by Szego, Faber, Erdós and Turan.
- Classical weight on the real line.
- Leja L^* condition.
- Systematic study for general measures in the plane early '90s Stahl, Totik [9]. Regular measures.
- Determining measure: Widom and Ullman.

Here we follow the approach of Berman, Boucksom, Nymstrom [6], Bloom and Levenberg [3], which is more adapted to the scv context and pluripotential theory.

Some Examples

1. $K = \overline{\mathbb{D}}$, $\mu = \delta_0$. This is not a BM couple.
   It should be that μ defines at least a norm.
2. $K := \overline{\mathbb{D}} \times \overline{\mathbb{D}}$, $\operatorname{supp} \mu = S(K)$ the Šilov boundary and $\mu := ds \otimes ds$. Instead it is. Monomials are orthonormal...

$$\|p\|_K = \left|p(z_0)\right| \leq \sqrt{\sum_{|\alpha|\leq m} |c_\alpha|^2} \sqrt{\sum_{|\alpha|\leq m} |z_0|^{2\alpha}} = \sqrt{\frac{(m+2)(m+1)}{2}} \|p\|_{L^2_\mu}$$

   μ should be thick on $S(K)$...
3. It has been shown that there exists a BM measure for $\overline{\mathbb{D}}$ with discrete carrier in the interior of the disk.
   In general we find out only sufficient conditions.
4. But the weight $w(z) = \exp(-|z|^2)$ makes $(\overline{\mathbb{D}}, ds, w)$ not a WBM triple.

Bergman Function

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$\mathscr{P}_\mu^m := (\mathscr{P}^m, \langle \cdot, \cdot \rangle_{L^2_\mu(K)})$ is a reproducing kernel Hilbert space, being the kernel

$$K_m^\mu(z,\zeta) := \sum_{|\alpha|\leq m} q_\alpha(z,\mu)\bar{q}_\alpha(\zeta,\mu). \ \ \{q_\alpha\}_{|\alpha|\leq m} \text{ o.n.b.}$$

$$B_m^\mu(z) := K_m(z,z) = \langle K_m^\mu(z,\zeta); K_m^\mu(z,\zeta)\rangle_{L^2_{\mu(\zeta)}(K)}.$$

- Let $\delta_z \in L(\mathscr{P}_\mu^m, \mathbb{C})$ be the point-wise evaluation, for any $z \in K$ we have $\|\delta_z\| = \sqrt{B_m^\mu(z)}$.
- *The best possible constant in* (1) *is* $\sqrt{\|B_m^\mu\|_K}$.

The dd^c operator

For an open set $\Omega \subset \mathbb{C}^n$ and $u \in \mathcal{C}^2(\Omega)$, one defines first $\bar{\partial} := \sum_{j=1}^{n} \frac{\partial(\cdot)}{\partial \bar{z}_j} d\bar{z}_j$, $d := (\partial + \bar{\partial})$ and $d^c := i(-\partial + \bar{\partial})$,

$$\mathrm{dd^c}\, u := 2i \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) dz_j \wedge d\bar{z}_k.$$

For any *plurisubharmonic* u we can define by smoothing

$$\boxed{\mathrm{dd^c}\, u \text{ as a positive } (1,1) \text{ current}}$$

i.e., an element of the dual of the test forms of bidegree $(n-1, n-1)$ such that

$$\langle \mathrm{dd^c}\, u, \theta \rangle > 0 \quad \forall \theta \in SP^{(n-1,n-1)}(\Omega).$$

Defining $(dd^c)^2$

Let $\Omega \subset \mathbb{C}^2$ be a domain

1. if $u \in \mathrm{PSH}$ and θ is a positive $(1,1)$ form

$$\langle dd^c u \wedge \theta, \psi\rangle := \langle dd^c u, \theta \wedge \psi\rangle \ \ \forall \psi \in \mathcal{D}^{(n-2,n-2)}(\Omega) = \mathcal{C}^\infty_c(\Omega).$$

2. if $u \in \mathcal{C}^2(\Omega)$ then we have $\forall \varphi \in \mathcal{C}^\infty_c(\Omega)$

$$\int_\Omega \varphi (dd^c u)^2 = \int_\Omega u \, dd^c \varphi \wedge dd^c u.$$

...but the r.h.s. takes sense even for $u \in \mathrm{PSH} \cap L^\infty_{loc}$...

Extending to plurisubharmonic u

Theorem [Chern Levine Nirenberg]

For any $K \subset\subset \Omega$ there exist $C > 0$ and compact set $L \subset \Omega \setminus K$ such that
$$\int_K (dd^c u)^n \leq C \|u\|_L^n \quad \forall u \in \mathcal{C}^2(\Omega).$$

Combining C.L.N. estimate and

Proposition

Any (p, p) positive current has measure coefficients.

Bedford and Taylor find out that

Monge Ampere Operator

For $u \in \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}$ one can iteratively define $(\mathrm{dd^c}\, u)^n$ as a positive measure by

$$\langle (\mathrm{dd^c}\, u)^{k+1}, \theta \rangle := \langle \mathrm{dd^c}\, \theta \wedge (\mathrm{dd^c}\, u)^k, u \rangle. \;\; \forall \theta \in \mathcal{D}^{n-k-1,n-k-1}(\Omega)$$

Theorem [Bedford Taylor][2]

The operator $(\mathrm{dd^c})^n$ is continuous w.r.t. the weak * topology under any point-wise converging decreasing sequence of functions.

This is a fully non linear partial diff operator that in the case $n = 1$ corresponds to the distributional Laplacian.

Dirichlet problem

For "nice" compact set K we can solve the Dirichlet problem

$$\begin{cases} (dd^c\, u)^n = 0 & \text{in } \Omega := \mathbb{C}^n \setminus K \\ u \equiv_{\text{q.e.}} 0 & \text{in } \partial K \\ u \in \mathcal{L} \text{ Lelong class.} \end{cases}$$

The unique solution V_K^* is said **Extremal Function**.

The extremal function

The solution is in the sense of Perron-Bremermann

$$V_K^*(z) := \limsup_{\zeta \to z} \left(\sup\{u(\zeta) \in \mathcal{L}\,,\; u|_K \leq 0\} \right)$$

If it is continuous the compact set is said $\mathcal{L}$ **regular**.
By Siciak and Zaharyuta's results we have

$$V_K = \log \Phi_K := \log^+ \sup\{|p|^{1/\deg p} : \|p\|_K \leq 1\}.$$

And **Bernstein-Walsh-Siciak Inequality** follows

$$|p(z)| \leq \|p\|_K \exp(\deg p V_K^*(z)).$$

The measure

$$\mu_K := (dd^c\, V_K^*)^n$$

is said the **equilibrium measure**.

Motivations: from Approx. Theory

- **LS asymptotic.** If (K,μ) has the BMP and K is $\mathcal{L}$ regular then

$$\limsup_m d_\infty(f, \mathscr{P}^m)^{1/m} = \frac{1}{R} \text{ (i.e., } f \in hol(\{V_K < \log R\}) \text{)}$$

$$\Downarrow$$

$$\limsup_m \|f - \mathscr{L}_m f\|_K^{1/m} = \frac{1}{R} \text{ (here } \mathscr{L}_m \text{ is the LS projection).}$$

Moreover

$$\limsup_m \|f - \mathscr{L}_m f\|_{L^2_\mu}^{1/m} \leq \frac{1}{R} \Rightarrow f \in hol(\{V_K < \log R\}).$$

- *m*-**th roots asymptotic.** For regular compact set K

$$(K,\mu) \quad \text{has} \quad \text{BMP}$$
$$\Updownarrow$$
$$\lim_m \frac{1}{2m} \log B_m^\mu \quad = \quad V_K \;\; \text{loc. uniformly in } \mathbb{C}^n.$$

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- **Free energy asymptotic.** If (K,μ) has the BMP then we have

$$\limsup_m \left(\int_{K^{N_m}} \left|\det \underset{m}{\mathrm{VDM}}(z_1,\dots,z_{N_m})\right|^2 d\mu(z_1)\dots d\mu(z_{N_m})\right)^{\frac{n+1}{2nmN_m}} = \delta(K)$$

i.e., the l.h.s. is maximal among $\{\nu \in \mathcal{M}^+(K) : \nu(K) = \mu(K)\}$.
This is the main tool for proving

1. Strong Bergman Asymptotic.

$$\boxed{\frac{B_m^\mu}{N_m}\mu \rightharpoonup^* \mu_K.}$$

2. Large Deviation Principle.

..and all these results go straightforward into the *weighted* setting.

Mass Density Condition in $\mathbb{C}$

From the example we guess μ should be *thick* on $S(K)$...
We had $\lim_{r\to 0^+} \mu(B(z,r))/r = 1 \ \forall z \in S(K)$.

Theorem [Stahl Totik]

Let μ be positive Borel measure with compact support $K = \operatorname{supp}\mu$ in $\mathbb{C}$, suppose that K is a non-polar regular set w.r.t. the Dirichlet problem for the Laplace operator and there exists $t > 0$ such that

$$\lim_{r\to 0^+} \operatorname{cap}\left(\{z \in K : \mu(B(z,r)) > r^t\}\right) = \operatorname{cap}(K).$$

Then (K,μ) has the BMP.

Here $\mathrm{cap}(K)$ is the logarithmic capacity of the set K,

$$\mathrm{cap}(K) := \max_{\nu \in \mathcal{M}_1(K)} \exp\left(\int \int \log |z - \zeta| d\nu(z) d\nu(\zeta) \right).$$

That is the (inverse of the exponential) of the minimum of the logarithmic energy functional: the variational formulation of the Dirichlet problem for the Laplacian in $\mathbb{C} \setminus K$.

Mass Density Condition in $\mathbb{C}^n$

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Theorem [Bloom Levenberg]

Let μ be positive Borel measure with compact support $K := \operatorname{supp} \mu \subset B(0,1)$ in $\mathbb{C}^n$, suppose that K is a non-pluripolar $\mathcal{L}$-regular set and there exists $t > 0$ such that

$$\lim_{r \to 0^+} \operatorname{Cap}\Big(\{z \in K : \mu(B(z,r)) > r^t\}, B(0,1)\Big) = \operatorname{Cap}(K, B(0,1)). \tag{2}$$

Then (K, μ) has the BMP.

But here $\mathrm{Cap}(K,\Omega)$ is capacity in the **non-linear** pluripotential theory in $\mathbb{C}^n$ related to the Monge Ampere complex operator, namely the **relative capacity** $\mathrm{Cap}(K,\Omega)$ w.r.t. a hyperconvex sup-set Ω of K.

Relative Capacity in $\mathbb{C}^n$

$$\mathrm{Cap}(K,\Omega) := \sup\left\{ \int_K (dd^c u)^n : u \in \mathrm{PSH}(\Omega), 0 \leq u \leq 1 \right\}$$

Core of the proof

The proof of these results relays on the following facts

(A) The regularity assumption on the set: V_K^* is continuous.

(B) Bernstein Walsh Siciak lemma.

$$|p(z)| \leq \|p\|_K \exp(\deg(p) V_K(z)). \quad (3)$$

(C) The following theorem

Capacity Convergence [Bloom Levenberg]

For any sequence of compact subsets of the compact non pluripolar $\mathcal{L}$–regular set K the following facts are equivalent

(i) $\lim_j \mathrm{Cap}(K_j, B(0,1)) = \mathrm{Cap}(K, B(0,1))$.

(ii) $\lim_j V_{K_j^*} = V_K^*$ locally uniformly in $\mathbb{C}^n$.

RBM mass density condition

Motivation: LDP for vector energy problems [4].

Theorem [P.]

Let K be a regular non polar compact set in the complex plane, $\Omega := \mathbb{C}_\infty \setminus \hat{K}$ and $P \subset \Omega$ a compactum. Let $\mu \in \mathcal{M}(K)$, $\operatorname{supp} \mu = K$ and suppose there exists a positive t such that

$$\lim_{r \to 0^+} \operatorname{cap}\left(\{z : \mu(B(z, r)) \geq r^t\}\right) = \operatorname{cap}(K). \tag{4}$$

Then μ enjoys the Bernstein Markov property on K for the rational functions with poles in P.

Idea of the proof

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- We replace the Bernstein Walsh Siciak Inequality by

$$|r_m(z)| \leq \|r\|_K \exp\left(\sum_{z_j \in \mathrm{Poles}(r_m)} g_{\Omega_K}(z, z_j)\right) \quad \forall r_m \in \mathscr{R}_m(K, P).$$

  Here $g_{\Omega_K}(z, z_j)$ is the generalized Green function
- We recover a modified capacity convergence result.

Proposition [P.]

Let $K \subset \mathbb{C}$ be a regular non polar compact set, let Ω_K be the unbounded component of $\mathbb{C} \setminus K$ and P a compact subset of Ω_K such that $P \cap K = \emptyset$. Then there exist a domain D such that $K \subset\subset D$ and $P \cap \overline{D} = \emptyset$, such that for any sequence $\{K_j\}$ of compact subsets of K the following are equivalent (here Ω_{K_j} is defined similarly to Ω_K).

$$\lim_j \mathrm{cap}(K_j) = \mathrm{cap}(K).$$

$$\lim_j g_{\Omega_j}(z, a) = g_{\Omega}(z, a) \text{ loc. unif. for } z \in D \text{ unif. for } a \in P.$$

In the case of a closed unbounded set K and an admissible weight function $w : \mathbb{C} \to [0, +\infty[$, can we do something?

Idea:

- **Compactification**, we look at the real sphere.
- search for **Strong BMP**, leads to
- **complexification**: $\mathcal{A} := \{z \in \mathbb{C}^3 : \sum z_j^2 = 1\}$ of the sphere
- use Pluripotential Theory for **Algebraic Submanifold**.

In such a setting the proof of an adapted formulation of sufficient mass density condition works provided an adapted version of the capacity convergence result.

The adapted setting

There is a specific $\mathbb{C}$–linear change of (Rudin) coordinates [7] such that

$$\boxed{\mathcal{A} \subset \{(z,w) \in \mathbb{C}^m \times \mathbb{C}^{n-m} : |w|^2 \leq C(1+|z|^2)\}.}$$

Sadullaev [8] defined $V_K^*(\cdot, \mathcal{A})$ and gave sense to pluripotential theory on algebraic sets.
Notation: for $R >> 1$ we use the pseudo-ball

$$\boxed{\Omega(r) := \{(z,w) \in \mathcal{A} : |z|^2 - R^2 < r\}.}$$

$\Omega := \Omega(-\sqrt{R^2-1})$.

Capacity convergence on algebr. man.

Theorem [P.]

Let $\mathcal{A} \subset \mathbb{C}^n$ be an algebraic variety of pure dimension $m < n$, $\mathcal{A}_{\text{reg}} \supset \Omega_0 \supset K$ where K is a compact $\mathcal{L}$ regular nonpluripolar set. Let $\{K_j\}_{j\in\mathbb{N}}$ be a sequence of compact subsets of K, then the following are equivalent.

(i) $\lim_j \operatorname{Cap}(K_j, \Omega) = \operatorname{Cap}(K, \Omega)$.

(ii) $V^*_{K_j}(\cdot, \mathcal{A}) \to V^*_K(\cdot, \mathcal{A})$ locally uniformly on $\mathcal{A}$.

The proof is similar to the original one . . .

if

we provide a modified version of the *Capacities Comparison Theorem* of Alexander and Taylor [1].

To do it ,first we re-defined the (modified) Tchebyshev constant as

$$m_\nu(K,\mathcal{A}) := \inf\{\|p\|_K :\ p \in \mathscr{P}_{\mathbb{C}}(K), \deg p \le \nu, \|p\|_{|z|\le 1} \ge 1\},$$

$$T(K,\mathcal{A}) := \inf_{\nu>0} m_\nu^{1/\nu}(K,\mathcal{A}).$$

Theorem [P.]

Let $\mathcal{A}$ be a m-dimensional algebraic variety of $\mathbb{C}^n$, such that for an $R > 1$ $\mathcal{A}_{\text{reg}} \supset \Omega_0$, then for any $r < -\sqrt{R^2 - 1}$ there exist two positive constants c_1, c_2 such that for any compact set $K \subset \Omega_r$

$$\exp\left[-\left(\frac{c_1}{\mathrm{Cap}(K,\Omega)}\right)^{1/m}\right] \geq T(K,\mathcal{A}),$$

$$T(K,\mathcal{A}) \geq \exp\left(-\frac{c_2}{\mathrm{Cap}(K,\Omega)}\right).$$

In particular for any $E \subseteq K$ we have

$$\|V_E(z,\mathcal{A})\|_\Omega \leq \frac{c_2}{\mathrm{Cap}(E,\Omega)}.$$

Among other issues we need a further sharpening of Chern Levine Nirenberg Estimate for $u \leq 0$

CLN $\int_K (dd^c u)^n \leq C\|u\|_L^m \quad \forall u \in \mathcal{C}^2(B)$

AT $\int_K (dd^c u)^m \leq C(-u(0))\|u\|_B^{m-1} \quad u \in \mathrm{PSH}(B)$

New $\int_K (dd^c u)^m \leq C \int_{\Omega(z_0,r)} -u\, (dd^c \rho)^m \, \|u\|_\Omega^{m-1} \; u \in \mathrm{PSH}(\Omega)$
$\Omega \subset \Omega(z_0, r) \subset \mathcal{A}.$

The main tool here is the Lelong Jensen Formula proven by Demailly [5].
For the case of the relative extremal function the r.h.s. integral can be dominated by the same function for the projected set, i.e.,

$$\boxed{\int_{\Omega(z_0,r)} -U_{K,\Omega(z_0,r)}\, (dd^c \rho)^m \leq C'(r)\left(-U_{\pi K, B_{\mathbb{C}^m}}(z_0)\right).}$$

Thank you for the attention.

H. J. Alexander and B. A. Taylor.
Comparison of two capacities in $\mathbb{C}^n$.
Math. Z., **186**:407–414, 1984.

E. Bedord and B. A. Taylor.
A new capacity for plurisubharmonic functions.
Acta Mathematica, **149**[1]:1–40, 1982.

T. Bloom and N. Levenberg.
Capacity convergence results and applications to a Bernstein Markov Inequality.
Trans. of AMS, **351**[12]:4753–4767, 1999.

T. Bloom, N. Levenberg, and F. Wielonsky.
Vector energy and large deviations.
J. Anal. Math, **preprint**, 2013.

J. P. Demailly.
Complex Analytic and Differential Geometry.
Universit e de Grenoble I Institut Fourier, UMR 5582 du CNRS, 2012.

R.Berman, S. Boucksom, and D.W.Nymstrom.
Fekete points and convergence toward equilibrium on complex manifolds.
Acta. Mat., **207**:1–27, 2011.

W. Rudin.
A geometric criterion for algebraic varieties.
J. Math. Mech., **17**:671–693, 1967-1968.

A. Sadullaev.
An estimates for polynomials on analytic sets.
Math. URSS Izvestiya, **20**[3]:493–502, 1982.

H. Stahl and V. Totik.
General Orthogonal Polynomials.
Cambridge Univ. Press, 1992.