THE BERSTEIN-MARKOV PROPERTY AND APPLICATIONS IN PLURIPOTENTIAL THEORY

FEDERICO PIAZZON[1]

ABSTRACT. The Bernstein Markov Property (BMP) is a comparability conditions of the L^2_μ and $\max_K |\cdot|$ norms of polynomials for a given a compact set $K \subset \mathbb{C}^n$ and a measure μ with $\operatorname{supp} \mu \subseteq K$. Several variants (i.e., L^p, weighted, ...) of this property has been introduced.

Bernstein Markov property arises as a key tool in the proofs of some fundamental results in (weighted) Pluripotential Theory [10, 11] and random polynomials [4]. More recently, it has been shown that such results can be reinterpreted in a probability fashion proving a Large Deviation Principle.

We recall [3, 5] the best-known sufficient condition for the standard BMP and present two new results. Namely, a sufficient mass density condition for the BMP for rational functions and a sufficient mass density condition for the weighted BMP on unbounded closed sets in the complex plane.

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1. First Definitions.

Let $\mathscr{P}^m(K)$ the space of holomorphic polynomials restricted to the compact set $K \subset \mathbb{C}^n$. Since $\dim \mathscr{P}^m < \infty$ any norm on it is comparable, in particular for any positive Borel measure $\mu \in \mathcal{M}^+(K)$ supported on K there exists $0 < C(\mu, K, m) < \infty$

$$\frac{1}{\sqrt{\mu(K)}}\|p\|_{L^2_\mu} \leq \|p\|_{C(K)} \leq C(\mu, K, m)\|p\|_{L^2_\mu} \quad \forall p \in \mathscr{P}^m(K).$$

The Bernstein Markov Property is a quantitative requirement on the asymptotic of the $m-$th root of the comparability constant $C(\mu, K, m)$.

Definition 1.1 (Bernstein Markov Property (BMP))**.** *Let $K \subset \mathbb{C}$ be compact and $\mu \in \mathcal{M}_+(K)$ then the (K, μ) is said to enjoy the* ***Bernstein Markov Property*** *if exists a sequence $\{C_m\}_{m\in\mathbb{N}}$ such that*

$$\begin{aligned} &\|p\|_K \leq C_m\|p\|_{L^2_\mu} \quad \forall p \in \mathscr{P}^m(K), \\ &\limsup_m C_m^{1/m} \leq 1. \end{aligned} \tag{1}$$

There are several variants of such a property.

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Definition 1.2. *Let $K \subset \mathbb{C}$ be compact and $\mu \in \mathcal{M}_+(K)$ If for a given function $w : K \to [0, +\infty[$ there exists a sequence $\{C_m^{(w)}\}_{m\in\mathbb{N}}$ such that*

$$\begin{aligned} &\|pw^m\|_K \le C_m^{(w)} \|pw^m\|_{L^2_\mu} \;\; \forall p \in \mathscr{P}^m(K),\\ &\limsup_m (C_m^{(w)})^{1/m} \le 1. \end{aligned} \tag{2}$$

Then we say that the triple (K, μ, w) enjoy the **Weighted Bernstein Markov Property.** *If for any $w \in C(K)$ there exists such a $\{C_m^{(w)}\}_{m\in\mathbb{N}}$, then (K, μ) is said to enjoy the* **Strong Bernstein Markov Property.** *For a given compact set P, $K \cap P = \emptyset$ we set*

$$\mathscr{R}^m := \{p_m/q_m \;,\; p_m, q_m \in \mathscr{P}^m(K),\; Z(q_m) \subset P\},$$

the triple (K, μ, P) enjoys the **Rational Bernstein Markov Property** *[respectively the* **Weighted Rational Bernstein Markov Property***] if there exists a sequence $\{C_m^{(w)}\}_{m\in\mathbb{N}}$ such that* (1) *(resp* (2) *for a given w) hold when $\mathscr{P}^m(K)$ is substituted by $\mathscr{R}^m$.*

2. Historical Remarks.

The first motivations to such a study comes from Approximation Theory and Analytic Function Theory and goes back to Szego Faber and immediatly after Erdós and Turan. Properties of this kind for general measures has been more intensively studied and developed during the late 80's and 90's.

The most important contributes are due to Widom andUllman who introduced *determining measures* and Stahl and Totik [14] who mostly studied *regular measures* essentially in the complex plane. Under the additional standard hypothesis of $K = \operatorname{supp} \mu$ being regular (see Section (5) below) the second class of measures is precisely the Bernstein Markov ones, while the first is smaller.

The present setting is instead the one more related (and adapted) to Several Complex Variables and Pluripotential Theory, it is due to Bloom Levenberg Berman and Boucksom.

3. Examples.

Example 1. Let $K = \overline{\Delta}$, $\mu = \delta_0$. This is not a BM couple.

This tells us that μ needs at least to induce a *norm* on $\mathscr{P}^m(K)$ for any $m \in \mathbb{N}$.

Example 2. Let $K := \overline{\Delta} \times \overline{\Delta}$, $\operatorname{supp} \mu = S(K)$ the Šilov boundary and $\mu := ds \times ds$. Then monomials $z^\alpha, |\alpha| \le m$ are an o.n basis of $\mathscr{P}^m(K)$ and for $z_0 \in S(K)$ extremal point for $p(z) := \sum_{|\alpha|\le m} c_\alpha z^\alpha$ we can compute

$$\begin{aligned} \|p\|_K = |p(z_0)| \le \sum_{|\alpha|\le m} |c_\alpha||z_0^\alpha| \le \sqrt{\sum_{|\alpha|\le m} |c_\alpha|^2} \sqrt{\sum_{|\alpha|\le m} |z_0|^{2\alpha}} = \\ \sqrt{\dim \mathscr{P}^m(K)} \|p\|_{L^2_\mu} = \sqrt{\frac{(m+2)(m+1)}{2}} \|p\|_{L^2_\mu} \end{aligned}$$

Notice that, being z_0 arbitrary, the constant in the inequality is exactly $\max_K \sqrt{\sum_{|\alpha|\le m} |z|^{2\alpha}}$. Looking at this example one can notice some things. First if $(S(K), \mu)$ has BMP then (K, μ)

has. We are induced to think that a BM measure for a given K should be thick on $S(K)$. Actually here we have

$$\lim_{r\to 0^+}\frac{\mu(B(z,r))}{r}=1\ \ \forall z\in S(K).$$

Example 3. In [6] authors build an example of BM measure for a given (real, initially) compact set $K\subset\mathbb{C}^n$ taking $\mu_m:=\sum\frac{\delta_{z_j^m}}{N_m}$ where $N_m=\dim\mathscr{P}^m$ and $(z_1^{(m)},\dots,z_{N_m}^{(m)})$ is a Fekete sett of degree m for K and setting $\mu:=c\sum_{m=1}^{\infty}\mu_m$, $c>0$ such that $\mu(K)=1$. Notice that the support of μ is discrete (and hence (pluri-)polar), thus is small on the potential theoretic point of view. This is an example of the following fact, in general we can find only *sufficient* conditions for BMP.

Example 4. Let $K=\overline{\Delta}\ \mu=ds$ and consider the weight function $w(z):=\exp-|z|^2$. Taking the monomials as a counterexample we can see that $z^m w(z)^m$ is achieving its maximum modulus at any $z:|z|=2^{-1/2}$ with the value $\|z^m w^m\|_K=\exp m(1/2\log 2-2)$, while the L^2_μ norms are $\|z^m w^m\|_{L^2_\mu}=e^{-m}$. Therefore we have $\left(\frac{\|z^m w^m\|_K}{\|z^m w^m\|_{L^2_\mu}}\right)^{1/m}=\frac{e}{\sqrt{2}}>1$. This shows that the weighted case is essentially different, but some measures are good for all weights.

4. Bergman Function.

Let us go back to Example 2. We calculated the constant for the inequality as

$$C_m=\max_K\sqrt{\sum_{|\alpha|\le m}|z|^{2\alpha}}=\max_K\sqrt{\sum_{|\alpha|\le m}|q_\alpha(z,\mu)|^2}. \tag{3}$$

Where we indicated by $q_\alpha(z,\mu)$ the orthogonal polynomial of degree α obtained by Gram-Schmidt ortogonalization with lexicographical ordering.

More generally we can notice that $\mathscr{P}^m_\mu:=(\mathscr{P}^m,\langle\cdot;\cdot\rangle_{L^2_\mu(K)})$ is a reproducing kernel Hilbert space, being the kernel

$$K^\mu_m(z,\zeta):=\sum_{|\alpha|\le m}q_\alpha(z,\mu)\bar{q}_\alpha(\zeta,\mu).$$

It is customary to denote by B^μ_n the *diagonal* of such a kernel, namely the Bergman function

$$B^\mu_m(z):=K_m(z,z)=\langle K^\mu_m(z,\zeta);K^\mu_m(z,\zeta)\rangle_{L^2_{\mu(\zeta)}(K)}.$$

It is not difficult to prove that this is actually the general case:

- Let $\delta_z\in L(\mathscr{P}^m_\mu,\mathbb{C})$ be the point-wise evaluation, for any $z\in K$ we have $\|\delta_z\|=\sqrt{B^\mu_m(z)}$.
- *The best possible constant in* (1) *is* $\sqrt{\|B^\mu_m\|_K}$.

To go on we need to introduce some concepts from Pluripotential Theory.

5. Rawly concise survey on Pluripotential Theory.

Pluripotential theory is a non linear potential theory arising from the study of the Monge Ampere operator and the plurisubharmonic functions.

For an open set $\Omega \subset \mathbb{C}^n$ and $u \in C^2(\Omega)$ one defines first

$$\mathrm{dd^c}\, u := \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) dz_j \wedge d\bar{z}_k. \tag{4}$$

Then by smoothing it turns out that for any plurisubharmonic u the matrix $[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}]_{j,k}$ can be defined as a positive definite Hermitian form in the sense of distributions. That is $\forall v \in \mathbb{C}^n$ and any test function φ we set

$$\left\langle v^T [\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}](z) v \, ; \, \varphi \right\rangle := \int u\, v^T [\frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}](z) v \, d \operatorname*{Vol}_{\mathbb{C}^n}(z).$$

Even more interesting is that it can be defined as a positive $(1,1)$ current (i.e. an element of the dual of the test forms of bidegree $n-1, n-1$)).

In a fundamental paper [2] Bedford and Taylor found out how to define the wedge product $(\mathrm{dd^c}\, u)^n = \wedge^n \mathrm{dd^c}\, u$ for any plurisubharmonic locally bounded function, enjoying some continuity property and such that for C^2 functions we have

$$(\mathrm{dd^c}\, u)^n = c(n)\, \det[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}]_{j,k} d \operatorname*{Vol}_{\mathbb{C}^n}. \tag{5}$$

We denote by $\mathcal{L}$ the Lelong class of plurisubharmonic functions having logarithmic pole at infinity. Given a nonpluripolar (see[9][pg. 67]) compact set K we set

$$V_K^*(z) := \limsup_{\zeta \to z} \left(\sup\{u(\zeta) \in \mathcal{L}\, , \; u|_K \leq 0\} \right) \tag{6}$$

the Plurisubharmonic (locally bounded) Extremal Function which is solving the differential problem $(\mathrm{dd^c}\, u)^n = 0$ on $\mathbb{C}^n \setminus K$ among all plurisubharmonic function having logarithmic pole at ∞.

It turns out that $\mu_K := (\mathrm{dd^c}\, V_K^*(z))^n$ is a Borel probability measure on K, we call it the pluripotential equilibrium measure for K.

6. Motivations for BMP.

Why should be worth to study the BMP? There are at least three good different reasons.

- *DLS asymptotic.* There exists a sort of L^2 Bernstein Walsh Lemma for regular compact sets K. Let $\mathscr{L}_m : C(K) \to \mathscr{P}^m_\mu$ be the o.n. projection. If (K, μ) has the BMP then if $d_\infty(f, \mathscr{P}^m)^{1/m} = \frac{1}{R}$ (i.e. $f \in hol(\{V_K < \log R\})$) then $\|f - \mathscr{L}_m f\|_K^{1/m} = \frac{1}{R}$. On the other hand, if $d_2(f, \mathscr{P}^m)^{1/m} = \frac{1}{R}$, then $d_\infty(f, \mathscr{P}^m)^{1/m} \leq \frac{1}{R}$ and thus $f \in hol(\{V_K < \log R\})$.
- *m-th roots asymptotic.* For any regular compact set $K \subset \mathbb{C}^n$ and any admissible weight $w = -\log Q$ such that (K, μ, w) has the WBMP

$$\lim_m \frac{1}{2m} \log B_m^{\mu,w} = V_{K,Q} \;\; \text{loc. uniformly in } \mathbb{C}^n. \tag{7}$$

This result is achieved combining the reproducing property and the BMP to relate $B_m^{\mu,w}$ to the weighted Siciak extremal function[13] $\Phi_{m,K,Q}$ in the inequality

$$\left(\frac{\Phi_{m,K,Q}}{N_m}\right)^2 \leq \frac{B_m^{\mu,w}}{N_m} \leq C_m^2 \Phi_{m,K,Q}^2.$$

- *Free energy asymptotic.* If (K,μ) has the BMP then we have

$$\limsup_m \left(\int \ldots \int \left|\mathrm{VDM}_m(z_1,\ldots,z_{N_m})\right|^2 d\mu(z_1)\ldots d\mu(z_{N_m})\right)^{\frac{n+1}{2nmN_m}} = \delta(K) \tag{8}$$

Where $\delta(K) = \lim_m \delta_m(K)^{\frac{n+1}{nN_m}}$ is the transfinite diameter, VDM is the Vadermonde matrix and $\delta_m = \max_{K^m} |\mathrm{VDM}_m(z_1,\ldots,z_{N_m})|$. This statement goes directly to the weighted case. Equation (8) is exactly what is needed for the so called *Strong Bergman Asymptotic*, i.e.we have

$$\overset{*}{\lim_m} \frac{B_m^{\mu,Q}}{N_m} = \mu_{K,Q}. \tag{9}$$

Moreover is the starting point to develop a sophisticate probabilistic machinery which leads to introduce a sequence of measures and prove [7], [8] a Large Deviation Principle where the good rate functional is the *primitive* of the Monge Ampere, namely the *pluripotential energy.*

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7. Sufficient Condition for BM.

How to find a measure satisfying a BMP for a given compact set K?

Let us go back again to Example 2, we have

$$K = \overline{\Delta}\times\overline{\Delta},\quad \operatorname{supp}\mu = S(K) \text{ and}$$
$$\lim_{r\to 0^+} \frac{\mu(B(z,r))}{r} = 1 \quad \forall z \in S(K).$$

This is somehow a prototypical case, is not difficult to move to any $K = Cl_{\mathbb{C}}(D) \subset\subset \mathbb{C}$ for some open set D and μ absolutely continuous w.r.t. the arc-length measure on the outher boundary of K.

To deal with general sets and measures and to grasp the core of the problem one should move from looking at *geometrical/analytical* properties to use purely *potential theoretic means.* Working in this spirit Stahl and Totik proved the following theorem working in one complex variable.

Theorem 7.1 (Mass-density sufficient condition in the plane.)**.** *Let μ be positive Borel measure with compact support $K = \operatorname{supp}\mu$ in $\mathbb{C}$, suppose that K is a non-polar regular set w.r.t. the Dirichlet problem for the Laplace operator and there exists $t > 0$ such that*

$$\lim_{r\to 0^+} \operatorname{cap}\left(\{z \in K : \mu(B(z,r)) > r^t\}\right) = \operatorname{cap}(K). \tag{10}$$

Then (K,μ) has the BMP.

Here cap() is the logarithmic capacity of the set K,

$$\operatorname{cap}(K) := \max_{\nu\in\mathcal{M}_1(K)} \exp\left(\int\int \log|z-\zeta| d\nu(z)d\nu(\zeta)\right).$$

This techniques are working in the complex plane, thinking to the distributional Laplacian and the *linear* potential theory related to it.

There is a capacity in the non-linear pluripotential theory in $\mathbb{C}^n$ related to the Monge Ampere complex operator, namely the **relative capacity** $\operatorname{Cap}(K,\Omega)$ w.r.t. a hyperconvex supset Ω of K.

In this setting Bloom and Levenberg [5] have extended Theorem 7.1 above that reads as follows.

Theorem 7.2 (Mass-density sufficient condition in $\mathbb{C}^n$.)**.** *Let μ be positive Borel measure with compact support $K := \operatorname{supp}\mu \subset B(0,1)$ in $\mathbb{C}^n$, suppose that K is a non-pluripolar $\mathcal{L}$-regular set and there exists $t > 0$ such that*

$$\lim_{r\to 0^+} \operatorname{Cap}\left(\{z \in K : \mu(B(z,r)) > r^t\}, B(0,1)\right) = \operatorname{Cap}(K, B(0,1)). \tag{11}$$

Then (K,μ) has the BMP.

The proof of these results relays on th following facts

(A) One side of the Bernstein Walsh and Bernstein Walsh Siciak lemmas respectively. That is, for any regular non pluripolar polynomially convex compact K and $p \in \mathscr{P}(K)$ we have

$$|p(z)| \leq \|p\|_K \exp(\deg(p)V_K(z)). \tag{12}$$

(B) Polynomials are holomorphic functions in a neighborhood of K, we can use a Cauchy type estimate.

(C) A part of a five conditions equivalence theorem due (in the $\mathbb{C}^n$ case) to Bloom and Levenberg.

Theorem 7.3. *For any sequence of compact subsets of the compact non pluripolar regular set K the following facts are equivalent*

(i) $\lim_j \operatorname{cap}(K_j) = \operatorname{cap}(K)$*, if* $n = 1$*, otherwise* $\lim_j \operatorname{Cap}(K_j, B(0,1)) = \operatorname{Cap}(K, B(0,1))$.

(ii) $\lim_j V_{K_j} = V_K$ *locally uniformly in* $\mathbb{C}^n$.

What follows is the work I made here with my co-advisor N. Levenberg and has been deeply stimulated/motivated by his work with F. Wielonsky.

8. Rational BMP

The Large Deviation Principle (LDP) proved in [8] applies to sequences of (vector of) probability measures which are defined starting from a rational Bernstein Markov measure.

We searched for a nice potential theoretic sufficient condition by the same argument of theorems 7.1 and7.2. We proved the following.

Theorem 8.1. *Let K be a regular non polar compact set in the complex plane, $\Omega := \mathbb{C}_\infty \setminus \hat{K}$ and $P \subset \Omega$ a compactum. Let $\mu \in \mathcal{M}(K)$ and suppose there exists a positive T such that*

$$\lim_{r\to 0^+} \operatorname{Cap}\left(\{z : \mu(B(z,r)) \geq r^T\}\right) = \operatorname{Cap}(\operatorname{supp}\mu). \tag{13}$$

Then μ enjoys the Bernstein Markov property on K for the rational functions with poles in P.

The overall outline of the proof is similar, however we are no more dealing with polynomials, the Bernstein Walsh Inequality is replaced by the following estimate

$$|r_m(z)| \leq \|r\|_K \exp\left(\sum_{z_j \in \mathrm{Poles}(r_m)} g_{\Omega_K}(z, z_j)\right) \quad \forall r_m \in \mathscr{R}_m(K, P). \tag{14}$$

Here $g_{\Omega_K}(z, z_j)$ is the generalized Green function for the set $\Omega_K := \mathbb{C}_\infty \setminus K$ having pole at z_j. Therefore the Theorem 7.3 is no more useful, we need to state and prove a different one.

Theorem 8.2. *Let $K \subset \mathbb{C}$ be a regular non polar compact set, let Ω_K be the unbounded component of $\mathbb{C} \setminus K$ and P a compact subset of Ω_K such that $P \cap K = \emptyset$. Then there exist a domain D such that $K \subset\subset D$ and $P \cap \overline{D} = \emptyset$, such that for any sequence $\{K_j\}$ of compact subsets of K the following are equivalent (here Ω_{K_j} is defined similarly to Ω_K).*

$$\lim_j \operatorname{Cap}(K_j) = \operatorname{Cap}(K). \tag{15}$$

$$\lim_j g_{\Omega_j}(z, a) = g_\Omega(z, a) \text{ loc. unif. for } z \in D \text{ unif. for } a \in P. \tag{16}$$

9. The task of unbounded set in $\mathbb{C}$

We almost finished proving a couple of theorems... In the case of a closed unbounded set K and an admissible weight function $w : \mathbb{C} \to [0, +\infty[$ we address the task of providing a sufficient mass density condition for the Weighted Bernstein Markov Property.

We can proceed as follows.

- First we consider a compactification of the problem, this is done by means of inverse stereographic projection $\mathcal{S}$. Consequently we gain an extra term in the weight, namely $\log(1 + |z|^2)$. $\mathcal{S}(K)$ is now a compact subset of $\mathbb{S}^2 \subset \mathbb{R}^3$.
- We try to build a Strong Bernstein Markov Measure. We can use a generalization of the technique used in [8][Th. 4.6]. That is..
- We consider the complexification: $\mathcal{A} := \{z \in \mathbb{C}^3 : \sum z_j^2 = 1\}$ of the sphere, if we prove a Bernstein Markov property for holomorphic polynomials $\mathscr{P}(\mathcal{A})$ then it turns in a strong Bernstein Markov one on the *real points* where K lives.
- We notice that $\mathcal{A}$ is a smooth (unbounded!) algebraic subvariety of $\mathbb{C}^3$, there is a specific pluripotential Theory for such kind of sets [12],[15]. We denote by $V_E(z, \mathcal{A})$ the new extremal function.
- In such a setting the proof of an adapted formulation of 7.2 works provided an adapted version of 7.3.

There is a specific $\mathbb{C}$−linear change of coordinates such that $\mathcal{A} \subset \{(z, w) \in \mathbb{C}^2 \times \mathbb{C} : |w|^2 \leq C(1 + |z|^2)\}$. We fix it and we introduce for $R >> 1$ $\Omega(r) := \{(z, w) \in \mathcal{A} : |z|^2 - R < r\}$, then we the new theorem reads as follows.

Theorem 9.1. *Let $\mathcal{A} \subset \mathbb{C}^n$ be an algebraic variety of pure dimension $m < n$, $\mathcal{A}_{reg} \supset \Omega_0 \supset K$ where K is a compact $\mathcal{L}$ regular nonpluripolar set. Let $\{K_j\}_{j\in\mathbb{N}}$ be a sequence of compact subsets of K, then the following are equivalent.*

(i) $\lim_j \mathrm{Cap}(K_j, \Omega(-\sqrt{R^2-1})) = \mathrm{Cap}(K, \Omega(-\sqrt{R^2-1}))$.

(ii) $V^*_{K_j}(\cdot, \mathcal{A}) \to V^*_K(\cdot, \mathcal{A})$ *point-wise on* $\mathcal{A}$.

The proof (of even the other equivalences) is almost finished and the main difference with the original one is in providing the *Capacity comparison theorem* originally stated in $\mathbb{C}^n$ by Alexnder and Taylor [1], for which we give a new formulation and a new proof.

Theorem 9.2. *Let $\mathcal{A}$ be a m-dimensional algebraic variety of $\mathbb{C}^n$, such that for an $R > 1$ $\mathcal{A}_{reg} \supset \Omega_0$, then for any $r < -\sqrt{R^2-1}$ there exist two positive constants c_1, c_2 such that for any compact $K \subset \Omega_r$.*

$$\exp\left[-\left(\frac{c_1}{\mathrm{Cap}(K, \Omega(-\sqrt{R^2-1}))}\right)^{1/m}\right] \geq T^A(K), \tag{17}$$

$$T^A(K) \geq \exp\left(-\frac{c_2}{\mathrm{Cap}(K, \Omega(-\sqrt{R^2-1}))}\right). \tag{18}$$

In particular for any $E \subset K$ we have $\|V_E(z, \mathcal{A})\|_{\Omega(-\sqrt{R^2-1})} \leq \frac{c_2}{\mathrm{Cap}(K,\Omega(-\sqrt{R^2-1}))}$.

Here $T^A(K)$ is a specific variant of the Tchebyshev constant we introduce for replacing the standard one in this particular coordinates switching.

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room 712 Department of Mathematics, Universitá di Padova, Italy. Phone +39 0498271260

E-mail address: fpiazzon@math.unipd.it

URL: http://www.math.unipd.it/~fpiazzon/ (work in progress)