Monge-Ampere Capacities on Algebraic Varieties: Comparability and Convergence Results

Mini-courses in Mathematical Analysis

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Pluripotential Theory

Aim

Concise overview of a non linear extension of logarithmic potential theory to several complex dimensions.

psh functions

Pluripotential theory is the study of plurisubharmonic (psh) functions in several complex variables.

- $u : \mathbb{C}^n \supseteq \Omega \to \mathbb{R} \cup \{-\infty\}$ is upper semi-continuous,
- u is shm along each complex line.

We introduce

$$\partial := \sum_{j=1}^{n} \frac{\partial}{\partial z_j} dz_j, \quad \bar\partial := \sum_{i=j}^{n} \frac{\partial}{\partial \bar z_j} d\bar z_j$$

$$d := \partial + \bar\partial, \quad d^c := i(\bar\partial - \partial)$$

such that $dd^c = 2i\partial\bar\partial$.

$u \in \mathcal{C}^2$ is psh iff $dd^c u$ is a positive $(1,1)$ form.

more generally (using the theory of distributions)

$u \in L^1_{\text{loc}}$ usc is psh iff $dd^c u$ is a positive $(1,1)$ *current*.

Complex Monge Ampere operator I

If $u \in \mathcal{C}^2$ we can consider

$$(\mathrm{dd^c}\, u)^n := \mathrm{dd^c}\, u \wedge \mathrm{dd^c}\, u \wedge \cdots \wedge \mathrm{dd^c}\, u = c_n \det[\partial\bar\partial u] dVol_{\mathbb{C}^n},$$

but defining the product of currents may be complicated.

Theorem [Bedford and Taylor (1981)]

If $u_1, \ldots, u_n$ are locally bounded psh functions then

$$\mathrm{dd^c}\, u_1 \wedge \mathrm{dd^c}\, u_2 \wedge \cdots \wedge \mathrm{dd^c}\, u_n := \overset{*}{\lim_{\epsilon \to 0^+}} \mathrm{dd^c}\, u_1^\epsilon \wedge \mathrm{dd^c}\, u_2^\epsilon \wedge \cdots \wedge \mathrm{dd^c}\, u_n^\epsilon$$

is a well defined locally bounded positive measure. Here u^ϵ is any radial smooth mollification.

Inductive definition, for a test form θ:

$$\int \mathrm{dd^c}\, u \wedge \mathrm{dd^c}\, u \wedge \theta := \int u\, \mathrm{dd^c}\, u \wedge \mathrm{dd^c}\, \theta.$$

Complex Monge Ampere operator II

- approximation property of psh functions
- locally boundedness (for a good definition)
- positivity (distribution $\rightsquigarrow$ measure)

Rmk: continuity under decreasing limits is part of the statement.

Counterexample to L^1 continuity

Warning: the monotonicity is a important assumption!

General procedure for counterexamples:

- **Fact:** if $v = (c \log |f|)^+$ for a positive c and a never vanishing holomorphic f, then $(\mathrm{dd^c}\, v)^n = 0$
- Let (the limit is in L^1_{loc})

$$u(z) := \log \max\{|f_1(z)|, \ldots, |f_m(z)|\} = \lim_{p \to \infty} \left(\frac{1}{p} \log \left| \sum_{j=1}^{m} f_j(z) \right| \right)^+$$

in general $(\mathrm{dd^c}\, u)^n \neq 0$.

- but we always have

$$\left(\mathrm{dd^c} \left(\frac{1}{p} \log \left| \sum_{j=1}^{m} f_j(z) \right| \right)^+ \right)^n = 0$$

Maximal psh functions and MA

If v is psh in Ω and for any domain $\Omega_1 \subset \Omega$ and u psh with $u \leq v$ in $\partial\Omega_1$ we have $u \leq v$ in Ω_1, we say that v is maximal.

v is maximal iff it has minimal MA, i.e., $(\mathrm{dd^c}\, v)^n = 0$.

Geometrically

$$sh \rightsquigarrow psh$$
$$h \rightsquigarrow \text{maximal } psh.$$

Example: $u := \log(|z|^2 + 1)$, $v := \log^+ |z| + c$.

Extremal psh Functions I

Global. Lelong class $\mathcal{L}(\mathbb{C}^n)$, i.e., psh with logarithmic pole at infinity. For any $E \subset \mathbb{C}^n$ compact the extremal plurisubharmonic function

$$V_E(z) := \sup\{u(z) : u \in \mathcal{L}(\mathbb{C}^n), u|_E \leq 0\}$$
$$V_E^*(z) := \overline{\lim}_{\zeta \to z} V_E(\zeta).$$

is the counterpart of the Green function for $\mathbb{C}^n \setminus E$ with pole at infinity.
It turns out (Siciak and Zaharjuta) that $\tilde{V}_E^*(z) \equiv V_E^*(z)$, where

$$\tilde{V}_E(z) := \sup\left\{\frac{1}{\deg p} \log^+ |p(z)| : \|p\|_E \leq 1\right\}.$$

Local. For open $\Omega \supset E$ compact

$$U_{E,\Omega}(z) := \sup\{u(z) : u \in psh(\Omega), u|_E \leq -1, u \leq 0\}$$
$$U^*_{E,\Omega}(z) := \overline{\lim}_{\zeta \to z} U_{E,\Omega}(\zeta).$$

Is *close* to the potential of a condenser...

pluripolar sets are subsets of the $-\infty$ level sets of psh functions.

Facts:

- V_E^* is either $+\infty$ (off a *pluripolar set*) or is a locally bounded psh function such that the equilibrium measure $\mu_E := (dd^c V_E^*)^n$ is supported on E. That is $\mu_E = 0$ on $\mathbb{C}^n \setminus E$.
- $U_{E,\Omega}^*$ is either 0 (off a *pluripolar set*) or is a locally bounded psh function such that the relative equilibrium measure $\mu_{E,\Omega} := (dd^c U_{E,\Omega}^*)^n$ is supported on E.

Many (non equivalent but comparable) capacities have been introduced in pluripotential theory.
Bedford and Taylor introduced the relative MA capacity

$$\mathrm{Cap}(E,\Omega) := \sup\left\{\int_E (dd^c u)^n, u \in psh(\Omega), 0 \leq u \leq 1\right\}$$

and showed that

$$\mathrm{Cap}(E,\Omega) = \int_E (dd^c U_{E,\Omega})^n.$$

Siciak used the Chebyshev constant

$$T(E) := \lim_k \left(\inf_{\deg p \leq k} \{\|p\|_E : \|p\|_\Omega \geq 1\}\right)^{1/k}$$

we have $\sup_{|z|\leq 1} V_E^*(z) = -\log T(E)$.

Motivations and Applications

Many related field

- almost no ”practical” applications so far
  - ongoing research in superstring theory
- plenty of theoretical motivations and applications
  - Approximation Theory
  - Complex Analysis
  - Complex Geometry
  - Probabillity

In particular…

- V_E^* is related to the maximal growth of polynomials (Bernstein Walsh Inequality) and control the relationship among the radius of maximum holomorphic extension and the approximation numbers of a function.
- V_E^* is related to the asymptotic behaviour of general orthogonal polynomials
- $(dd^c V_E^*)^n$ is the asymptotic of good interpolation points (Berman and Boucksom)
- Applications to the Bergman metric
- Calabi Conjecture, Kahler-Einstein metrics
- Large deviation principles for random arrays, polynomials and matrices. Determinantal point processes.

New Results

Our variety setting

- We assume $A \subset \mathbb{C}^m$ to be a algebraic irreducible variety of *pure dimension* $1 \le m < n$.
- We use Rudin coordinates $z = (z', z'') \in \mathbb{C}^m \times \mathbb{C}^{n-m}$: A is an analytic covering.
- We define $\mathcal{L}(A) := \{u \in psh(A) : u - \log |z'| \text{ bounded at } \infty\}$
- psh and weakly psh functions almost coincide (Demailly).

This setting was first considered by Zeriahi and Bedford: *extension of m-dimensional pluripotential theory from A_{reg} to A.*

Sadullaev algebraicity condition

Consider any irreducible variety A and $E \subset A$ compact and define

$$\tilde{V}_E(\zeta) := \sup\{\frac{1}{\deg p} \log |p(\zeta)| : \|p\|_E \leq 1\}.$$

Two situation may occur

- $\tilde{V}_E(\zeta) = +\infty$ for some set E such that $E \cap A_{reg}$ is not pluripolar
- $\tilde{V}_E$ is locally bounded for any such set E and $\tilde{V}_E^* \equiv V_E^*$ on A_{reg},

the second scenario occurs if and only if A is algebraic.

Aims

Extend to A the results holding in $\mathbb{C}^n$

- Comparability of $\mathrm{Cap}(E, \Omega)$ with $T(E)$ (Alexander and Taylor 1984)
- Continuity properties under convergence in capacity (Bloom and Levenberg 2009)
- Application to general orthogonal polynomials (Bloom 1996)

Notation:

$$\Omega(r) := \{z \in A : |z'| < r\}, \quad \Omega := \Omega(1).$$

Comparability result

Theorem 1 [P., Levenberg]

Let A be a irreducible pure m-dimensional algebraic subset of $\mathbb{C}^n$. For any $0 < r < 1$ there exist two positive constants c_1, c_2 (depending only on A and r) such that for any compact non pluripolar $E \subset \Omega(r)$ we have

$$\exp\left[-\left(\frac{c_1}{\operatorname{Cap}(E,\Omega)}\right)^{1/m}\right] \geq T(E,A), \tag{1}$$

$$T(E,A) \geq \exp\left(-\frac{c_2}{\operatorname{Cap}^2(E,\Omega)}\right). \tag{2}$$

In particular

$$\max_{\overline{\Omega}} V_E(\cdot,A) \leq \frac{c_2}{\operatorname{Cap}^2(E,\Omega)}. \tag{3}$$

Elements of the proof

- relation among $V_E(\cdot, A)$ and $T(E, A)$ still valid (Zeriahi)
- combine the original technique of Alexander and Taylor with properties of proper coordinate projections and extremal functions of images and pre-images under projections.
- provide a particular Chern Levine type estimate for

$$\int_E (dd^c u)^m \leq C^{m-1} \|u\|_{D'}^{m-1} \int_{D'} dd^c u \wedge \beta_m^{m-1}, \;\; E \subset D' \subset D$$

  Need to prove integration by parts formulas for currents.
- use the Poisson Jensen Lelong Formula for algebraic varieties proved by Demailly $\mu_r := (dd^c \varphi_r)^m - \chi_{A\setminus\Omega_r} (dd^c \varphi)^m$

$$\begin{aligned}\int u d\mu_r &= \int_{\Omega_r} u\,(dd^c \varphi_r)^m + \int_{-\infty}^{r} \int_{\Omega_t} dd^c u \wedge (dd^c \varphi)^{m-1} dt \\ &= \int_{\Omega_r} u\,(dd^c \varphi_r)^m + \int_{\Omega_r} (r-\varphi)\, dd^c u \wedge (dd^c \varphi)^{m-1}.\end{aligned}$$

Problem

Let $E_j \subset E \subset \Omega \subset A$ and set

- $u_j(z) := U^*_{E_j,\Omega}(z)$
- $v_j(z) := V^*_{E_j}(z, A)$

what can we say about $u_j \to u := U^*_{E,\Omega}$ and $v_j \to V^*_E(\cdot, A)$?

In general convergence holds if $E_j \uparrow E$.

Convergence result

Theorem 2 [P., Levenberg]

Let $A \subset \mathbb{C}^n$ be a pure m-dimensional irreducible algebraic subset of $\mathbb{C}^n$, and $E \subset \Omega$ be a compact set. Let $\{E_j\}$ be a sequence of Borel subsets of E. Then the following are equivalent.

i) $\lim_j \mathrm{Cap}(E_j, \Omega) = \mathrm{Cap}(E, \Omega)$

ii) $\lim_j u_j = u$ in capacity and $(dd^c u_j)^m \rightharpoonup^* (dd^c u)^m$.

iii) $\lim_j u_j = u$ point-wise on Ω.

iv) $\lim_j v_j = v$ point-wise on A.

If we furthermore suppose E to be regular and E_j to be compact for any j, then equations (iii) and (iv) can be replaced by

v) $\lim_j u_j = u$ uniformly on Ω.

vi) $\lim_j v_j = v$ uniformly on A.

Elements of the proof

- technical adjustment of the proof by Bloom and Levenberg
- back to the original reasoning by Bedford and Taylor (proof of quasicontinuity)
- $A_{sing} \neq \emptyset$ implies a longer reasoning...
- bounds for $||V_{E_j}(\cdot, A)||_{\Omega}$ given by Theorem 1
- Integration by parts for currents
- improving of the mode of convergence: Hartog's Lemma.

Application of our results

Stähl-Totik regular measures

Borel Measure $\mu \in \mathcal{M}^+(E)$, $E \subset \mathbb{C}$ compact, with *regular asymptotic behaviour*:

$$\overline{\lim}_k \left(\sup\left\{|p(z)| : \deg p \leq k, \|p\|_{L^2_\mu} \leq 1\right\}\right)^{1/k} =_{q.e.} 1 \text{ on } E.$$

Set $B_k^\mu(z) := \sum_{j=0}^k |q_j(z,\mu)|^2$ (o.n. poly), it follows that

$$\lim_k \frac{1}{2k} \log B_k^\mu(z) =_{q.e.} g_E(z)$$

They provide an almost sharp sufficient condition fot the regular asymptotic behaviour...

Bernstein Markov property

Borel Measure $\mu \in \mathcal{M}^+(E)$, $E \subset \mathbb{C}^n$ compact such that

$$\overline{\lim}_k \left(\|B_k^\mu\|_E\right)^{1/k} = 1.$$

Important consequences

- $\lim_k \frac{1}{2k} \log B_k^\mu(z) = V_E^*(z)$ uniformly,
- $\lim_k \frac{B_k^\mu}{\dim \mathscr{P}^k(\mathbb{C}^n)}\mu = \mu_E$ weak*

Sufficient mass density condition: $\exists t > 0$

$$\lim_{r\to 0^+} \operatorname{Cap}\left(\{z \in E : \mu(B(z,r)) > r^t\}, \Omega\right) = \operatorname{Cap}(E,\Omega).$$

Application of Theorem 2

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Theorem 3 [P., Levenberg]

Let A be a pure m dimensional irreducible algebraic set in $\mathbb{C}^n$, $n > m$. Let E be a compact regular subset of Ω and $\mu \in \mathcal{M}^+(E)$ such that $\operatorname{supp} \mu = E$. Suppose that there exists $t > 0$ such that the following mass density condition holds

$$\operatorname{Cap}(E, \Omega) = \lim_{r \to 0^+} \operatorname{Cap}\left(\left\{z \in E : d(z', Y) > 2r \text{ and } \mu(\Omega_{j(z)}(z, r)) > r^t\right\}, \Omega\right).$$

Then (E, μ) has the Bernstein Markov property for the restriction of polynomials to A.

Here $\Omega_{j(z)}$ is the leaf of $\pi^{-1}\pi(\Omega)$ containing z.

Corollary : the standard volume measure has the Bernstein Markov Property.

Main References I

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Thank You!