Discussione Titoli RTDB Mat/08

An Overview of my Recent Research

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Some qualifications I

1. **PhD in Mathemathics**. University of Padova 2016, under the supervision of Prof. M. Vianello.
   Thesis "*Bernstein Markov Properties and Applications*".
   Thesis Advisor: Prof. N. Levemberg (Indiana University)
2. **ASN** MAT/08 *professore seconda facia*
3. **Teaching/Advising:**
   - advisor of 1 PhD student (since Jan. 2023)
   - advisor of 1 master thesis in Mathematics
   - co-advisor of 3 master thesis in Mathematics and 1 in Mathematical Engeneering
   - 2 crash courses (12h) and 1 invited lecture for PhD students,
   - 3 bachelor courses (with course responsability) 160h of lecture,
   - 11 bachelor or master (without course responsability) ≈200h.
4. **Previous research positions:**
   - 4 years of post-doctoral research fellowships (3 contracts),
   - 9 months post-lauream research fellowship.
5. **Research projects:**
   - participant of a EC project founded by horizon 2020
   - participant of a 18 months research project founded by ENIprogetti
   - participant of 5 national research projects

Some qualifications II

6 **Conferences:**

- 16 talks given at international conferences,
- 4 posters presented at international conferences.

7 **Others:**

- Member of doctoral school council of Math. Dept.
- Member of the commitee of 3 reaserch fellowships
- 2021 Best Paper Award in *Optimization Letters*
- 2 invited seminars (international) and 3 visiting periods
- Author of 3 deliverables of EC project, 2 reports for ENIprogetti
- member of the ”Commissione di Laurea” for the bachelor and master in Mathematics
- member of sci.comm. or organizer of international conferences
- editor of 2 special issues

Bibliometrics

Author of

- 21 published articles (5 as unique author)
- 3 conference proceedings
- 1 arχiv preprint (50 pg., submitted to ESAIM: Mathematical Modelling and Numerical Analysis)
- 1 arχiv preprint 45 pg.

	Documents	Citations	h-index
Scopus	24	186	9
WoS	24	179	8
Scholar	55	338	11

NB: data are updated at 15/06/2023

Overview of my research

- since 2021: Numerical methods for viscoelastic models and parametric PDEs
- since 2019: Numerical Methods for Optimal Transport
- since 2013: Pluripotential Theory and applications
- since 2013: *multivariate* Approximation Theory

Pol. Samp., Approx.	**Pluripot. Theory**	**Optimal Transport**	**PDE and Modelling**
ONP and LS behaviour Low card. meshes Mesh stability Meas. and DLS compress. Optimal Design Polynomial optim.	L^2 methods PPT on alg. manif. Compar. of Cap Explicit formulas Numer. approx. alg.	Energy for L^1 tran. GF of energy FEM approximation spectral approx. related models	Spec. meth. viscoelast. Mix. meth. viscoelast. Variational deduction of Mix form. dissip. prob. Param./stoc. PDEs
13 papers and 3 proc.	7 papers	1 paper, 2 prepr.	3 prep.

This talk: selected contributions to the third topic

Ongoing research projects:

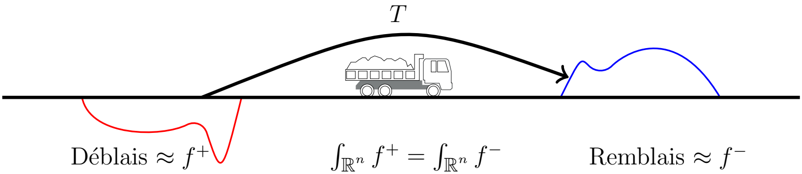
1. Numerical Approximation of Optimal Transport, with E. Facca (Bergen) and M. Putti, and applications, with G. Santin (FBK Trento),
2. Mathematical and numerical (pure spectral) modelling of (linearized) visco-elastic waves propagating in a etherogeneous domain, with N. Crescenzio, A. Larese, G. Giusteri and M. Putti. Project supported by private company (covered by non-disclosure agreement).
3. Total variation: mixed formulation, numerical approximation, and application to denoising, with N. Segala.
4. Mixed formulation of dissipative models: a variational deduction, with F. Fantin.
5. Optimal polynomial approximation of parametric PDEs solution maps, with M. Putti.

Let us focus on the first topic. . .

Morally speaking

Optimal Transport

Search for the strategy of moving mass from a given configuration to a target one minimizing the total cost.



apart from *soil dragging*, there are plenty of applications

- Probability theory
- Cosmology
- Image recovery/classification

Mathematical models of OT

Given $\nu^+, \nu^- \in \mathcal{M}^+(\mathbb{R}^n)$ with equal mass and $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$

Monge formulation

Find a minimizer (*optimal transport map*) T^* of the transport cost

$$T \mapsto \int c(x, T(x)) d\nu^+$$

among the set of transport maps $\{T : \mathbb{R}^n \to \mathbb{R}^n, \text{ Borel} : \ T_* \nu^+ = \nu^-\}$.

Kantorovich formulation

Find a minimizer (*optimal transport plan*) γ^* of

$$\gamma \mapsto \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x, y) d\gamma(x, y),$$

subject to $\gamma(\cdot, \mathbb{R}^n) = \nu^+, \gamma(\mathbb{R}^n, \cdot) = \nu^-$.

Our setting: L^1 optimal transport

Assumptions

1. $c(x, y) := |x - y|$.
2. $\nu^{\pm}$ are absolutely continuous w.r.t. Lebesgue with densities $f^{\pm} \in L^{\infty}(\mathbb{R}^n)$.
3. $f := f^{+} - f^{-}$ is compactly supported.

Disclamer: the famous works by Benamou, Brenier, and Otto are not concerned on L^1 OT, their results do not apply to this setting.

Search for a transport density

Monge-Kantorovich Eqs. (Evans-Gangbo '99)

Let Ω be an open bounded Lipschitz domain with $co(\operatorname{supp} f) \subset\subset \Omega$. Find a non-negative function $\mu^* \in L^\infty(\Omega)$ for which the following system of PDEs admits a "weak" solution u^*

$$\begin{cases} -div(\mu^* \nabla u^*) = f, & \text{in } \Omega \\ |\nabla u^*| \leq 1, & \text{in } \Omega \\ |\nabla u^*| = 1 & \mu^* \text{ a.e. in } \Omega \end{cases} . \qquad \text{(MK-eqs)}$$

μ^* : optimal *transport density*

u^* : *Kantorovich potential*

$\rightsquigarrow$ Can be used to find T^*

New approach: L^1 Transport Energy

E. Facca, F. Piazzon, M. Putti. *L^1 Transport Energy*. *Appl. Math. Opt. 2022*

Definition (Facca, P., Putti)

We denote by $\mathcal{E} : \mathcal{M}_+(\Omega) \to [0, +\infty]$ the L^1 transport energy functional defined by

$$\mathcal{E}(\mu) := \sup_{u \in \mathscr{C}^1(\overline{\Omega}), \int_\Omega u dx = 0} \left(2 \int_\Omega f u \, dx - \int_\Omega |\nabla u|^2 d\mu \right) + \int_\Omega d\mu.$$

Proposition (Facca, P., Putti)

The transport energy admits a unique minimizer and

$$\operatorname*{argmin}_{\mu \in \mathcal{M}_+(\Omega)} \mathcal{E}(\mu) = \mu^*.$$

Our strategy

Discretize-Minimize approach

- **Discretize:** construct (by a suitable FEM discretization) a sequence $\{\mathcal{E}_n\}$ of ”good” *variational approximations* of $\mathcal{E}$
- **Minimize:** design a global minimization scheme for $\mathcal{E}_n$

- ”good”:
  - finite dimensional
  - smooth, convex, ”easy” to compute and differentiate
  - possibly with unique minimizer
  - robust global minimization can be devised
- ”variational”:
  - sequence of minimizers μ_n^* of $\mathcal{E}_n$ converging to a minimizer of $\mathcal{E}$.

"good" properties of $\mathcal{E}_n$:

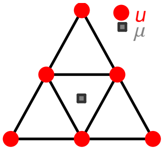
Defining $\mathcal{E}_n$

FEM discretization on nested grids + evanescent viscosity $\delta_n \downarrow 0$

- Convex
- real analytic on $\{\mu \in \mathbb{R}_n^N : \mu_i > -\delta_n \, \forall i\}$ (δ_n reg. parameter $\to 0^+$)
- computing $\mathcal{E}_n(\mu)$ requires the solution of a FEM linear system
- derivatives are computed similarly
- a geometric hypothesis (GH) (a posteriori check) provides uniqueness of minimizer

(GH) and FEM

(GH) is a sort of non-linear BLB condition. Experimentally it always holds when **P1-P0** double grid FEM discretization is used for defining $\mathcal{E}_n$.



"variational" property of $\mathcal{E}_n$

Theorem (P., Facca, Putti)

If $\mu_n^* = \operatorname{argmin} \mathcal{E}_n$, then

$$\mu_n^* \rightharpoonup \mu^*.$$

- Γ-*convergence* w.r.t weak* topology to a "relaxed functional"

$$\mathcal{E}_n \xrightarrow{\Gamma} \tilde{\mathcal{E}}$$

- An *energy integral continuity property* implies

$$\tilde{\mathcal{E}}|_{L^\infty} \equiv \mathcal{E}|_{L^\infty}$$

- *Regularity of minimizer*

$$\mu^* \in L^\infty(\Omega)$$

Now we want to **minimize** $\mathcal{E}_n$...

Idea of minimizing algorithm

New objective

Consider $\mathcal{F}_n(\sigma) := \mathcal{E}_n(\sigma^2)$, $\sigma \in \mathbb{R}^N$.

+ remove positivity constraint of min. problem
+ gain global real analiticity
− loose convexity

Idea: numerically integrate the gradient flow:

Theorem (P., Facca, Putti)

Let $\sigma^0 \in \mathbb{R}^N$, $\nabla\mathcal{F}_n(\sigma^0) \neq 0$, then $\exists!$ real analytic $[0, +\infty[\ni t \mapsto \sigma(t; \sigma^0)$ solving

$$\begin{cases} \sigma' = -\nabla\mathcal{F}_n(\sigma), \quad t > 0 \\ \sigma(0) = \sigma^0 \end{cases} \tag{$\mathcal{F}_n$-GF}$$

Moreover

$$\left(\lim_{t\to+\infty} \sigma(t; \sigma^0)\right)^2 = \mu_n^*.$$

Numerical integration of ($\mathcal{F}_n$-GF)

Derive the algorithm **BEN-GF** using:

- **B**ackward **E**uler scheme for the **G**radient **F**low
- **N**ewton solver with prescribed initial guess for each time step
- special "a-posteriori" stopping criterion for the Newton's method

Features

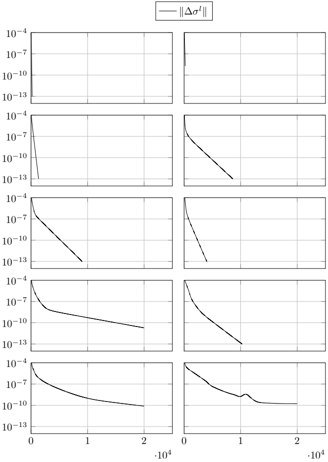
- Backward Euler scheme is variational for $\tau < \tau^*(\sigma^0)$
- the "initial guess = previous step" policy avoids oscillations from basin to basin
- stopping criterion prevents saddle points to become attractors, and provide a stability estimate.

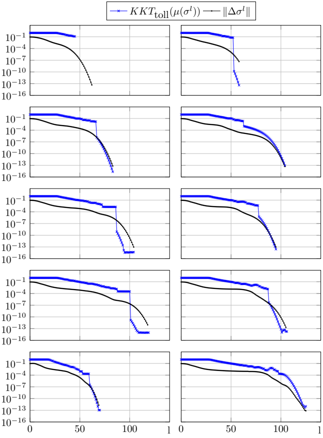
Convergence analysis

Theorem (P., Facca, Putti)

- If $\nabla\mathcal{F}(\sigma_n^{(0)}) \neq 0$ and the Newton tollerance is sufficiently small, then **BEN-GF** converges at least polynomially to $\sigma_n^* \in \operatorname{argmin} \mathcal{F}_n$, i.e., $(\sigma_n^*)^2 \in \operatorname{argmin} \mathcal{E}_n$.
- If in addition (GH) holds, then **BEN-GF** converges at least geometrically.
- In the latter case, we can include adaptive time stepping in **BEN-GF** to achieve super-geometric convergence.

Experimental convergence

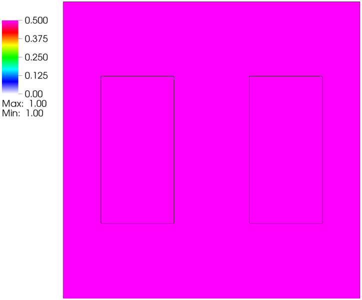




Simple test case

$$f = \chi_{[\frac{1}{8},\frac{3}{8}]\times[\frac{1}{4},\frac{3}{4}]} - \chi_{[\frac{5}{8},\frac{7}{8}]\times[\frac{1}{4},\frac{3}{4}]}$$

$$\mu^* = \chi_{[\frac{1}{8},\frac{3}{8}]\times[\frac{1}{4},\frac{3}{4}]} \cdot (x - 1/8) + \frac{1}{4}\chi_{[\frac{3}{8},\frac{7}{8}]\times[\frac{1}{4},\frac{3}{4}]} - \chi_{[\frac{5}{8},\frac{7}{8}]\times[\frac{1}{4},\frac{3}{4}]} \cdot (x - 7/8)$$



$$G_h(\sigma;\eta,\tau) := \mathcal{F}_n(\sigma) + \frac{\|\sigma-\eta\|^2}{2\tau}$$

Algorithm BEN-GF

Input $\sigma_h^0 \in \mathbb{R}^N$, $\tau > 0$, $n_{step} \in \mathbb{N}$, $toll > 0$, $\epsilon > 0$
Set $k := 0$
Compute $res = |\nabla F_h(\sigma_h^0)|$
if $res = 0$ **then**
    Exit with error.
end if
while $k < n_{step}$ **and** $res > toll$ **do**
    Set $k = k + 1$, $\sigma^{new} := \sigma^{old}$
    Compute $res_{Newton} := \nabla G_h(\sigma^{new}; \sigma^{old}, \tau)$
    while $|(res_{Newton})_i| > \epsilon|(\sigma^{new} - \sigma^{old})_i|$ for some i **or** sign $\sigma^{new} \neq$ sign σ^{old} **do**
        Compute $\sigma^{new} = \sigma^{new} - [\text{Hess}\, G_h(\sigma^{new}; \sigma^{old}, \tau)]^{-1} \nabla G_h(\sigma^{new}; \sigma^{old}, \tau)$
        Compute $res_{Newton} := \nabla G_h(\sigma^{new}; \sigma^{old}, \tau)$
    end while
    Compute $res = |\nabla F_h(\sigma^{new})|$
end while
return σ^{new}

From linear to superlinear convergence

Note that:

- the stab. bound $\tau < \tau^*(\sigma^k)$ on time step depends only on $\mathcal{F}_n(\sigma^k)$
- under the above condition $\tau^*(\sigma^k) \to +\infty$.

Idea of modified algorithm:

- Set $\tau_{k+1} = \alpha\tau_k$ ($\alpha > 1$) start Newton solver
- **if** Newton reaches the exit criterion in few step, **go**
- **else** set $\tau_{k+1} = \tau_k$ and restart

Proposition

If $\alpha > 1$ is sufficiently small, then the modified algorithm has

superlinear convergence rate α

.

Why we use a $\mathcal{P}_0(\mathcal{T}^n) - \mathcal{P}_1(\mathcal{T}^{2n})$ discretization

Experiments show that

- Our FEM discretization with $\mu \in \mathcal{P}_0(\mathcal{T}^n)$, $u \in \mathcal{P}_1(\mathcal{T}^{2n})$ provides a well-posed and well-conditioned problem:

$$\begin{cases} [\partial^2_{i,j}\mathcal{E}_n(\mu^*_n)]_{i,j:(\mu^*_n)_i \neq 0,\,(\mu^*_n)_j \neq 0} \succ 0 \\ \partial_i \mathcal{E}_n(\mu^*_n) \neq 0, \ \forall i : (\mu^*_n)_i = 0 \end{cases} \quad \Leftrightarrow \quad \mathrm{Hess}\,\mathcal{F}_n(\sigma^*_n) \succ 0 \quad \Rightarrow \quad \vartheta = 1/2$$

- other (e.g., $\mathcal{P}_0(\mathcal{T}^n) - \mathcal{P}_1(\mathcal{T}^n)$) classical FEM discretization do not satisfy the above conditions.