Relative Capacity on Algebraic Sub-varieties of $\mathbb{C}^n$ and application to the Bernstein Markov Property

Constructive Approximation of Functions

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Notations

Let $E \subset \mathbb{C}^n$ be a compact set, $\mathcal{M}^+(E)$ denotes the set of positive finite Borel measures μ such that $\operatorname{supp}\mu \subseteq E$.
We denote by $\mathscr{P}^k(E)$ complex polynomials restricted to E of total degree not exceeding k and by $N = N(k,n) := \dim \mathscr{P}^k(\mathbb{C}^n)$.
We use complex exterior differentiation

$$d := \partial + \bar\partial,\ d^c := i(-\partial + \bar\partial),\ \text{thus } dd^c = 2i\partial\bar\partial.$$

For $u \in \mathscr{C}^2$ the complex Monge Ampere operator is defined as

$$(dd^c u)^n = dd^c u \wedge dd^c u \cdots \wedge dd^c u = \det\left[\frac{\partial^2}{\partial z_i \partial \bar z_j} u\right] dVol_{\mathbb{C}^n}$$

it extends as a positive measure to the class $\operatorname{psh} \cap L^\infty_{\text{loc}}$ of bounded plurisubharmonic functions.
$\mathcal{L}$ is the Lelong class of psh function of logarithmic growth.

Bernstein Markov Property

Let $\mu \in \mathcal{M}^+(E)$. Suppose that there exists a positive sequence $\{C_k\}$ such that

$$\begin{aligned} &\overline{\lim}_k C_k^{1/k} \leq 1 \\ &\|p\|_E \leq C_k \|p\|_{L^2_\mu} \quad \forall p \in \mathscr{P}^k(E). \end{aligned} \tag{BMP}$$

Then (E, μ) is said to have the Bernstein Markov Property (BMP).

Weighted Bernstein Markov Property

Let $\mu \in \mathcal{M}^+(E)$ and let $w : E \to [0, +\infty[$ be an usc function. Suppose that there exists a positive sequence $\{C_k\}$ such that

$$\begin{aligned} &\overline{\lim}_k C_k^{1/k} \leq 1 \text{ and} \\ &\|pw^k\|_E \leq C_k \|pw^k\|_{L^2_\mu} \quad \forall p \in \mathscr{P}^k(E). \end{aligned} \tag{WBMP}$$

Then (E, μ) is said to have the Weighted Bernstein Markov Property (WBMP).

Example

$E = [-1,1]$ $\mu := \frac{1}{\sqrt{1-x^2}} m\llcorner_{[-1,1]}$ the Chebyshev polynomials $T_k(x)$ are an orthonormal basis

$$T_k(x) := \begin{cases} \sqrt{\frac{2}{\pi}}, & k = 0 \\ \frac{\cos(k \arccos x)}{\sqrt{\pi}}, & k > 0. \end{cases}$$

Take $p(x) := \sum_{j=0}^{k} a_j T_j(x)$, by Cauchy-Schwarz Ineq. and Parseval Id.

$$|p(x)| \leq \left(\sum_{j=0}^{k} |a_j|^2\right)^{1/2} \left(\sum_{j=0}^{k} T_j^2(x)\right) =: \|p\|_{L^2_\mu} \sqrt{B_k^\mu(x)}$$

Here $B_k^\mu := \sum_{j=1}^{\dim \mathscr{P}^k} |q_j(z,\mu)|^2$, $\{q_j\}$ o.n.b., is the **Bergman Function**.

$$\left(\frac{\|p\|_E}{\|p\|_{L^2_\mu}}\right)^{1/k} \leq \left(\|B_k^\mu\|_E\right)^{1/2k} \leq \left(\frac{k+2}{\pi}\right)^{1/2k} \to 1.$$

Applications&Motivations

1. Good behavior of LS approximation.
   Least squares projection in L^2_μ and best polynomial approximation have the same asymptotic.

Applications&Motivations

1. Good behavior of LS approximation
2. Approximating the extremal fiunction V_K^*. For any regular non pluripolar compact set $K \subset \mathbb{C}^n$ we have

$$\frac{1}{2k} \log B_k^\mu(z) \longrightarrow V_K^* \text{ loc. unif.}$$

Here

$$\boxed{V_K := \sup\{u \in \mathcal{L} : \max_K u \leq 0\} \text{ is the extremal psh function.}}$$

1. Good behavior of LS approximation
2. Approximating the extremal function V_K^*.
3. *normalization constant/free energy* asymptotic:

$$\overline{\lim}_k \left(\left\| \mathrm{VDM}_k(z_1, \dots, z_{N_k}) \right\|^2_{L^2(\mu^N)} \right)^{\frac{n+1}{nkN}} = \delta(K)$$

where VDM_k is the Vandermonde matrix of order k and $\delta(K)$ is the transfinite diameter.

$$\delta(K) := \lim_k \max_{(z_1, \dots, z_{N_k}) \in K^N} |\mathrm{VDM}_k(z_1, \dots, z_{N_k}))|^{\frac{d+1}{dN}}$$

This is the key for LDP results[3, 4].

1. Good behavior of LS approximation
2. Approximating the extremal function V_K^*.
3. Free energy asymptotic.
4. Strong Bergman asymptotic:

$$\frac{B_k^\mu}{N}\mu \rightharpoonup^* \mu_K \text{ as } k \to \infty.$$

where $\mu_K := (dd^c V_K^*)^n$ is the pluripotential equilibrium measure.

1. Good behavior of LS approximation
2. Approximating the extremal function V_K^*.
3. Free energy asymptotic.
4. Strong Bergman asymptotic.

...these things work to the weighted setting as well.

Sufficient conditionfor BMP

The most powerful tool for proving BMP is the following theorem (due to Stahl and Totik [10] when $n = 1$).

Mass density sufficient condition [Bloom-Levenberg] [2]

Let $K \subset B(0,1) \subset \mathbb{C}^n$ be a regular compact set and μ a finite Borel measure with $\operatorname{supp} \mu = K$, the following condition is sufficient for (μ, K) to have the BM property.
There exists $t > 0$ such that

$$\operatorname{Cap}(K, B(0,1)) = \lim_{r \to 0^+} \operatorname{Cap}\Big(\{z \in K : \mu(B(z,r)) \geq r^t\}, B(0,1)\Big).$$

here we used the relative Monge Ampere capacity

$$\boxed{\operatorname{Cap}(K,\Omega) := \sup\{\textstyle\int_K (dd^c u)^n : \ u \in \operatorname{psh}(\Omega),\, 0 \leq u \leq 1\}}$$

- The Bernstein Walsh Siciak Inequality

$$|p(z)| \leq \|p\|_K \exp(\deg p V_K(z)).$$

- K is regular, thus V_K is continuous.
- Cauchy Inequality for derivatives of polynomials.
- The following *capacity convergence theorem*.

Theorem [Bloom Levenberg] [2]

Let $K \subset B(0,1) \subset \mathbb{C}^n$ be a regular compact set and $\{K_j\}$ a sequence of compact subsets of it, then

$$V_{K_j}^* \rightrightarrows_{\text{loc}} V_K \;\Leftrightarrow\; \lim_j \operatorname{Cap}(K_j, B(0,1)) = \operatorname{Cap}(K, B(0,1)).$$

Motivating question

Problem

Let $K \subset \mathbb{C}$ be a closed set and w an admissible weight, find a sufficient condition for WBMP on it.

Overview of the strategy

1. Compactification by projection on $\mathbb{S}^2$.

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3. Adapting the setting. . .
4. A modified versoin of *Capacity convergence theorem.*
5. A new *Comparison Theorem for realtive capacity and Chebyshev constant.*

Step 1: Compactification

In place of considering an unbounded set E in the plane we can consider its compact image K under stereographic projection

$$\begin{array}{rccc} S: & \mathbb{C} \cup \{\infty\} & \longrightarrow & B \cong \mathbb{S}^2 \subset \mathbb{R}^3 \\ & z & \longmapsto & \left(\frac{\Re z}{1+|z|^2}, \frac{\Im z}{1+|z|^2}, \frac{|z|^2}{1+|z|^2} \right) \quad = (x_1, x_2, x_3) \end{array}$$

Where B is the ball of radius $1/2$ centered at $(0, 0, 1/2)$.
The inverse is $z = S^{-1}(x_1, x_2, x_3) = \frac{x_1 + i x_2}{1 - x_3}$.

The polynomials are transformed as follows

$$\boxed{p(z) := \sum_{j=0}^{k} a_j z^j \longmapsto p(S(z)) =: \frac{1}{(1-x_3)^k}\tilde{p}(x_1, x_2, x_3).}$$

Where we define

$$\tilde{p}(x_1, x_2, x_3) := \sum_{j=0}^{k} a_j (x_1 + ix_2)^j (1 - x_3)^{k-j}$$

Thus we introduce the weight on $\mathbb{R}^3$

$$\tilde{w}(x_1, x_2, x_3) := \frac{w\left(\frac{x_1+ix_2}{1-x_3}\right)}{(1-x_3)}$$

such that

$$w(z)^{\deg p} p(z) = \tilde{w}(S(z))^{\deg \tilde{p}} \tilde{p}(S(z))$$

We look at $B \cong \mathbb{S}^2$ as the real points of the algebraic variety

$$\boxed{\mathcal{A} := \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^2 + (z_3 - 1/2)^2 - 1/4 = 0\right\}}$$

and use the following

Theorem [Bloom Levenberg Wielonsky][3]

If $K \subset \mathbb{R}^d$ is a compact set and μ is a positive Borel measure supported on K having the Bernstein Markov property for holomorphic polynomials of d complex variables, then it has the Weighted Bernstein Markov property for any continuous admissible weight w.

To reduce the problem to an unweighted one.

The original theorem was for bounded w, usc admissible works as well.

Step 3: Adapting the setting

There are some issues

(A) $\mathscr{A}$ is a pluripolar subset of $\mathbb{C}^n$, which Potential Th. to use?

(B) What could replace V_K?

(C) What should be the analogous of $B(0,1)$?

Some results come in play...

(A) Pluripotential Theory on (any) algebraic sub-variety

- $\mathcal{A}_{\text{reg}}$ is a m–dimensional sub-manifold of $\mathbb{C}^n$, we define ***weakly psh*** functions $\widetilde{\text{psh}}(\mathcal{A})$ as all $u : \mathcal{A} \to [-\infty, +\infty[$:
  - $u \in \text{psh}(\mathcal{A}_{\text{reg}})$
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  - $u \in \text{psh}(\mathcal{A}_{\text{reg}})$
  - u bounded above on $\mathcal{A}$.
- $(dd^c)^m$ is a well defined positive (m, m)-current on $\mathcal{A}_{\text{reg}}$ (thus positive measure),

$$(dd^c u)^m = \det \text{Hess}(u) dVol_{\mathcal{A}_{\text{reg}}}(z) \ \forall u \in \mathscr{C}^2(\mathcal{A}_{\text{reg}}) \cap L^\infty(\mathcal{A}).$$

Thanks to [13] we can extend it to $\mathcal{A}$ by

$$\int_E (dd^c u)^m := \int_{E \cap \mathcal{A}_{\text{reg}}} (dd^c u)^m \quad \forall E \subset \mathcal{A}, u \in \widetilde{\text{psh}} \cap L^\infty_{\text{loc}}(\mathcal{A}).$$

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- The set $K \subset \mathcal{A}$ is said **non pluripolar** if it is at any regular point.
- K is **regular** if the **relative extremal function**

$$\boxed{U^*_{K,\Omega_{\mathcal{A}}}(z, \mathcal{A}) := (\sup\{u \in \text{psh}(\mathcal{A} \cap \Omega_{\mathcal{A}}) : u \leq 0, u|_K \leq -1\})^*}$$

  is continuous in $\Omega_{\mathcal{A}} \cap \mathcal{A}$.

(B) Extremal psh function on algebraic sub-variety

We use the Siciak-Zaharjuta Function Φ_K ([8, 12]) to define V_K

$$\begin{aligned} V_K(z) &:= \log \Phi_K(z) = \sup \left\{ \log |p(z)|^{1/\deg p} : \|p\|_K \leq 1 \right\} \\ V_K^*(z, \mathcal{A}) &:= \overline{\lim}_{\mathcal{A} \ni \zeta \to z} V_K(\zeta). \end{aligned}$$

Algebraicity characterization of Sadullaev [7]

Let $\mathcal{A} \subset \mathbb{C}^n$ be an analytic set, the following are equivalent

1. $\mathcal{A}$ is a piece of an algebraic set.
2. There exists a set $K \subset \mathcal{A}$ such that V_K is locally bounded on $\mathcal{A}$.

In such a case (2) holds for any non pluripolar compact $K \subset \mathcal{A}$ and $V_K^*(z, \mathcal{A})$ is a maximal psh i.e., $\left(dd^c\, V_K^*(z, \mathcal{A})\right)^m \llcorner_{\mathcal{A} \setminus K} = 0$.

(C) Rudin coordinates and pseudo-balls

Rudin introduced [6] an affine change of coordinates such that

$$\zeta \in \mathbb{C}^n = \mathbb{C}^m \times \mathbb{C}^{n-m} \ni (z, w)$$
$$\mathcal{A} \subset \{|w|^2 \leq C(1 + |z|^2)\}.$$

we define the pseudo-ball

$$\boxed{\begin{aligned} \Omega_{\mathcal{A}}(z_0, r) &:= \{(z, w) \in \mathcal{A} : |z - z_0| \leq r\} \\ \Omega_{\mathcal{A}} &:= \Omega_{\mathcal{A}}(0, 1). \end{aligned}}$$

that is an hyper-convex set exhausted by

$$\rho_r(z, w) := |(z - z_0)|^2 - r^2$$

Step 4: Capacity convergence on $\mathcal{A}$

We re-formulate the result [2, Th. 3.1] for $K \subset \Omega_{\mathcal{A}} \subset \mathcal{A}_{\text{reg}}$ using

$$\boxed{\operatorname{Cap}_{\mathcal{A}}(K,\Omega_{\mathcal{A}}) := \sup\left\{\int_K (dd^c u)^m : u \in \widetilde{\text{psh}}(\Omega_{\mathcal{A}}), 0 \le u \le 1\right\}}$$

Theorem (P.)

Let $\mathcal{A}$ be an algebraic m-dimensional sub-variety of $\mathbb{C}^n$, $K \subset \Omega_{\mathcal{A}} \subset \mathcal{A}_{\text{reg}}$ be a compact regular set and $\{K_j\}$ a sequence of compact subsets of it, then the following are equivalent

- $\lim_j \operatorname{Cap}_{\mathcal{A}}(K_j,\Omega_{\mathcal{A}}) = \operatorname{Cap}_{\mathcal{A}}(K,\Omega_{\mathcal{A}})$.
- $V_{K_j}(z,\mathcal{A})^* \to V_K(z,\mathcal{A})$.

The Bloom-Levenberg proof

(1) Integration by parts of currents (D. Coman [5]).
(2) Continuity of Monge Ampere under convergence in capacity (Xing [11]).
(3) Convergence in capacity is more restrictive than convergence a.e. .
(4) A specific lemma on shm functions in $\mathbb{R}^{2n}$.

Moreover...

(5) **Theorem (Siciak [9])**

Let $K \subset \mathbb{C}^n$ be a compact subset of the unit ball, then the Chebyshev constant $T(K)$ satisfies

$$T(K) = \exp\left(- \sup_{B(0,1)} |V_K^*| \right).$$

where

Chebyshev constant

Let $K \subset B(0,1)$ then the Chebyshev constant of K is

$$T(K) := \lim_{j \to \infty} \left(\inf\{ \|p\|_K : p \in \mathscr{P}^j(K), \|p\|_{\overline{B}} = 1 \} \right)^{1/j}.$$

(6) **Comparison Theorem (Alexander and Taylor)[1]**

Let $0 < r < 1$ and $K \subset B(0, r)$ be a compact subset of $\mathbb{C}^n$, then there exists two positive constant c_1, c_2 such that

$$\exp\left(\frac{-c_1(n,r)}{\mathrm{Cap}(K, B(0,1))}\right) \leq T(K) \leq \exp\left[-\left(\frac{c_2(n)}{\mathrm{Cap}(K, B(0,1))}\right)^{1/n}\right].$$

Modified setting

- Consider $\Omega_{\mathcal{A}}$ in place of $B(0,1)$
- $V_K^*(z,\mathcal{A})$ in place of $V_K^*(z)$
- $\mathrm{Cap}_{\mathcal{A}}(K,\Omega_{\mathcal{A}})$ in place of $\mathrm{Cap}(K,B(0,1))$
- We re-define the Chebyshev constant

Chebyshev constant on $\mathcal{A}$

Let $K\subset\Omega_{\mathcal{A}}\subset\mathcal{A}$ then the Chebyshev constant of K in $\mathcal{A}$ is

$$T(K,\mathcal{A}) := \lim_{j\to\infty}\left(\inf\{\|p\|_K : p\in\mathscr{P}^j(K), \|p\|_{\overline{\Omega}_{\mathcal{A}}}=1\}\right)^{1/j}.$$

What does tun out?

- Siciak result (5) in this setting holds by def. :

$$T(K,\mathcal{A}) = \exp\left(-\sup_{z\in\Omega_{\mathcal{A}}} |V_K^*(z,\mathcal{A})|\right)$$

- Results (1)-(4) hold if $\Omega_{\mathcal{A}} \subset \mathcal{A}_{\text{reg}}$.
- The proof of the analogous of Bloom-Levenberg result is similar to the original one.

We only miss an adapted version of Comparison Theorem.

Step 5: A new comparison theorem

Comparison Theorem (P.)

Let $\mathcal{A}$ be a m-dimensional algebraic variety of $\mathbb{C}^n$ such that $\Omega_{\mathcal{A}} \subseteq \mathcal{A}_{\text{reg}}$, then there exist two positive constants c_1, c_2 such that for any compact $K \subset \Omega_{\mathcal{A}}(0, r)$.

$$\exp\left(\frac{-c_1(n,r)}{\mathrm{Cap}_{\mathcal{A}}(K, \Omega_{\mathcal{A}})}\right) \le T(K, \mathcal{A}) \le \exp\left[-\left(\frac{c_2(n)}{\mathrm{Cap}_{\mathcal{A}}(K, \Omega_{\mathcal{A}})}\right)^{1/n}\right].$$

In particular the above theorem gives that

$$\|V_K(\cdot, A)\|_{\Omega_{\mathcal{A}}} \le \frac{c}{\mathrm{Cap}_{\mathcal{A}}(K, \Omega_{\mathcal{A}})} \quad \forall K \subset \Omega_{\mathcal{A}}(0, r).$$

Tools for the proof

Theorem: Chern Levine Nirenberg type estimate (P.)

(1) $\displaystyle\int_K (dd^c u)^m \leq C(r)\|u\|^{m-1}_{\Omega_{\mathcal{A}}(z_0,r)} \int_{\Omega_{\mathcal{A}}(z_0,r)} dd^c u \wedge (dd^c \rho_R)^{m-1}$

(2) $\displaystyle\int_{\Omega_{\mathcal{A}}(z_0,r)} -U_{K,\Omega_{\mathcal{A}}}(z,\mathcal{A})(dd^c \rho(z))^m \leq C'(r)\left(-U_{\pi_m K, B_{\mathbb{C}^m}(0,1)}(z_0)\right).$

$$\text{Thus } \operatorname{Cap}_{\mathcal{A}}(K,\Omega_{\mathcal{A}}) \leq C'(r)\left(-U_{\pi_m K, B_{\mathbb{C}^m}(0,1)}(z_0)\right)$$

Where π_m is the projection on the first m coordinates.
Lelong-Jensen formula.

$$\int u\, d\mu_{r_1,\epsilon} + \int_{\Omega(z_0,r_1,\epsilon)} -u\, (\mathrm{dd^c}\, \rho_\epsilon)^m = \int_{\Omega(z_0,r_1,\epsilon)} (r_1 - \rho_\epsilon)\, \mathrm{dd^c}\, u \wedge (\mathrm{dd^c}\, \rho_\epsilon)^{m-1}. \tag{1}$$

where $\mu_{r_1,\epsilon} := (\mathrm{dd^c} \max\{\rho_\epsilon, r_1\})^m - \chi_{A\setminus\Omega(z_0,r_1,\epsilon)} (\mathrm{dd^c}\, \rho_\epsilon)^m$, and
$\rho_\epsilon(z,w) := |z|^2 + \epsilon|w|^2 - R^2$

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$$(1)\ \int_K (dd^c u)^m \leq C(r)\|u\|^{m-1}_{\Omega_{\mathcal{A}}(z_0,r)} \int_{\Omega_{\mathcal{A}}(z_0,r)} dd^c u \wedge (dd^c \rho_R)^{m-1}$$

$$(2)\ \int_{\Omega_{\mathcal{A}}(z_0,r)} -U_{K,\Omega_{\mathcal{A}}}(z,\mathcal{A})(dd^c\rho(z))^m \leq C'(r)\left(-U_{\pi_m K, B_{\mathbb{C}^m}(0,1)}(z_0)\right).$$

$$\text{Thus } \operatorname{Cap}_{\mathcal{A}}(K,\Omega_{\mathcal{A}}) \leq C'(r)\left(-U_{\pi_m K, B_{\mathbb{C}^m}(0,1)}(z_0)\right)$$

Where π_m is the projection on the first m coordinates.

Further investigations: It seems possible to remove the hypotesis $\Omega_{\mathcal{A}} \subseteq \mathcal{A}_{\text{reg}}$.

BMP mass density sufficent condition on algebraic variety

Theorem [P.]

Let $\mathcal{A}$ be a m dimensional algebraic variety of $\mathbb{C}^n$ such that $\Omega_{\mathcal{A}} \subseteq \mathcal{A}_{\text{reg}}$, $K \subset \Omega_{\mathcal{A}}$ a regular non pluripolar compact set and $\mu \in \mathcal{M}^+(K)$ $\operatorname{supp} \mu = K$.. Suppose that there exists $t > 0$ such that

$$\lim_{r \to 0^+} \text{Cap}_{\mathcal{A}}\left(\left\{(z,w) \in K : \mu(B((z,w),r)) \geq r^t\right\}, \Omega_{\mathcal{A}}\right) = \text{Cap}_{\mathcal{A}}(K, \Omega_{\mathcal{A}}),$$

then

$$\overline{\lim}_k \left(\max_{\deg_{\mathcal{A}} p \leq k} \frac{\|p\|_K}{\|p\|_{L^2_\mu}} \right)^{1/k} \leq 1.$$

WBMP sufficient condition

Here we choose Rudin coordinates for the complexified sphere $\mathcal{A}$

$$z = (z_1, z_2), \quad w = z_3 - 1/2.$$

Theorem [P.]

Let $E \subset \mathbb{C}$ be a closed set, $\mu \in \mathcal{M}^+(E)$ and let $r_0 > 0$ such that $K := S(E)$ is a regular subset of $\Omega_{\mathcal{A}}(0, r_0)$, suppose furthermore that there exists $t > 0$ such that

$$\lim_{r \to 0^+} \mathrm{Cap}_{\mathcal{A}}\left(\{(z, w) \in K : S_* \mu(B((z, w), r)) \geq r^t\right) = \mathrm{Cap}_{\mathcal{A}}(K),$$

then (E, μ, w) have the weighted Bernstein Markov Property for any continuous admissible weight w.

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Thank You!