WHY SHOULD PEOPLE IN APPROXIMATION THEORY CARE ABOUT (PLURI-)POTENTIAL THEORY?

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Abstract. We give a summary of results in (pluri-)potential theory that naturally come into play when considering classical approximation theory issues both in one and (very concisely) in several complex variables. We focus on Fekete points and the asymptotic of orthonormal polynomials for certain L^2 counterpart of Fekete measures.

1. One variable case

1.1. **Interpolation and Fekete Points.** Let $K \subset \mathbb{C}$ be a compact set such that $\mathbb{C}_\infty \setminus K$ is connected and $f : K \to \mathbb{C}$ be a continuous function that is holomorphic in the interior of K, then Mergelyan Theorem ensures the existence of sequences of polynomials uniformly approximating f on K.

Two reasonable questions are whether we can compute such a sequence by interpolation or not, and how the interpolation nodes should be chosen. It turns out that these questions become much more difficult when suitably translated in the several variables context.

From here on we suppose K to be a polynomial determining set, that is if a polynomial vanish on K then it is the zero polynomial. If we consider the monomial basis $\{z^j\}_{j=1,2,\dots,k}$ for the space $\mathscr{P}^k$ of polynomials of degree at most k and we pick $k+1$ distinct points $\mathbf{z}_k := (z_0, z_1, \dots, z_k) \in K^{k+1}$ the interpolation problem can be written as $VDM_k(z_0, \dots, z_k)c = (f(z_0), \dots, f(z_k))^t$ where $c \in \mathbb{C}^{k+1}$ and

$$\underset{k}{\mathrm{VDM}}(z_0, \dots, z_k) := [z_i^j]_{i,j=0,\dots,k}$$

is the Vandermonde Matrix. Notice that for each $k+1$-tuple of distinct points $\det \mathrm{VDM}_k(\mathbf{z}_k) \neq 0$ so the problem is well posed.

It is a classical result that the norm of the operator $I_k : (\mathscr{C}(K), \|\cdot\|_K) \to \left(\mathscr{P}^k, \|\cdot\|_K\right)$ is the *Lebesgue Constant* $\Lambda_k := \max_{z \in K} \sum_{m=0}^k \left|l_{m,k}(z)\right|$, where $l_{m,k}(z) := \frac{\det \mathrm{VDM}_k(z_0,\dots,z_{l-1},z,\dots,z_k)}{\det \mathrm{VDM}_k(z_0,\dots,z_k)}$.

Minimizing $\Lambda_k(\mathbf{z}_k)$ is an extremely hard task so one can consider the *simplified* problem of maximizing $|\det \mathrm{VDM}(\mathbf{z}_k)|$, though it is still a very hard one.

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Definition 1.1 (Fekete Points)**.** *Let $K \subset \mathbb{C}$ be a compact set and $\boldsymbol{z}_k := (z_0, \dots, z_k) \in K^{k+1}$. If we have $|\det \mathrm{VDM}_k(\boldsymbol{z}_k)| = \max_{\zeta \in K^{k+1}} |\det \mathrm{VDM}_k(\boldsymbol{\zeta})|$, then $\boldsymbol{z}_k$ is said to be a Fekete array of order k, its elements are said Fekete points.*

The relevance of Fekete points is easy to see: one has $\Lambda_k(\boldsymbol{F}_k) \leq k+1$ for any Fekete array $\boldsymbol{F}_k$; moreover notice that in general $\boldsymbol{F}_k$ is not unique.

An interesting property of Vandermonde determinants is that for any array of points $\boldsymbol{z}_k$ we have

$$|\det \underset{k}{\mathrm{VDM}}(\boldsymbol{z}_k)| = \prod_{i<j\leq k} |z_i - z_j|$$

that is the product of their distances; this is one of the main points in connecting interpolation with logarithmic potential theory. We introduce the number $d_k(K) := |\mathrm{VDM}_k(\boldsymbol{z}_k)|^{1/\binom{k}{2}}$ for one (thus any) Fekete array $\boldsymbol{z}_k$ of order k. In analogy with the case $k = 0$, $d_k(K)$ is called *k-th diameter* of K, it is a decreasing positive sequence, its limit $d(K)$ is called *transfinite diameter* of K.

Example 1.2 (Fekete points on the unit disc)**.** *Let $\mathbb{D}$ be the unit circle. Take any set of distinct points $\boldsymbol{z} = \{z_0, \dots, z_k\}$ and consider the determinant of the matrix $V(\boldsymbol{z}) := \mathrm{VDM}_k(\boldsymbol{z})$. We have*

$$\|V_{:,j}(\boldsymbol{z})\|_2 = \left\| \begin{pmatrix} z_0^j \\ z_1^j \\ \vdots \\ z_k^j \end{pmatrix} \right\|_2 = \sqrt{k+1}, \textit{ for any } j = 0, 1, \dots k.$$

Therefore Hadamard Inequality for determinants implies $|\det \mathrm{VDM}_k(\boldsymbol{z})| \leq \prod_{j=0}^k \|V_{:,j}(\boldsymbol{z})\|_2 = (k+1)^{\frac{k+1}{2}}$. This upper bound is achieved if and only if the columns of $V(\boldsymbol{z})$ are orthonormal and this condition is satisfied for $\boldsymbol{z}$ a set of $k+1$ roots of unity. Therefore $\{\frac{2i\pi j}{k+1}\}_{j=0,\dots,k}$ is a Fekete set for $\mathbb{D}$.

1.2. **logarithmic Potential Theory.** It is customary to introduce the differential operators $d := \partial + \bar{\partial}$ and $d^c := i(\bar{\partial} - \partial)$ such that (passing to real coordinates) one has $\Delta = 2i\partial\bar{\partial} = dd^c$.

We recall that using the Green Identity it is possible to show that $E(z) := \frac{1}{2\pi} \log |z|$ is a *fundamental solution* for the Laplacian, i.e., $\mathrm{dd^c}\, E(z) = \delta_0$ in the sense of distributions. We also recall that a real valued function $u \in \mathscr{C}^2$ on a domain Ω is said to be *harmonic* if $\Delta u = 0$ in Ω. A upper semi-continuous function v is said to be *subharmonic* in Ω if for any open relatively compact subset $\Omega_1 \subset \Omega$ and any harmonic function u on Ω such that $v \leq u$ on $\partial\Omega_1$ we have $v \leq u$ on Ω_1. It follows that, given any compactly supported finite Borel measure μ, the function

$$U^\mu(z) := \mu * E(z) = \int \log \frac{1}{|z-\zeta|} d\mu(\zeta),$$

said the *logarithmic potential* of μ, is a superharmonic function (i.e. $-U^\mu$ is subharmonic) on $\mathbb{C}$ and in particular harmonic in $\mathbb{C} \setminus \operatorname{supp} \mu$, moreover we have $\mathrm{dd^c}\, U^\mu = \mu$.

To the Laplace operator is naturally attached an energy minimization problem; we refer the reader to [21], [23] and [22] for details.

Problem 1.1 (Logarithmic Energy Minimization)**.** *Let K be a compact subset of $\mathbb{C}$, minimize*

$$I[\mu] := \int\int \log \frac{1}{|z-\zeta|} d\mu(\zeta) d\mu(z) = \int U^{\mu}(z) d\mu(z), \tag{LEM}$$

among $\mu \in \mathcal{M}_1(K)$, the set of Borel probability measures on K endowed with the weak* topology.

We notice that if we consider the electrostatic interaction between $k+1$ unitary charges in the plan, the force acting on the i-th particle is $\sum_{j\neq i} \frac{(x_i, x_j, y_i - y_j)}{|z_i - z_j|^2} = -\nabla U^{\mu}$, where μ denotes the uniform probability measure associated to the charges distribution. The electrostatic energy associated to this charges distribution is precisely $\tilde{U}^{\mu} = \sum_{i\neq j} \sum \log \frac{1}{|z_i - z_j|}$, this minimization problem is very close to (LEM), we will see that in particular (LEM) it is, in a suitable sense, its limit.

It turns out that $I[\cdot]$ is a lower semi-continuous functional on a locally compact space, one can use the Direct Methods of Calculus of Variation to prove that two situations may occur. Either $I[\mu] = +\infty$ for all $\mu \in \mathcal{M}_1(K)$, either there exists a unique minimizer that is the *equilibrium measure* of K and is usually denoted by μ_K.

In the latter case one has $U^{\mu_K}(z) = -\log c(K) - g_K(z)$, where $g_K := G_{\mathbb{C}_\infty \setminus K}$ is the generalized (e.g. possibly not continuous) Green function with logarithmic pole at ∞ for the complement of K. The number $c(K)$ called *logarithmic capacity* of K and is defined as $c(K) := \exp(-\inf_{\mu \in \mathcal{M}_1(K)} I[\mu])$, thus is non zero precisely when the minimization problem is well posed. It turns out that $U^{\mu_K}(z) = -\log c(K)$ quasi everywhere on K, that is for any $z \in K$ but for set of zero logarithmic capacity.

Sets of zero capacity are, roughly speaking, too small for logarithmic potential theory, they are termed *polar* and it can been shown that $c(K) = 0$ if and only if K is the $-\infty$ set of some subharmonic function defined in a neighbourhood of K.

The following result is due to Szego, Leja and Fekete.

Theorem 1.3 (Fundamental Theorem of Logarithmic Potential Theory)**.** *Let $K \subset \mathbb{C}$ be a compact non polar set, we have*

$$c(K) = d(K).$$

Therefore, for any sequence of Fekete arrays $\{\boldsymbol{F}_k\}$, setting $\mu_k := \frac{1}{k+1} \sum_{j=0}^{k} \delta_{F_k^j}$, we have

$$\mu_k \rightharpoonup^* \mu_K.$$

Moreover locally uniformly on $\mathbb{C} \setminus K$ we have

$$\lim_k -U^{\mu_k}(z) := \lim_k \frac{1}{k+1} \log \prod_{j=0}^{k+1} |z - F_k^j| = g_K(z) - \log c(K) = -U^{\mu_K}.$$

The proof is based on lower semi-continuity and strict convexity of the energy functional and on the extremal property of Fekete points. Moreover the result holds true for any sequence of so called *asymptotically Fekete arrays* that is, arrays $\boldsymbol{L}_k$ such that $\lim_k |\operatorname{VDM}_k(\boldsymbol{L}_k)|^{1/\binom{n}{2}} = d(K)$.

1.3. **Back to Approximation.** Other deep connections between interpolation and logarithmic potential theory are given by the following two results. We recall that a compact set K is said to be *regular* if g_K is a continuous function.

Theorem 1.4 (Bernstein Walsh [25])**.** *Let K be a compact polynomially determining non polar set, then we have*

$$g_K(z) = \limsup_{\zeta \to z} \left(\left\{ \frac{1}{\deg p} \log^+ |p(\zeta)|, \|p\|_K \leq 1 \right\} \right).$$

Moreover

$$\text{(Bernstein Wals Ineq.)} \qquad |p(z)| \leq \|p\|_K \exp(\deg p \, g_K(z)) \ \ \forall p \in \mathscr{P}(\mathbb{C}).$$

The approximation theorem comes as an application of Hermite reminder formula and the previous theorem.

Theorem 1.5 (Bernstein-Walsh [25])**.** *Let $K \subset \mathbb{C}$ be a compact regular polynomially convex non polar set and $f : K \to \mathbb{C}$ be a bounded function. Let $d_k(f, K) := \inf\{\|f - p\|_K : p \in \mathscr{P}^k\}$, then for any real number $R > 1$ the following are equivalent*

(1) $\lim_k d_k(f, K)^{1/k} < 1/R$

(2) f is the restriction to K of $\tilde{f} \in hol(D_R)$, where $D_R := \{g_K < \log R\}$.

1.4. L^2 **theory.** We want to show that some analogues of results for Fekete points (that are L^∞ maximizers, in some sense) holds for particular measures in a L^2 fashion.

Definition 1.6 (Bernstein Markov Measures)**.** *Let $K \subset \mathbb{C}$ be a compact set and μ be a Borel measure such that* $\operatorname{supp} \mu \subseteq K$*, assume that*

$$\limsup_k \left(\frac{\|p_k\|_K}{\|p_k\|_{L^2_\mu}} \right)^{1/\deg(p_k)} \leq 1,$$

for any sequence of non zero polynomials $\{p_k\}$. Then we say that (K, μ) has the Bernstein Markov property, BMP for short, or equivalently μ is a Bernstein Markov measure on K.

Example 1.7. *We claim that $d\mu := \frac{1}{2\pi} d\theta$ is a Bernstein Markov measure for $\mathbb{S}^1$. To show that, one first notice that the monomials (up to degree k) are a orthonormal basis of $\left(\mathscr{P}^k, \langle \cdot ; \cdot \rangle_{L^2_\mu}\right)$. Therefore, for any $p \in \mathscr{P}^k$ we have*

$$|p(z)| = \left| \sum_{j=0}^k \langle p; z^j \rangle_{L^2_\mu} z^j \right| \leq \|p\|_{L^2_\mu} \left(\sum_{j=0}^k |z^j|^2 \right)^{1/2} \leq \sqrt{k+1} \|p\|_{L^2_\mu},$$

hence $\limsup_k \left(\frac{\|p_k\|_K}{\|p_k\|_{L^2_\mu}}\right)^{1/\deg(p_k)} \leq \limsup_k (k+1)^{1/2k} = 1.$

Bernstein Markov measures on a given K are in general a very large set as the following sufficient condition shows. For a exhaustive treatment of *regular measures* (e.g. a weakened version of Bernstein Markov property) the reader is referred to [24]; generalization to the $\mathbb{C}^n$ version can be found in [5]. A survey with further development is [10].

Theorem 1.8 (Sufficient condition for BMP [24])**.** *Let K be a compact non polar regular set and μ a Borel measure such that* $\operatorname{supp}\mu = K$*, assume that there exists a positive number* $t > 0$ *such that*

$$\lim_{r \to 0^+} c\left(\{z \in K : \mu(B(z,r)) \geq r^t\}\right) = c(K). \tag{1}$$

Then (K, μ) satisfies the Bernstein Markov property.

Finding *necessary condition* is a relevant open question in the general theory of orthogonal polynomials, a necessary condition was stated as a conjecture by Erdös.

The first interest on BMP is from the approximation point of view. Let us take a orthonormal system $\{q_j\}_{j=0,1,\ldots}$ for $\mathscr{P}$, then each $\mathscr{P}^k$ is a *Reproducing Kernel Hilbert Space*, being $K_k^\mu(z,\zeta) := \sum_{j=0}^k q_j(z)\bar{q}_j(\zeta)$ the kernel. We denote by $B_k^\mu(z)$ the diagonal of the kernel, say the *Bergman Function* $B_k^\mu(z) = K_k^\mu(z,z) = \sum_{j=0}^k |q_j(z)|^2$. It is not hard to see that the Bergman function represent the worst possible case for the l.h.s. of the definition of BMP, that is $\sup_{\deg p \leq k} \frac{\|p\|_K}{\|p\|_{L^2_\mu}} = \sqrt{\|B_k^\mu\|_K}$.

We consider natural the projection operator $L_k^\mu : \left(\mathscr{C}^0(K), \|\cdot\|_K\right) \to \left(\mathscr{P}^k, \|\cdot\|_K\right)$ defined by embedding the two spaces in L^2_μ, $\mathcal{L}_k^\mu[f](z) := \sum_{j=0}^k \langle f, q_j\rangle q_j(z)$. It follows that

$$\|\mathcal{L}_k^\mu[f]\|_K \leq \left(\sum_{j=0}^k |\langle f, q_j\rangle|^2\right)^{1/2} \left\|\left(\sum_{j=0}^k |q_j(z)|^2\right)^{1/2}\right\|_K \leq \|f\|_{L^2_\mu} \sqrt{\|B_k^\mu(z)\|_K} \leq \|f\|_K \sqrt{\mu(K)\|B_k^\mu(z)\|_K}.$$

Therefore we have $\|\mathcal{L}_k^\mu\| \leq \sqrt{\mu(K)\|B_k^\mu(z)\|_K}$. This can be used to bound the error of polynomial approximation by least square projection, let p_k be the best uniform polynomial approximation of degree at most k to f

$$\|f - \mathcal{L}_k^\mu[f]\|_K = \|f - p_k + p_k - \mathcal{L}_k^\mu[f]\|_K \leq d_k(f,K) + \|\mathcal{L}_k^\mu[f - p_k]\|_K \leq d_k(f,K)\left(1 + \sqrt{\mu(K)\|B_k^\mu(z)\|_K}\right).$$

This allow us to state a version of the Bernstein Walsh Lemma in a L^2 fashion for Bernstein Markov measures.

Theorem 1.9 (Bernstein Walsh L^2 version [17])**.** *Let K be a compact polynomially convex regular non polar set and μ a Borel probability measure such that* $\operatorname{supp}\mu = K$ *satisfying the Bernstein Markov property. Then the following are equivalent.*

(1) $\lim_k d_k(f,\mu)^{1/k} < 1/R$

(2) *f is the restriction to K of $\tilde f \in hol(D_R)$, where $D_R := \{g_K < \log R\}$.*

Here $\lim_k d_k(f,\mu)$ is the error of best L^2_μ polynomial approximation to f of degree not greater than k.

There are other interesting analogies between BM measures and measures associated to Fekete points. First we notice that if we pick a Fekete array F_k for K and we compute the squares of the modulus of the Vandermonde determinant on such points we can rewrite it as a L^2 norm w.r.t. the associated Fekete measure μ_k, that is

(2)

$$(k(k+1))!|\det \mathrm{VDM}(F_k)|^2 = \int \dots \int |\det \mathrm{VDM}(\zeta_0,\dots,\zeta_k)|^2 d\mu_k(\zeta_0)\dots d\mu_k(\zeta_k) := Z_k(\mu_k),$$

notice that the right hand side can be generalized to any measure on E. On the other hand if we perform the Gram-Schmidt ortogonalization of the Vandrmonde matrix one the right hand side of (2) we obtain (up to a normalizing constant $(k(k+1))!$) the product of L^2_μ norms of the monic orthonormal polynomials relative to μ and this is precisely the determinant of the Gram-matrix $G_k^{\mu_k}$ w.r.t. $\left(\mathscr{P}^k, \|\cdot\|_{L^2_{\mu_k}}\right)$ in the standard basis, that is $G_k^{\mu_k} = [\langle z^i \bar z^j\rangle_{L^2_{\mu_k}}]_{i,j}$.

This observation leads to a generalization of asymptotically Fekete points to any measure, namely $\mu \in \mathcal{M}_1(E)$ is *asymptotically Fekete* for E if $\lim_k Z_k(\mu,E)^{1/(k(k+1))} = d(E)$.

The following result, despite a not very difficult proof is fundamental, especially in more general contexts: Bernstein Markov measures are asymptotically Fekete; see [8, 6].

Proposition 1.1. *Let K be a compact non polar set and μ a Borel probability measure such that* $\operatorname{supp}\mu \subset K$ *satisfying the Bernstein Markov property. We have*

$$\lim_k \left(\int \dots \int |\det \mathrm{VDM}(\zeta_0,\dots,\zeta_k)|^2 d\mu(\zeta_0)\dots d\mu(\zeta_k)\right)^{\frac{1}{k(k+1)}} = \lim_k \left[(k(k+1))! \det G_k^\mu\right]^{\frac{1}{k(k+1)}} = d(K).$$

Morally speaking, Fekete points are L^∞ maximizers, while BM measures are L^2 maximizers.

Also we have that other interesting properties of Fekete points can be translated in this fashion.

Theorem 1.10. *Let K be a compact non polar set and μ a Borel probability measure such that* $\operatorname{supp}\mu \subset K$ *satisfying the Bernstein Markov property. We have*

i) $\lim_k \frac{1}{2k(k+1)} \log B_k^\mu(z) = g_K(z)$ *point-wise, locally uniformly if K is regular.*

ii) $\lim_k \frac{B_k^\mu}{k(k+1)}\mu = \mu_K$ *in the weak* sense.*

Notice that for Fekete measures $B_k^{\mu_k}$ is the sum of the squared modulus of Lagrange polynomials and $\frac{B_k^{\mu_k}}{k(k+1)} \equiv 1$ on the support of μ_k.

There are other applications of Bernstein Markov measures and potential theory tools concerning for instance random polynomials ensambles generalizing the classical result on Kac polynomials and asymptotic of zeroes of orthogonal polynomials; see for instance [11], [5], [9].

2. Several Variables Case

The extension of what we saw in the case of one complex variable is much more difficult and technical, but still the most of the relation in the previous section have their scv counterpart, provided a correct "translation".

The first difficulty is in defining the n-dimensional transfinite diameter for a given compact set, in particular showing the existence of the limit and its independence by the ordering of the monomial basis. The solution has been given by Zaharjuta [26] by a sophisticate procedure comparing the k-th diameter with certain integral mean of *directional Chebyshev constants*.

The second problem is that logarithmic energy is not related to maximization of Vandermonde determinants when $n > 1$. As a consequence, subharmonic functions are no more the "correct space" to look at; they are replaced in this context by plurisubharmonic ones.

Plurisubharmonic functions, PSH for short, are upper semi continuous functions being subharmonic along each complex line. This property is invariant under any holomorphic mapping, moreover there is a differential operator (the complex Monge Ampere) playing a role with PSH function similar to the one of Laplacian with respect to subharmonic functions in $\mathbb{C}$.

Let $u \in \mathrm{PSH}(\mathbb{C}^n) \cap \mathscr{C}^2$, then one can consider the continuous $(1,1)$ form $\mathrm{dd^c}\, u$,

$$\mathrm{dd^c}\, u := \sum_{i=1}^{n} 2i \frac{\partial^2}{\partial z_i \partial \bar{z}_j} u(z) dz_i \wedge d\bar{z}_j$$

and then take the wedge powers of it

$$(\mathrm{dd^c}\, u)^n := \mathrm{dd^c}\, u \wedge \cdots \wedge \mathrm{dd^c}\, u = \det[\frac{\partial^2}{\partial z_i \partial \bar{z}_j} u(z)]_{i,j} dV_n,$$

where dV_n is the standard volume form on $\mathbb{C}^n$.

It is a classical result that $\mathrm{dd^c}\, u$ can be defined as a *positive current* (i.e., an element of the dual of test forms) for any PSH function and, due to the seminal works of Bedford and Taylor [1] [2], for locally bounded PSH function the operator $(\mathrm{dd^c}\, u)^n$ is well defined as a positive Borel measure. This extension is termed the generalized complex *Monge Ampere operator*, *Pluripotential Theory* is the study of plurisubharmonic functions and Monge Ampere operator; we refer the reader to [15] for a detailed treatment of the subject.

In this context the role of Harmonic function is replaced by *maximal plurisubharmonic functions* that are defined requiring precisely the domination property that harmonic functions enjoy with respect to subharmonic functions in $\mathbb{C}$; they are characterized by $(dd^c u)^n = 0$ in the sense of measures on the given domain. We denote by $L(\mathbb{C}^n)$ the class of plurisubharmonic functions with logarithmic pole at infinity, the Dirichlet problem for the Monge Ampere operator

$$\begin{cases} (dd^c u)^n = 0 & \text{in } \mathbb{C}^n \setminus K \\ u =_{\text{q.e.}} 0 & \text{on } K, u \in L(\mathbb{C}^n) \cap L^\infty_{\text{loc}} \end{cases}$$

enjoys the role of the Dirichlet problem for the Laplacian in $\mathbb{C}$, its solution V_K^* is called *plurisubharmonic extremal function* or, by analogy, *pluricomplex Green function*. Being V_K^* representable, precisely as g_K, with the (upper-semicontinuous regularization of the) upper envelope of function v in $L(\mathbb{C}^n)$ such that $v \leq 0$ on K.

$$V_K^*(z) := \limsup_{\zeta \to z} V_K(\zeta)$$
$$V_K(\zeta) := \sup\{u(\zeta),\ u \in L(\mathbb{C}^n),\ u|_K \leq 1\}.$$

Again, as in the one dimension, one has $\Delta g_K = \mu_K$ here one has a *pluripotential equilibrium measure* $\mu_K := (dd^c V_K^*)^n$, thus it is supported on K by definition.

Pluripotential theory has analogies with potential theory but also differences, first the Monge Ampere operator if fully non linear, there is no notion of potential, a suitable energy functional has been found only recently and there is no direct connection between polynomials and finitely supported measures. However there are plenty of good news as well.

First, the (Bernstein Wals Ineq.) goes precisely to $\mathbb{C}^n$ replacing g_K by V_K^*, due to that and a density result one has the scv counterpart (again replacing g_K by V_K^*) of the Bernstein Walsh Lemma 1.4, usually referred as the *Bernstein Walsh Siciak Theorem*.

For years the extension of the asymptotic property of Fekete points to $\mathbb{C}^n$ has been only conjectured. The work (see [4], [3]) of Berman Boucksom and Nymstrom finally proved that, despite the strong differences between potential and pluripotential theory, one has the same L^∞ and L^2 asymptotic results. More precisely, for a non pluri-polar (i.e., not contained in the $-\infty$ set of a plurisubharmonic function) compact set K the following holds.

(1) Fekete measures for K converge weakly* to μ_K.

(2) The same remains true for any sequence of asymptotically Fekete arrays.

(3) For any Bernstein Markov measure μ $\lim_k \frac{B_k^\mu}{\dim \mathscr{P}^k(\mathbb{C}^n)}\mu = \mu_K$.

(4) For any Bernstein Markov measure μ $\lim_k \frac{1}{2k} \log B_k^\mu = V_K^*$ locally uniformly if K is L-regular, e.g. V_K^* is continuous.

Moreover, the sufficient condition for the Bernstein Markov property can be translated to $\mathbb{C}^n$ by replacing the logarithmic capacity by a non linear "local" (e.g., relative to a open hold all set) capacity associated with the Monge Ampere operator, say the *relative capacity*; see [7].

3. A discrete approach

Admissible meshes, shortly AM, are sequences $\{A_k\}$ of finite subsets of a given compact set K such that

- there exists a positive real constant C such that for any $p \in \mathscr{P}^k$ we have
$$\max_K |p| \leq C \max_{A_k} |p|.$$
- $\operatorname{Card} A_k$ increase at most polynomially.

They have been first introduced [14] as good sampling sets for uniform polynomial approximation by discrete least squares. The construction of such subsets has been studied for several cases, with emphasis in holding the cardinality growth rate; see for instances [12, 18, 20, 16].

Let associate the uniform probability measure μ_k to A_k, then we can see that, picking an orthonormal system $q_1, \dots, q_{N_k}$ of $\mathscr{P}^k$ we have
$$\sqrt{\|\sum_{j=1}^{N_k} |q_j|\|_K} = \sqrt{\|B_k^{\mu_k}\|_K} \leq C \sup_{p \in \mathscr{P}^k} \frac{\|p\|_{A_k}}{\|p\|_{L^2_{\mu_k}}} \leq C \sqrt{\operatorname{Card} A_k}.$$
As a consequence the error of uniform polynomial approximation by DLS on an AM has the (far to be sharp) upper bound $\|f - \mathcal{L}_k^{\mu_k}[f]\|_K \leq (1 + C \sqrt{\operatorname{Card} A_k}) d_k(f, K)$.

Notice that in particular we shown that $\limsup_k \left(\frac{\|p_k\|_K}{\|p_k\|_{L^2_{\mu_k}}} \right)^{1/k} \leq \left(C \sqrt{\operatorname{Card} A_k}\right)^{1/k} = 1$ for any sequence of polynomials p_k, $\deg p_k \leq k$. In this sense *admissible meshes are a kind of discrete model of Bernstein Markov measures* suitable for applications since for each finite degree they are finitely supported, moreover in a variety of cases we can explicitly compute an admissible mesh for the given K.

Another analogy of these sequences of finitely supported measures is that it still hold true that *the sequence of uniform probability measures μ_k associated to an admissible mesh for K* s an asymptotically Fekete sequence of measures, namely
$$\lim_k \left(\int \dots \int |\det \mathrm{VDM}(z_1, \dots, z_{N_k})| d\mu_k(z_0) \dots d\mu_k(z_{N_k}) \right)^{\frac{n+1}{2nkN_k}} = d(K), \text{ where } N_k := \binom{k+d}{d}.$$
As a consequence it is possible to prove (following the case of a fixed Bernstein Markov measure; see [13] [19]) that one has

- $\lim_k \frac{B_k^{\mu_k}}{N_k} \mu_k = \mu_K$.

- $\lim_k \frac{1}{2N_k} \log B_k^{\mu_k} = V_K^*$ locally uniformly if K is L-regular.

Lastly we can extract (by numerical linear algebra) from an admissible mesh its Fekete points $F_k \subset A_k$, it turns out that they are asymptotically Fekete for K and thus

- $\lim_k \mu_{F_k} = \mu_K$ in the weak* sense; see [12].

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