PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Special issue dedicated to Norm Levenberg on the occasion of his 60th birthday, Volume 11 · 2018 · Pages 62–74

Extremal Measures for Bounding Orthogonal Polynomials

Doron S. Lubinsky[a]

Dedicated to the 60th Birthday of Norman Levenberg

Communicated by F. Piazzon

Abstract

Let ν be a positive measure supported on $[-1,1]$, with infinitely many points in its support. What are the largest possible values taken at a given point, by orthonormal polynomials of degree n, arising from a measure μ such that the restriction of μ to $(-1,1)$ is ν? In a recent paper we addressed this issue, providing bounds. Here we solve the problem for ν symmetric about 0, n even, and $x=0$ as our given point. We also find the extremal measures when the measure μ has mass at most S, for a given $S>0$. For ultraspherical weights, the formulae take an explicit form.

1 Results

Let μ be a finite positive Borel measure on the real line with infinitely many points in its support, and all finite moments

$$\int t^j d\mu(t),\ j=0,1,2,\dots .$$

Then we may define orthonormal polynomials

$$p_n(\mu,x)=\gamma_n(\mu)x^n+\dots,\gamma_n(\mu)>0,$$

$n=0,1,2,\dots$ satisfying the orthonormality conditions

$$\int p_n(\mu,x)\,p_m(\mu,x)\,d\mu(x)=\delta_{mn}.$$

The zeros of $p_n(\mu,x)$ are denoted by

$$x_{nn}(\mu)<x_{n-1,n}(\mu)<\dots<x_{2n}(\mu)<x_{1n}(\mu).$$

The nth reproducing kernel for μ is

$$K_n(\mu,x,t)=\sum_{j=0}^{n-1}p_j(\mu,x)\,p_j(\mu,t)=\frac{\gamma_{n-1}}{\gamma_n}(\mu)\,\frac{p_n(\mu,x)\,p_{n-1}(\mu,t)-p_{n-1}(\mu,x)\,p_{n-1}(\mu,t)}{x-t}.$$

The three term recurrence relation has the form

$$(x-b_n(\mu))\,p_n(\mu,x)=\frac{\gamma_n}{\gamma_{n+1}}(\mu)\,p_{n+1}(\mu,x)+\frac{\gamma_{n-1}}{\gamma_n}(\mu)\,p_{n-1}(\mu,x).$$

A central problem in the theory of orthonormal polynomials is to establish bounds on $p_n(\mu,x)$, and there is an extensive literature. See for example [1], [3], [5], [8], [13], [14]. In a recent paper [12], we proved that if μ is absolutely continuous on some subinterval $[a,b]$ of the support, and μ' satisfies a Dini-Lipschitz condition in $[a,b]$, then $\{p_n(\mu,x)\}_{n\geq 0}$ are uniformly bounded in compact subsets of (a,b). The main tool in the proof of this was the solution to the following variational problem:

Consider a fixed positive measure ν supported on $[-1,1]$ with infinitely many points in its support, and that does not have mass points at ± 1. Fix $S>0$. We let $\mathcal{M}(\nu,S)$ denote the class of all positive measures μ supported on the real line, with all finite moments, such that $\mu=\nu$ in $(-1,1)$, while

$$\mu(\mathbb{R}\backslash(-1,1))\leq S.$$

Our key result was:

[a] School of Mathematics Georgia Institute of Technology Atlanta, GA 30332-0160 USA. Research supported by NSF grant DMS1800251

Theorem A

Let ν be a positive measure with support in $[-1,1]$ and with infinitely many points in its support, and that does not have mass points at ± 1. Let $n \geq 1, S > 0, x \in (-1,1)$ and let $\mathcal{M}(\nu, S)$ denote the class of measures defined above. Then there exists a measure μ^* of the form

$$\mu^* = \nu + M\delta_{-1} + N\delta_1$$

where $M, N \geq 0$ and $M + N \leq S$, such that

$$|p_n(\mu^*, x)| = \sup\{|p_n(\mu, x)| : \mu \in \mathcal{M}(\nu, S)\}.$$

Thus in seeking to maximize $|p_n(\mu, x)|$ amongst all measures supported on the real line, and of total mass at most S, whose restriction to $(-1,1)$ is ν, it suffices to consider measures formed from μ by adding mass points to ± 1. In this paper, we shall analyze the parameters M and N in the special case where ν is a symmetric measure about 0 (so that if μ is absolutely continuous, μ' is even) while n is even and $x = 0$. We shall also resolve the role of the total mass S.

We shall see that either $M = N = 0$ or $M = N = S/2$, so that μ^* is symmetric. Moreover, except for a single possible value of S, μ^* is unique. This also raises the question of just how large $|p_n(\mu, 0)|$ can be amongst all measures whose restriction to $(-1,1)$ is ν, without any restriction on the total mass of μ. Accordingly, we define $\mathcal{M}(\nu)$ to be the class of all positive measures μ supported on the real line, with all finite moments, such that $\mu = \nu$ in $(-1,1)$. We shall need some auxiliary parameters that depend only on n and ν:

$$r_n = \frac{\gamma_{n-1}}{\gamma_n}(\nu)\frac{p_{n-1}(\nu,1)}{p_n(\nu,1)} = -\frac{K_n(\nu,-1,1)}{p_n^2(\nu,1)}. \tag{1}$$

The second formula for r_n follows from the Christoffel-Darboux formula, and symmetry of ν.

$$\begin{aligned} U_n &= K_n(\nu,1,1) - K_n(\nu,-1,1); \\ V_n &= K_n(\nu,1,1) + K_n(\nu,-1,1). \end{aligned} \tag{2}$$

We note that it follows from the recurrence relation that $0 < r_n < 1$, while the symmetry of ν and Cauchy-Schwarz show that $U_n, V_n > 0$.

We prove:

Theorem 1.1

Let ν be a positive measure with support in $[-1,1]$ and with infinitely many points in its support, and that does not have mass points at ± 1. Assume also that ν is symmetric, so that $\nu([-b,-a]) = \nu([a,b])$ for all subintervals $[a,b]$ of $[-1,1]$. Let $n \geq 2$ be even. Then

$$\sup_{\mu \in \mathcal{M}(\nu)} \left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2 = \max\left\{1, \frac{U_n^2}{V_n V_{n+1}}\right\}. \tag{3}$$

Moreover,

$$\sup_{\mu \in \mathcal{M}(\nu)} \left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2 = \frac{U_n^2}{V_n V_{n+1}} > 1 \tag{4}$$

iff

$$\frac{2p_n^2(\nu,1)}{V_n} > \frac{1-2r_n}{r_n^2}. \tag{5}$$

Remarks

(a) We have been unable to find a measure for which (1.5) fails, but nor have we been able to prove that it is always true. It is true for all even Jacobi weights and large enough n, as we shall see below.

(b) Interestingly enough, the supremum in (1.4) is not attained. It occurs as $M = N \to \infty$. However, we note that for very large classes of measures, the last right-hand side decays to 1 as $n \to \infty$, as is evident from the case of ultraspherical weights considered below.

Next, we consider the case where we maximize over the class $\mathcal{M}(\nu, S)$. Here the maximum is attained, as was shown in [12]. For a given $S > 0$, let

$$X_S = p_n^2(\nu,1)\frac{S + S^2 U_n/2}{S^2 U_n V_n/4 + S K_n(\nu,1,1) + 1}. \tag{6}$$

In the course of our proofs, we shall show that X_S is an increasing function of $S > 0$, and its limit as $S \to \infty$ coincides with the left-hand side of (1.5). We prove:

Theorem 1.2

Let ν be a positive measure with support in $[-1,1]$ and with infinitely many points in its support, and that does not have mass points at ± 1. Assume also that ν is symmetric, so that $\nu([-b,-a]) = \nu([a,b])$ for all subintervals $[a,b]$ of $[-1,1]$. Let $n \geq 2$ be even and $S > 0$ and let $\mathcal{M}(\nu, S)$ denote the class of measures defined above. Let $\mu^* = \nu + M^*\delta_{-1} + N^*\delta_1 \in \mathcal{M}(\nu, S)$ satisfy

$$|p_n(\mu^*, 0)| = \sup\{|p_n(\mu, 0)| : \mu \in \mathcal{M}(\nu, S)\}. \tag{7}$$

(a) If $X_S < \frac{1-2r_n}{r_n^2}$*, then* $M^* = N^* = 0$*,* $\mu^* = \nu$*, and*

$$|p_n(\mu^*, 0)| = |p_n(\nu, 0)|. \tag{8}$$

(b) If $X_S > \frac{1-2r_n}{r_n^2}$*, then* $M^* = N^* = \frac{S}{2}$*,* $\mu^* = \nu + \frac{S}{2}(\delta_{-1} + \delta_1)$*, and*

$$\begin{aligned} & \left(\frac{p_n(\mu^*, 0)}{p_n(\nu, 0)}\right)^2 \\ = \quad & \frac{\left(S^2 U_n^2/4 + SU_n + 1\right)^2}{(S^2 U_n V_n/4 + SK_n(\nu, 1, 1) + 1)(S^2 U_n V_{n+1}/4 + SK_{n+1}(\nu, 1, 1) + 1)} > 1. \end{aligned} \tag{9}$$

(c) If $X_S = \frac{1-2r_n}{r_n^2}$*, then there are two extremal measures, namely* $\mu^* = \nu$*, and* $\mu^* = \nu + \frac{S}{2}(\delta_{-1} + \delta_1)$*, and (1.8) holds.*

Thus the extremal measure is always symmetric. It is also unique, except when $X_S = \frac{1-2r_n}{r_n^2}$. For even Jacobi weights (or equivalently ultraspherical weights), we obtain more explicit results:

Theorem 1.3
Let $\alpha > -1$ *and*

$$\nu'(t) = \left(1 - t^2\right)^\alpha, t \in (-1, 1). \tag{10}$$

For even $n \geq 2$*, the inequality (1.5) holds, and*

$$\begin{aligned} & \sup_{\mu \in \mathcal{M}(\nu)} \left(\frac{p_n(\mu, 0)}{p_n(\nu, 0)}\right)^2 \\ = \quad & 1 + \frac{\left(\frac{1}{n+\alpha}\right)^2 2(\alpha+1)\{1 + \frac{2\alpha+1}{n}\}}{1 + 2\frac{\alpha+1}{n+\alpha} + \frac{\alpha+1}{(n+\alpha)^2}\left\{\alpha - 1 - \frac{2(2\alpha+1)}{n}\right\}} \\ = \quad & 1 + \frac{2(\alpha+1)}{(n+\alpha)^2} + O\left(n^{-3}\right). \end{aligned} \tag{11}$$

Thus for all $\alpha > -1$, the supremum exceeds 1 for large enough n, but decays to 1 as $n \to \infty$. For fixed S, we prove:

Theorem 1.4
Let ν, n *be as in Theorem 1.3 and let* $S > 0$*. Let* $\mu^* = \nu + M^*\delta_{-1} + N^*\delta_1 \in \mathcal{M}(\nu, S)$ *be an extremal measure satisfying (1.7).*
(a) Suppose $-1 < \alpha < -\frac{1}{2}$*. Then there exists* $n_0(\alpha)$ *such that for* $n \geq n_0(\alpha)$*,* $r_n > \frac{1}{2}$*. Moreover, for* $n \geq n_0(\alpha)$ *and for all* $S > 0$*,* $M^* = N^* = \frac{S}{2}$ *and* $\mu^* = \nu + \frac{S}{2}(\delta_{-1} + \delta_1)$*.*
(b) Suppose $\alpha > -\frac{1}{2}$*. Then there exists* $n_0(\alpha)$ *such that for* $n \geq n_0(\alpha)$*,* $r_n < \frac{1}{2}$*. Then for* $n \geq n_0(\alpha)$ *and* $S > 0$ *so small that* $X_S < \frac{1-2r_n}{r_n^2}$*,* $M^* = N^* = 0$ *and* $\mu^* = \nu$*. For* $n \geq n_0(\alpha)$ *and* $X_S = \frac{1-2r_n}{r_n^2}$*, we may take* $\mu^* = \nu$*, or* $\mu^* = \nu + \frac{S}{2}(\delta_{-1} + \delta_1)$*. For* $n \geq n_0(\alpha)$ *and* $X_S > \frac{1-2r_n}{r_n^2}$*,* $M^* = N^* = \frac{S}{2}$ *and* $\mu^* = \nu + \frac{S}{2}(\delta_{-1} + \delta_1)$*.*
(c) Suppose $\alpha = -\frac{1}{2}$*. Then* $r_n = \frac{1}{2}$*. For* $n \geq 2$*,* $M^* = N^* = \frac{S}{2}$ *and* $\mu^* = \nu + \frac{S}{2}(\delta_{-1} + \delta_1)$*.*

Observe that if $\alpha > -\frac{1}{2}$, the extremal measure is $\mu^* = \nu$ for small enough S, but once S increases beyond a certain threshold, $\mu^* = \nu + \frac{S}{2}(\delta_{-1} + \delta_1)$. It is possible to give a more explicit form to the expression for the sup in (1.9) for ultraspherical weights, but it is messy and so omitted.

There is an extensive literature on the effect on orthogonal polynomials of adding mass points to a given weight. Differential equations and other identities have been obtained, asymptotics as $n \to \infty$ have been established, and Sobolev analogues have been investigated. See [2], [4], [7], [10], [11] for some references.

This paper is organized as follows: In Section 2, we present a basic identity. In Section 3, we first prove Theorem 1.2 and then Theorem 1.1. In Section 4, we first prove Theorem 1.3 and then Theorem 1.4.

In the sequel $C, C_1, C_2, \ldots$ denote constants independent of n, x, t. The same symbol does not necessarily denote the same constant in different occurences.

2 The Basic Identity

Throughout this section, ν satisfies the hypotheses of Theorem 1.1. Recall that r, U_n, V_n and X_S are defined by (1.1), (1.2) and (1.6). Our analysis is based on the identity below. We do not claim that it is new, as identities of this type are commonly used in analyzing measures with masspoints added, but derive it in a form that we can apply it:

Theorem 2.1
Let $M, N \geq 0$ *and*

$$\mu = \nu + M\delta_1 + N\delta_{-1}.$$

Let

$$x = x(M,N) = p_n^2(\nu,1)\frac{2MNU_n + M + N}{MNU_nV_n + (M+N)K_n(\nu,1,1)+1}. \tag{12}$$

(a) Then

$$\left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2 = g(x) := \frac{(1+r_nx)^2}{1+x}.$$

(b) If $r_n < \frac{1}{2}$, *the function* g *is a strictly decreasing function of* $x \in (0, \frac{1-2r_n}{r_n})$ *and is a strictly increasing function of* $x \in (\frac{1-2r_n}{r_n}, \infty)$.
(c) If $r_n \geq \frac{1}{2}$, *the function* g *is a strictly increasing function of* $x \in (0,\infty)$.
(d) $g(x) > 1$ *iff*

$$x > \frac{1-2r_n}{r_n^2}. \tag{13}$$

while $g(x) = 1$ *iff* $x = \frac{1-2r_n}{r_n}$ *or* $x = 0$.

We begin the proof with

Lemma 2.2
(a) Let

$$\pi_{n-1}(y) = p_n(\mu,y) - \frac{\gamma_n(\mu)}{\gamma_n(\nu)}p_n(\nu,y); \tag{14}$$

$$A = \begin{bmatrix} 1+MK_n(\nu,1,1) & -MK_n(\nu,1,-1) \\ -NK_n(\nu,1,-1) & 1+NK_n(\nu,1,1) \end{bmatrix}; \tag{15}$$

and

$$d = MNU_nV_n + (M+N)K_n(\nu,1,1)+1. \tag{16}$$

(a) Then

$$p_n(\mu,y) = \frac{\gamma_n(\mu)}{\gamma_n(\nu)}\left\{p_n(\nu,y) + \frac{p_n(\nu,1)}{d}\begin{bmatrix} -NK_n(\nu,y,-1) \\ -MK_n(\nu,y,1)\end{bmatrix}^T A \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}. \tag{17}$$

(b)

$$\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2\left\{1 + \frac{p_n^2(\nu,1)}{d}\begin{bmatrix} 1 \\ 1 \end{bmatrix}^T A^T \begin{bmatrix} N \\ M \end{bmatrix}\right\} = 1. \tag{18}$$

Proof
(a) Using orthogonality, we see that

$$\begin{aligned}\pi_{n-1}(y) &= \int_{-1}^{1} K_n(\nu,y,t)\pi_{n-1}(t)\,d\nu(t) \\ &= \int_{-1}^{1} K_n(\nu,y,t)p_n(\mu,t)\,d\nu(t) \\ &= -MK_n(\nu,y,1)p_n(\mu,1) - NK_n(\nu,y,-1)p_n(\mu,-1).\end{aligned} \tag{19}$$

Taking $y=-1$ and $y=1$, and gathering the terms involving $p_n(\mu,\pm 1)$, gives the matrix equation

$$\begin{bmatrix} 1+NK_n(\nu,-1,-1) & MK_n(\nu,-1,1) \\ NK_n(\nu,1,-1) & 1+MK_n(\nu,1,1)\end{bmatrix}\begin{bmatrix} p_n(\mu,-1) \\ p_n(\mu,1)\end{bmatrix} = \frac{\gamma_n(\mu)}{\gamma_n(\nu)}\begin{bmatrix} p_n(\nu,-1) \\ p_n(\nu,1)\end{bmatrix}.$$

The determinant d of the matrix can be put into the form in (2.4), if we take account of the definition (1.2) of U_n, V_n. Solving the matrix equation and using the symmetry of ν gives

$$\begin{aligned}\begin{bmatrix} p_n(\mu,-1) \\ p_n(\mu,1)\end{bmatrix} &= \frac{\gamma_n(\mu)}{\gamma_n(\nu)}\frac{1}{d}\begin{bmatrix} 1+MK_n(\nu,1,1) & -MK_n(\nu,1,-1) \\ -NK_n(\nu,1,-1) & 1+NK_n(\nu,1,1)\end{bmatrix}\begin{bmatrix} p_n(\nu,1) \\ p_n(\nu,1)\end{bmatrix} \\ &= \frac{\gamma_n(\mu)}{\gamma_n(\nu)}p_n(\nu,1)\frac{A}{d}\begin{bmatrix} 1 \\ 1\end{bmatrix}.\end{aligned} \tag{20}$$

From (2.8) and this last identity,

$$\pi_{n-1}(y) = \frac{\gamma_n(\mu)}{\gamma_n(\nu)}\frac{p_n(\nu,1)}{d}\begin{bmatrix} -NK_n(\nu,y,-1) \\ -MK_n(\nu,y,1)\end{bmatrix}^T A\begin{bmatrix} 1 \\ 1\end{bmatrix}.$$

Then (2.6) follows from the definition of π_{n-1}.
(b) We obtain equations for $\frac{\gamma_n(\mu)}{\gamma_n(\nu)}$ in two ways:

$$\begin{aligned}&\int_{-1}^{1}\pi_{n-1}^2(y)\,d\nu(y)\\ =\;&\int_{-1}^{1}p_n^2(\mu,y)^2\,d\nu(y)-2\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2+\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2\\ =\;&1-Mp_n(\mu,1)^2-Np_n(\mu,-1)^2-\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2.\end{aligned}$$

Also, from (2.8),

$$\begin{aligned}&\int_{-1}^{1}\pi_{n-1}^2(y)\,d\nu(y)\\ =\;&\int_{-1}^{1}\left(-NK_n(\nu,y,-1)p_n(\mu,-1)-MK_n(\nu,y,1)p_n(\mu,1)\right)^2d\nu(y)\\ =\;&N^2p_n^2(\mu,-1)K_n(\nu,-1,-1)+M^2p_n^2(\mu,1)K_n(\nu,1,1)+2MNp_n(\mu,-1)p_n(\mu,1)K_n(\nu,-1,1).\end{aligned}$$

Then using the last two equations and solving for $1-\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2$,

$$\begin{aligned}&1-\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2\\ =\;&p_n^2(\mu,-1)\left\{N+N^2K_n(\nu,-1,-1)\right\}+p_n^2(\mu,1)\left\{M+M^2K_n(\nu,1,1)\right\}\\ &+2MNp_n(\mu,-1)p_n(\mu,1)K_n(\nu,-1,1)\\ =\;&\begin{bmatrix}p_n(\mu,-1)\\ p_n(\mu,1)\end{bmatrix}^T\begin{bmatrix}N&0\\0&M\end{bmatrix}\begin{bmatrix}1+NK_n(\nu,1,1)&MK_n(\nu,-1,1)\\ NK_n(\nu,-1,1)&1+MK_n(\nu,1,1)\end{bmatrix}\begin{bmatrix}p_n(\mu,-1)\\ p_n(\mu,1)\end{bmatrix}\\ =\;&d\begin{bmatrix}p_n(\mu,-1)\\ p_n(\mu,1)\end{bmatrix}^T\begin{bmatrix}N&0\\0&M\end{bmatrix}A^{-1}\begin{bmatrix}p_n(\mu,-1)\\ p_n(\mu,1)\end{bmatrix}\end{aligned}$$

Using (2.9) gives

$$\begin{aligned}&1-\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2\\ =\;&\frac{p_n^2(\nu,1)}{d}\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2\begin{bmatrix}1\\1\end{bmatrix}^T A^T\begin{bmatrix}N&0\\0&M\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\end{aligned}$$

and (2.7) follows.■

Proof of Theorem 2.1(a)
Setting $y=0$ in (2.6), squaring and using (2.7) gives

$$\begin{aligned}&p_n^2(\mu,0)\left\{1+\frac{p_n^2(\nu,1)}{d}\begin{bmatrix}1\\1\end{bmatrix}^T A^T\begin{bmatrix}N\\M\end{bmatrix}\right\}\\ =\;&\left\{p_n(\nu,0)+\frac{p_n(\nu,1)}{d}\begin{bmatrix}-NK_n(\nu,0,-1)\\-MK_n(\nu,0,1)\end{bmatrix}^T A\begin{bmatrix}1\\1\end{bmatrix}\right\}^2.\end{aligned}\tag{21}$$

Here from the Christoffel-Darboux formula and as $p_{n-1}(\nu,0)=0$,

$$K_n(\nu,0,\pm1)=-\frac{\gamma_{n-1}}{\gamma_n}(\nu)p_n(\nu,0)p_{n-1}(\nu,1)$$

so using Christoffel-Darboux again, and $p_{n-1}(\nu,-1)=-p_{n-1}(\nu,1)$

$$p_n(\nu,1)K_n(\nu,0,\pm1)=p_n(\nu,0)K_n(\nu,-1,1).$$

Thus (2.10) becomes

$$\begin{aligned}&\left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2\left\{1+\frac{p_n^2(\nu,1)}{d}\begin{bmatrix}1\\1\end{bmatrix}^T A^T\begin{bmatrix}N\\M\end{bmatrix}\right\}\\ =\;&\left\{1-\frac{K_n(\nu,-1,1)}{d}\begin{bmatrix}N\\M\end{bmatrix}^T A\begin{bmatrix}1\\1\end{bmatrix}\right\}^2.\end{aligned}\tag{22}$$

Here from (2.4) and (2.5), followed by (2.1),

$$\begin{aligned} & \frac{p_n^2(\nu,1)}{d}\begin{bmatrix} 1 \\ 1 \end{bmatrix}^T A^T \begin{bmatrix} N \\ M \end{bmatrix} \\ = \quad & p_n^2(\nu,1)\frac{N+M+2MNU_n}{MNU_nV_n+(M+N)K_n(\nu,1,1)+1} = x. \end{aligned}$$

Also, from (1.1),

$$K_n(\nu,-1,1) = -\frac{\gamma_{n-1}}{\gamma_n}(\nu)\, p_n(\nu,1)\, p_{n-1}(\nu,1) = -r_n p_n^2(\nu,1) \tag{23}$$

so (2.11) becomes

$$\left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2 \{1+x\} = \{1+r_n x\}^2.$$

■

Proof of Theorem 2.1 (b), (c), (d)
A calculation shows that

$$g(x) = r_n^2 x + \left(2r_n - r_n^2\right) + \frac{(r_n-1)^2}{1+x}$$

so

$$g'(x) = r_n^2\left\{1 - \frac{\left(1-\frac{1}{r_n}\right)^2}{(1+x)^2}\right\}.$$

Thus $g'(x)$ is an increasing function of $x \in [0,\infty)$, with limit $r_n^2 > 0$ as $x \to \infty$. Also

$$g'(x) = 0 \Leftrightarrow 1+x = \pm\left(1-\frac{1}{r_n}\right)$$

so as $x > 0$, and $r_n > 0$,

$$g'(x) = 0 \Leftrightarrow x = \frac{1-2r_n}{r_n}.$$

Then if $r_n < \frac{1}{2}$, it follows that $g(x)$ decreases in $\left(0, \frac{1-2r_n}{r_n}\right)$ and increases in $\left(\frac{1-2r_n}{r_n}, \infty\right)$. If $r_n \geq \frac{1}{2}$, it follows that $g(x)$ increases in $[0,\infty)$. Finally

$$\begin{aligned} g(x) \quad &> \quad 1 \Leftrightarrow 1 + 2r_n x + r_n^2 x^2 > 1 + x \\ &\Leftrightarrow \quad x > \frac{1-2r_n}{r_n^2}, \end{aligned}$$

as $x > 0$. Also $g(x) = 1$ iff $x = 0$ or $x = \frac{1-2r_n}{r_n^2}$.■

3 Proof of Theorems 1.1 and 1.2

Recall that $x = x(M,N)$ is given by (2.1). We begin with

Lemma 3.1
(a) For $M,N \geq 0$,

$$\frac{\partial x}{\partial M} > 0; \frac{\partial x}{\partial N} > 0.$$

(b) The maximum of $x = x(M,N)$ in the triangular region $T = \{(M,N) : 0 \leq M, N \text{and } M+N \leq S\}$ occurs when and only when

$$M = N = \frac{S}{2}.$$

(c) Moreover, the maximum is

$$x = X_S = p_n^2(\nu,1)\frac{S^2U_n/2+S}{S^2U_nV_n/4+SK_n(\nu,1,1)+1}.$$

(d)

$$X_\infty := \lim_{S\to\infty} X_S = \frac{2p_n^2(\nu,1)}{V_n}.$$

Proof
(a) Note that from Cauchy-Schwarz, and as $p_j(\nu,-1)=(-1)^j p_j(\nu,1)$,

$$\begin{aligned} |K_n(\nu,-1,1)| &= \left|\sum_{j=0}^{n-1} p_j(\nu,1)\,p_j(\nu,-1)\right| \\ &< \sum_{j=0}^{n-1}\left|p_j(\nu,1)\,p_j(\nu,-1)\right| \\ &\leq K_n(\nu,1,1)\,K_n(\nu,-1,-1)=K_n(\nu,1,1)^2 \end{aligned}$$

so that

$$U_n, V_n > 0. \tag{24}$$

Next, using $V_n - 2K_n(\nu,1,1) = -U_n$, and from (2.1),

$$\begin{aligned} &\frac{1}{p_n^2(\nu,1)}\left(MNU_nV_n+(M+N)K_n(\nu,1,1)+1\right)^2\left(\frac{\partial x}{\partial M}\right) \\ =\ & (2NU_n+1)\left(MNU_nV_n+(M+N)K_n(\nu,1,1)+1\right)-(2MNU_n+M+N)\left(NU_nV_n+K_n(\nu,1,1)\right) \\ =\ & MNU_n\left\{(2NU_n+1)V_n-2\left(NU_nV_n+K_n(\nu,1,1)\right)\right\} \\ & +(M+N)\left\{(1+2NU_n)K_n(\nu,1,1)-\left(NU_nV_n+K_n(\nu,1,1)\right)\right\}+2NU_n+1 \\ =\ & MNU_n\left\{V_n-2K_n(\nu,1,1)\right\}+(M+N)\left\{NU_n\left(2K_n(\nu,1,1)-V_n\right)\right\}+2NU_n+1 \\ =\ & MNU_n\{-U_n\}+(M+N)\left\{NU_n^2\right\}+2NU_n+1 \\ =\ & (NU_n+1)^2>0. \end{aligned}$$

Thus

$$\frac{\partial x}{\partial M}=p_n^2(\nu,1)\,\frac{(NU_n+1)^2}{d^2}.$$

Then as $U_n>0$, $\frac{\partial x}{\partial M}>0$ and similarly $\frac{\partial x}{\partial N}>0$.
(b) Since $\frac{\partial x}{\partial M}>0, \frac{\partial x}{\partial N}>0$ for all $M,N\geq 0$, so there are no critical points within the interior of the triangle. Moreover, it then follows that the maximum cannot occur on the axes $M=0$ or $N=0$, so occurs when $M+N=S$. Then on this line segment,

$$\begin{aligned} x &= p_n^2(\nu,1)\,\frac{2M(S-M)U_n+S}{M(S-M)U_nV_n+SK_n(\nu,1,1)+1} \\ &= \frac{p_n^2(\nu,1)}{V_n}\left\{2+\frac{SV_n-2SK_n(\nu,1,1)-2}{M(S-M)U_nV_n+SK_n(\nu,1,1)+1}\right\} \\ &= \frac{p_n^2(\nu,1)}{V_n}\left\{2-\frac{SU_n+2}{M(S-M)U_nV_n+SK_n(\nu,1,1)+1}\right\}. \end{aligned} \tag{25}$$

Here we have used the definition of U_n, V_n. Since $S\geq 0$ is fixed and $U_n>0$, this last expression is an increasing function of $M(S-M)$ and in turn that is maximized over $M\in[0,S]$ when and only when $M=\frac{S}{2}$.
(c) This follows by substituting $M=N=\frac{S}{2}$ into (2.1).
(d) This is immediate from (c). ■

Proof of Theorem 1.2(a)
We're assuming that $X_S<\frac{1-2r_n}{r_n^2}$. Of course this is possible only if $r_n<\frac{1}{2}$, since $X_S>0$. Let $0\leq M,N$ and $M+N\leq S$ and $\mu=\nu+M\delta_1+N\delta_{-1}$. By Theorem 2.1, if $x=x(M,N)$, we have

$$\left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2=\frac{(1+r_nx)^2}{1+x}=g(x).$$

Here by Lemma 3.1, $0\leq x\leq X_S<\frac{1-2r_n}{r_n^2}$, so Theorem 2.1(d) shows that

$$\left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2<1,$$

unless $x=0$. It follows that the maximum possible value of $\left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2$ for $\mu\in\mathcal{M}(\nu,S)$ occurs iff $M=N=0$. ■

Proof of Theorem 1.2(b)
We're assuming that $X_S>\frac{1-2r_n}{r_n^2}$. By Theorem 2.1, if $x=x(M,N)$, we have

$$\left(\frac{p_n(\mu,0)}{p_n(\nu,0)}\right)^2=\frac{(1+r_nx)^2}{1+x}=g(x)$$

is maximal when x is large as possible under the restrictions $0 \leq M, N$ and $M + N \leq S$. By Lemma 3.1, this occurs iff $M = N = \frac{S}{2}$, and then $x = X_S$. Here from (2.12) and (1.6),

$$r_n X_S = -SK_n(\nu, -1, 1) \frac{1 + SU_n/2}{S^2 U_n V_n/4 + SK_n(\nu, 1, 1) + 1}$$

so

$$\begin{aligned} & 1 + r_n X_S \\ = & \frac{S^2 (U_n V_n - 2U_n K_n(\nu, -1, 1))/4 + S(K_n(\nu, 1, 1) - K_n(\nu, -1, 1)) + 1}{S^2 U_n V_n/4 + SK_n(\nu, 1, 1) + 1} \\ = & \frac{S^2 U_n^2/4 + SU_n + 1}{S^2 U_n V_n/4 + SK_n(\nu, 1, 1) + 1} \end{aligned}$$

while

$$\begin{aligned} & 1 + X_S \\ = & \frac{S^2 U_n \left[V_n + 2p_n^2(\nu, 1)\right]/4 + S\left[K_n(\nu, 1, 1) + p_n^2(\nu, 1)\right] + 1}{S^2 U_n V_n/4 + SK_n(\nu, 1, 1) + 1} \\ = & \frac{S^2 U_n V_{n+1}/4 + SK_{n+1}(\nu, 1, 1) + 1}{S^2 U_n V_n/4 + SK_n(\nu, 1, 1) + 1} \end{aligned}$$

Then

$$\begin{aligned} & \left(\frac{p_n(\mu, 0)}{p_n(\nu, 0)}\right)^2 = \frac{(1 + r_n X_S)^2}{1 + r_n X_S} \\ = & \frac{\left(S^2 U_n^2/4 + SU_n + 1\right)^2}{(S^2 U_n V_n/4 + SK_n(\nu, 1, 1) + 1)(S^2 U_n V_{n+1}/4 + SK_{n+1}(\nu, 1, 1) + 1)}. \end{aligned}$$

By Theorem 2.1(d), and as $X_S > \frac{1-2r_n}{r_n^2}$, this exceeds 1. ■

Proof of Theorem 1.2(c)
Here as $X_S = \frac{1-2r}{r^2}$, we have $g(X_s) = 1 = g(0)$, and for any other value of $x = x(M, N)$ we have $g(x) < 1$. ■

Proof of Theorem 1.1
It follows from Theorem 2.1 and Lemma 3.1, that for a given $S > 0$,

$$\sup_{\mu \in \mathcal{M}(\nu, S)} \left(\frac{p_n(\mu, 0)}{p_n(\nu, 0)}\right)^2 = \max\left\{1, \frac{(1 + r_n X_S)^2}{1 + X_S}\right\}.$$

Indeed if $X_S \leq \frac{1-2r_n}{r_n^2}$, the maximum is 1, while if $X_S > \frac{1-2r_n}{r_n^2}$, the maximum is achieved when $M = N = \frac{S}{2}$. It also follows from Lemma 3.1, that $X_S = x\left(\frac{S}{2}, \frac{S}{2}\right)$ is an increasing function of S, while the function $g(x)$ has at most one local minimum in $(0, \infty)$, has $g(0) = 0$, and is increasing beyond that local minimum. Then from (1.9),

$$\begin{aligned} & \sup_{\mu \in \mathcal{M}(\nu)} \left(\frac{p_n(\mu, 0)}{p_n(\nu, 0)}\right)^2 \\ = & \lim_{S \to \infty} \sup_{\mu \in \mathcal{M}(\nu, S)} \left(\frac{p_n(\mu, 0)}{p_n(\nu, 0)}\right)^2 \\ = & \max\left\{1, \frac{\left(U_n^2/4\right)^2}{(U_n V_n/4) U_n V_{n+1}/4}\right\} \\ = & \max\{1, \frac{U_n^2}{V_n V_{n+1}}\}. \end{aligned}$$

Finally, the above considerations show that we can drop the 1 in the max, that is

$$\sup_{\mu \in \mathcal{M}(\nu)} \left(\frac{p_n(\mu, 0)}{p_n(\nu, 0)}\right)^2 = \frac{U_n^2}{V_n V_{n+1}} > 1$$

iff for large enough S, $X_S > \frac{1-2r_n}{r_n^2}$, which is true iff (recall Lemma 3.1(d))

$$\frac{2p_n^2(\nu, 1)}{V_n} = X_\infty > \frac{1 - 2r_n}{r_n^2}. \tag{26}$$

■

We have been unable to resolve if (3.3) is always true. Here is an equivalent form:

Lemma 3.2
The inequality (3.3) is equivalent to

$$\frac{-K_n(\nu,-1,1)}{K_{n+1}(\nu,-1,1)} > \frac{K_n(\nu,1,1)}{K_{n+1}(\nu,1,1)}.$$

Proof
From the second identity in (1.1),

$$\frac{1-2r_n}{r_n^2} = \frac{p_n^2(\nu,1)}{K_n(\nu,1,-1)^2}\left[2K_n(\nu,1,-1)+p_n^2(\nu,1)\right],$$

so (3.3) is equivalent to

$$\begin{aligned}
\frac{2p_n^2(\nu,1)}{V_n} &> \frac{p_n^2(\nu,1)}{K_n(\nu,-1,1)^2}\left(p_n^2(\nu,1)+2K_n(\nu,-1,1)\right)\\
\Leftrightarrow\quad & 2K_n(\nu,-1,1)^2 > (K_n(\nu,1,1)+K_n(\nu,-1,1))\left(p_n^2(\nu,1)+2K_n(\nu,-1,1)\right)\\
\Leftrightarrow\quad & 0 > (K_n(\nu,1,1)+K_n(\nu,-1,1))\,p_n^2(\nu,1)+2K_n(\nu,1,1)K_n(\nu,-1,1)\\
\Leftrightarrow\quad & 0 > \left(K_n(\nu,1,1)+p_n^2(\nu,1)\right)K_n(\nu,-1,1)+\left(K_n(\nu,-1,1)+p_n^2(\nu,1)\right)K_n(\nu,1,1)\\
\Leftrightarrow\quad & 0 > K_{n+1}(\nu,1,1)K_n(\nu,-1,1)+K_{n+1}(\nu,-1,1)K_n(\nu,1,1)\\
\Leftrightarrow\quad & \frac{-K_n(\nu,-1,1)}{K_{n+1}(\nu,-1,1)} > \frac{K_n(\nu,1,1)}{K_{n+1}(\nu,1,1)}.
\end{aligned}$$

Here we are using $K_n(\nu,-1,1) < 0 < K_{n+1}(\nu,-1,1)$. ■

4 Proof of Theorems 1.3 and 1.4

Let us first recall the values of some orthogonal polynomial quantities for the ultraspherical weight (or even Jacobi weight)

$$\nu'(t) = \left(1-t^2\right)^{\alpha},\ t\in(-1,1).$$

Here $\alpha > -1$ is fixed. Throughout this section, we drop the parameter ν in $p_n(\nu,x)$ etc. while n is even. The classical Jacobi polynomials $P_n^{(\alpha,\alpha)}$ are normalized by [16, p. 58]

$$P_n^{(\alpha,\alpha)}(1) = \binom{n+\alpha}{n}. \tag{27}$$

The leading coefficient of $P_n^{(\alpha,\alpha)}$ is [16, p. 63]

$$2^{-n}\binom{2n+2\alpha}{n}.$$

Also, the orthonormal polynomial is given by [16, p. 68]

$$p_n(x) = c_n P_n^{(\alpha,\alpha)}(x), \tag{28}$$

where

$$c_n = \left\{\frac{2n+2\alpha+1}{2^{2\alpha+1}}\frac{\Gamma(n+1)\Gamma(n+2\alpha+1)}{\Gamma(n+\alpha+1)^2}\right\}^{1/2}, \tag{29}$$

so that

$$p_n(1) = c_n\binom{n+\alpha}{n} \tag{30}$$

and

$$\gamma_n = c_n 2^{-n}\binom{2n+2\alpha}{n}. \tag{31}$$

Furthermore, taking account that our reproducing kernel sums to $n-1$ while that in [16] adds to n, [16, p. 71]

$$K_n(x,1) = 2^{-2\alpha-1}\frac{\Gamma(n+2\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+\alpha)}P_{n-1}^{(\alpha+1,\alpha)}(x) \tag{32}$$

so that

$$K_n(1,1) = 2^{-2\alpha-1}\frac{\Gamma(n+2\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+\alpha)}\binom{n+\alpha}{n-1} \tag{33}$$

while using that $P_{n-1}^{(\alpha+1,\alpha)}(-x) = (-1)^{n-1} P_{n-1}^{(\alpha,\alpha+1)}(x)$,

$$K_n(-1,1) = (-1)^{n-1} 2^{-2\alpha-1} \frac{\Gamma(n+2\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+\alpha)} \binom{n-1+\alpha}{n-1}. \tag{34}$$

The proofs of this section involve several straightforward calculations. We shall exclude some of the line by line computations.

Lemma 4.1
(a)

$$\frac{p_{n-1}(1)}{p_n(1)} = \left(1 - \frac{1+2\alpha}{n} + \eta_n\right)^{1/2}, \tag{35}$$

where

$$\eta_n = (2\alpha+1) \frac{n(4\alpha+1) + 2\alpha(2\alpha+1)}{(2n+2\alpha+1)(n+2\alpha)n}. \tag{36}$$

(b)

$$\begin{aligned} r_n &= \frac{1}{2}\left(1 + \frac{1-4\alpha^2}{4(n+\alpha)^2 - 1}\right)^{1/2} \left(1 - \frac{1+2\alpha}{n} + \eta_n\right)^{1/2} \\ &= \frac{1}{2}\left(1 - \frac{1+2\alpha}{2n} + O\left(n^{-2}\right)\right). \end{aligned} \tag{37}$$

(c)

$$\frac{1-2r_n}{r_n^2} = \frac{2(1+2\alpha)}{n}\left(1 + O\left(n^{-1}\right)\right). \tag{38}$$

(d) For $n \geq 2$,

$$X_\infty = \frac{2p_n^2(1)}{V_n} > \frac{1-2r_n}{r_n^2}. \tag{39}$$

(e)

$$\frac{2p_n^2(1)}{K_n(1,1)} = 4(\alpha+1)\left(1 + \frac{1}{2(n+\alpha)}\right)\frac{1}{n}. \tag{40}$$

(f)

$$\frac{-K_n(-1,1)}{K_n(1,1)} = \frac{\alpha+1}{n+\alpha}. \tag{41}$$

Proof
(a) Firstly using (4.3),

$$\begin{aligned} \frac{c_{n-1}}{c_n} &= \left(\frac{2n+2\alpha-1}{2n+2\alpha+1} \frac{\Gamma(n)\Gamma(n+2\alpha)}{\Gamma(n+1)\Gamma(n+2\alpha+1)} \frac{\Gamma(n+\alpha+1)^2}{\Gamma(n+\alpha)^2}\right)^{1/2} \\ &= \left(\frac{2n+2\alpha-1}{2n+2\alpha+1} \frac{(n+\alpha)^2}{n(n+2\alpha)}\right)^{1/2} \end{aligned}$$

so by (4.5), and a straightforward calculation,

$$\begin{aligned} \frac{\gamma_{n-1}}{\gamma_n} &= 2\frac{c_{n-1}}{c_n}\binom{2n-2+2\alpha}{n-1} \Big/ \binom{2n+2\alpha}{n} \\ &= \frac{1}{2}\left(1 + \frac{1-4\alpha^2}{4(n+\alpha)^2-1}\right)^{1/2}. \end{aligned} \tag{42}$$

Next, from (4.4),

$$\begin{aligned} &\frac{p_{n-1}(1)}{p_n(1)} \\ =\ & \frac{c_{n-1}\binom{n-1+\alpha}{n-1}}{c_n\binom{n+\alpha}{n}} \\ =\ & \left(\frac{2n+2\alpha-1}{2n+2\alpha+1} \frac{n}{n+2\alpha}\right)^{1/2} \\ =\ & \left(1 - \frac{1+2\alpha}{n} + \eta_n\right)^{1/2}, \end{aligned}$$

where

$$\begin{aligned}\eta_n &= -2\left[\frac{1}{2n+2\alpha+1}-\frac{1}{2n}+\frac{\alpha}{n+2\alpha}-\frac{\alpha}{n}\right]+\frac{4\alpha}{(2n+2\alpha+1)(n+2\alpha)} \\ &= (2\alpha+1)\frac{n(4\alpha+1)+2\alpha(2\alpha+1)}{(2n+2\alpha+1)(n+2\alpha)n},\end{aligned}$$

again, by a straightforward calculation.
(b) From (4.16) and (4.11),

$$\begin{aligned}r_n &= \frac{\gamma_{n-1}}{\gamma_n}\frac{p_{n-1}(1)}{p_n(1)} \\ &= \frac{1}{2}\left(1+\frac{1-4\alpha^2}{4(n+\alpha)^2-1}\right)^{1/2}\left(1-\frac{1+2\alpha}{n}+\eta_n\right)^{1/2} \\ &= \frac{1}{2}\left(1-\frac{1+2\alpha}{2n}+O\left(n^{-2}\right)\right).\end{aligned}$$

(c) This follows immediately from (b).
(d) Recall from Lemma 3.2 that

$$\frac{2p_n^2(\nu,1)}{V_n}=X_\infty>\frac{1-2r_n}{r_n^2}$$

is equivalent to

$$\frac{-K_n(-1,1)}{K_{n+1}(-1,1)}>\frac{K_n(1,1)}{K_{n+1}(1,1)}. \tag{43}$$

Now substitute in our values from (4.7) and (4.8):

$$\begin{aligned}\frac{-K_n(-1,1)}{K_{n+1}(-1,1)} &= \frac{\left(\frac{\Gamma(n+2\alpha+1)}{\Gamma(n+\alpha)}\right)\binom{n-1+\alpha}{n-1}}{\left(\frac{\Gamma(n+2\alpha+2)}{\Gamma(n+1+\alpha)}\right)\binom{n+\alpha}{n}} \\ &= 1-\frac{2\alpha+1}{n+2\alpha+1}. \end{aligned}\tag{44}$$

Also

$$\begin{aligned}\frac{K_n(1,1)}{K_{n+1}(1,1)} &= \frac{\left(\frac{\Gamma(n+2\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+\alpha)}\right)\binom{n+\alpha}{n-1}}{\left(\frac{\Gamma(n+2\alpha+2)}{\Gamma(\alpha+1)\Gamma(n+1+\alpha)}\right)\binom{n+1+\alpha}{n}} \\ &= \frac{n+\alpha}{n+2\alpha+1}\frac{n}{n+\alpha+1} \\ &= \left(1-\frac{\alpha+1}{n+2\alpha+1}\right)\left(1-\frac{\alpha+1}{n+\alpha+1}\right) \\ &= 1-(\alpha+1)\left[\frac{1}{n+2\alpha+1}+\frac{1}{n+\alpha+1}\right]+\frac{(\alpha+1)^2}{(n+2\alpha+1)(n+\alpha+1)} \\ &= 1-\frac{2(\alpha+1)}{n+2\alpha+1}-\frac{(\alpha+1)\alpha}{(n+\alpha+1)(n+2\alpha+1)}+\frac{(\alpha+1)^2}{(n+2\alpha+1)(n+\alpha+1)}\end{aligned}$$

so recalling (4.16) and (4.17), we want to check when

$$\frac{2\alpha+1}{n+2\alpha+1}<\frac{2(\alpha+1)}{n+2\alpha+1}+\frac{(\alpha+1)\alpha}{(n+\alpha+1)(n+2\alpha+1)}-\frac{(\alpha+1)^2}{(n+2\alpha+1)(n+\alpha+1)}$$

which is equivalent to

$$\begin{aligned}0 &< 1+\frac{(\alpha+1)\alpha}{(n+\alpha+1)}-\frac{(\alpha+1)^2}{(n+\alpha+1)} \\ &= 1-\frac{\alpha+1}{n+\alpha+1}.\end{aligned}$$

which is true for all even $n\geq 2$.
(e) From (4.4), (4.7), and then (4.3),

$$\begin{aligned}\frac{2p_n^2(1)}{K_n(1,1)} &= \frac{2\left\{c_n\binom{n+\alpha}{n}\right\}^2}{2^{-2\alpha-1}\frac{\Gamma(n+2\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+\alpha)}\binom{n+\alpha}{n-1}} \\ &= 4(\alpha+1)\left(1+\frac{1}{2(n+\alpha)}\right)\frac{1}{n}.\end{aligned}$$

(f)

$$\frac{-K_n(-1,1)}{K_n(1,1)} = \frac{\binom{n-1+\alpha}{n-1}}{\binom{n+\alpha}{n-1}} = \frac{\alpha+1}{n+\alpha}.$$

■

Proof of Theorem 1.3

As shown in the previous lemma, we have the inequality (4.13) for $n \geq 2$. For such n, we have from Theorem 1.1 that

$$\sup_{\mu \in \mathcal{M}(\nu)} \left(\frac{p_n(\mu,0)}{p_n(\nu,0)} \right)^2 = \frac{U_n^2}{V_n V_{n+1}}.$$

Here from (4.15),

$$\begin{aligned} U_n &= K_n(1,1)\left\{1 - \frac{K_n(-1,1)}{K_n(1,1)}\right\} = K_n(1,1)\left\{1 + \frac{\alpha+1}{n+\alpha}\right\}; \\ V_n &= K_n(1,1)\left\{1 + \frac{K_n(-1,1)}{K_n(1,1)}\right\} = K_n(1,1)\left\{1 - \frac{\alpha+1}{n+\alpha}\right\}; \end{aligned}$$

and from (4.14) and (4.15),

$$\begin{aligned} V_{n+1} &= K_n(1,1)\left\{1 + \frac{K_n(-1,1)}{K_n(1,1)} + \frac{2p_n^2(1)}{K_n(1,1)}\right\} \\ &= K_n(1,1)\{1 + 3\frac{\alpha+1}{n+\alpha} + \frac{2(\alpha+1)(2\alpha+1)}{n(n+\alpha)}\} \end{aligned}$$

so

$$\begin{aligned} V_n V_{n+1} &= K_n^2(1,1)\left\{ \begin{array}{c} 1 + 2\frac{\alpha+1}{n+\alpha} + \frac{2(\alpha+1)(2\alpha+1)}{n(n+\alpha)} \\ -3\left(\frac{\alpha+1}{n+\alpha}\right)^2 - \frac{2(\alpha+1)^2(2\alpha+1)}{n(n+\alpha)^2} \end{array} \right\} \\ &= K_n^2(1,1)\left\{ \begin{array}{c} 1 + 2\frac{\alpha+1}{n+\alpha} \\ +\frac{\alpha+1}{n+\alpha}\left\{\frac{\alpha-1}{n+\alpha} - \frac{2(2\alpha+1)}{n(n+\alpha)}\right\} \end{array} \right\}. \end{aligned}$$

Then by yet another calculation,

$$\begin{aligned} & \frac{U_n^2}{V_n V_{n+1}} \\ = \ & \frac{1 + 2\frac{\alpha+1}{n+\alpha} + \left(\frac{\alpha+1}{n+\alpha}\right)^2}{1 + 2\frac{\alpha+1}{n+\alpha} + \frac{\alpha+1}{n+\alpha}\left\{\frac{\alpha-1}{n+\alpha} - \frac{2(2\alpha+1)}{n(n+\alpha)}\right\}} \\ = \ & 1 + \frac{\left(\frac{1}{n+\alpha}\right)^2 2(\alpha+1)\{1 + \frac{2\alpha+1}{n}\}}{1 + 2\frac{\alpha+1}{n+\alpha} + \frac{\alpha+1}{(n+\alpha)^2}\left\{\alpha - 1 - \frac{2(2\alpha+1)}{n}\right\}}. \end{aligned}$$

■

Proof of Theorem 1.4

It follows from Lemma 4.1(b) that if $\alpha < -\frac{1}{2}$, then $r_n > \frac{1}{2}$ for $n \geq n_0(\alpha)$. If $\alpha > -\frac{1}{2}$, then $r_n < \frac{1}{2}$ for $n \geq n_0(\alpha)$. If $\alpha = -\frac{1}{2}$, then $r_n = \frac{1}{2}$ for all n. We can now apply Theorem 1.2, and also note that (as follows from Theorem 1.3 and Lemma 4.1(d)), we have $X_S > \frac{1-2r_n}{r_n^2}$ for large enough S. ■

References

[1] M.U. Ambroladze, *On the possible rate of growth of polynomials that are orthogonal with a continuous positive weight*. Math. USSR-Sb. 72 (1992), 311–331.

[2] R. Alvarez-Nodarse, F. Marcellán, J. Petronilho, *WKB approximation and Krall-type orthogonal polynomials,* Acta Applicandae Mathematicae, 54 (1998), no. 1, 27–58.

[3] A. Aptekarev, S. Denisov, D. Tulyakov, *On a problem by Steklov*, J. Amer. Math. Soc. 29 (2016), 1117–1165.

[4] J. Arvesu, F. Marcellan, R. Alvarez-Nodarse, *On a Modification of the Jacobi Linear Functional: Asymptotic Properties and Zeros of the Corresponding Orthogonal Polynomials*, Acta Applicandae Mathematicae, 71(2002), 127-158.

[5] V.M. Badkov, *The asymptotic behavior of orthogonal polynomials*. (Russian) Mat. Sb. (N.S.) 109(151) (1979), no. 1, 46–59.

[6] J. Breuer, Y. Last, B. Simon, *The Nevai Condition,* Constructive Approximation, 32(2010), 221–254.

[7] T.S. Chihara, *Orthogonal Polynomials and Meausres with End Point Masses*, Rocky Mountain Journal of Math., 15(1985), 705-712.

[8] S. Denisov, K. Rush, *Orthogonal polynomials on the circle for the weight w satisfying conditions* $w, w^{-1} \in BMO$, Constr. Approx. 46 (2017), no. 2, 285–303.

[9] G. Freud, *Orthogonal Polynomials*, Pergamon Press/ Akademiai Kiado, Budapest, 1971.

[10] H. Kiesel and J. Wimp, *A note on Koornwinder's polynomials with weight function* $(1-x)^{\alpha}(1+x)^{\beta}+M\delta(x+1)+N\delta(x-1)$, Numerical Agorithms, 11(1996), 229-241.

[11] T.H. Koornwinder, *Orthogonal Polynomials with Weight Function* $(1-x)^{\alpha}(1+x)^{\beta}+M\delta(x+1)+N\delta(x-1)$, Canad. Math. Bull., 27(1984), 205-214.

[12] D.S. Lubinsky, *Orthonormal Polynomials are Bounded in Subintervals where the Weight is Dini-Lipschitz*, submitted for consideration for publication.

[13] P. Nevai, *Geza Freud, Orthogonal Polynomials and Christoffel Functions: A Case Study*, J. Approx. Theory, 48(1986), 3-167.

[14] E.A. Rakhmanov, *Estimates of the growth of orthogonal polynomials whose weight is bounded away from zero,* (Russian) Mat. Sb. (N.S.) 114(156) (1981), 269–298.

[15] B. Simon, *Szegő's Theorem and its Descendants*, Princeton University Press, Princeton, 2011.

[16] G. Szegő, Orthogonal Polynomials, American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, 1975.