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Fekete Points as Norming Sets

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Abstract

Suppose that $K \subset \mathbb{R}^d$ is compact. Fekete points of degree n are those points $F_n \subset K$ that maximize the determinant of the interpolation matrix for polynomial interpolation of degree n. We discuss some special cases where we can show that Fekete points (of uniformly higher degree) are norming sets for K, i.e., for any $c > 1$, there exists a constant $C > 0$ such that $\|p\|_K \le C\|p\|_{F_{cn}}$, for all polynomials of degree at most n. It is conjectured that this is true for "general" K.

1 Introduction

Suppose that $K \subset \mathbb{R}^d$ is compact. We let $\mathcal{P}_n(K)$ denote the space of polynomials of degree $\le n$, *restricted* to K and $N_n(K) = \dim(\mathcal{P}_n(K))$. Often, when no ambiguity is possible, we will abbreviate, $N_n(K) = N_n$, or even $N_n(K) = N$. Also, in case $m \ge 0$ is *not* an integer, we will let

$$N_m(K) = N_m := N_{\lceil m \rceil}(K).$$

We note that if K is polynomially determining, i.e., $p(x) = 0$ for $\forall x \in K$ implies that $p \equiv 0$, then

$$N_n(K) = \binom{n+d}{d}.$$

Otherwise the dimension may be smaller than this binomial expression. Indeed, for for $K = S^{d-1} \subset \mathbb{R}^d$, the unit sphere $\mathcal{P}_n(K)$ is the space of spherical harmonics of degree at most n and then

$$N_n(K) = \binom{n+d}{d} - \binom{n-2+d}{d}.$$

The corresponding polynomial interpolation problem may be formulated as follows. Given $x_1, x_2, \ldots, x_N$ points in K and values $z_1, z_2, \ldots, z_N \in \mathbb{R}$, find $p \in \mathcal{P}_n(K)$ such that $p(x_i) = z_i$, $i = 1, \ldots, N$. Its solution is accomplished by choosing a basis $\{p_1, p_2, \ldots, p_N\}$ for $\mathcal{P}_n(K)$, writing $p = \sum_{j=1}^N a_j p_j$ and considering the associated linear system

$$\begin{bmatrix} p_1(x_1) & p_2(x_1) & \cdot & \cdot & p_N(x_1) \\ p_1(x_2) & p_2(x_2) & \cdot & \cdot & p_N(x_2) \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ p_1(x_N) & p_2(x_N) & \cdot & \cdot & p_N(x_N) \end{bmatrix} \vec{a} = \vec{z}$$

corresponding to $p(x_i) = z_i$, $1 \le i \le N$.

Hence, the interpolation problem has a unique solution for any set of values z_i iff the associated, so-called vandermonde determinant

$$\mathrm{vdm}(x_1, x_2, \cdots, x_N) := \begin{vmatrix} p_1(x_1) & p_2(x_1) & \cdot & \cdot & p_N(x_1) \\ p_1(x_2) & p_2(x_2) & \cdot & \cdot & p_N(x_2) \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ p_1(x_N) & p_2(x_N) & \cdot & \cdot & p_N(x_N) \end{vmatrix}$$

is non-zero. If this is the case then one may form the so-called fundamental (cardinal) Lagrange polynomials,

$$\ell_i(x) := \frac{\mathrm{vdm}(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_N)}{\mathrm{vdm}(x_1, \ldots, x_N)}, \quad 1 \le i \le N.$$

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These are cardinal in the sense that $\ell_i(x_j) = \delta_{ij}$. Further, the interpolation projection $\pi : C(K) \to \mathcal{P}_n(K)$ is given by

$$\pi(f)(x) = \sum_{i=1}^{n} f(x_i)\ell_i(x)$$

with operator norm

$$\|\pi\| = \max_{x\in K} \sum_{i=1}^{N} |\ell_i(x)|,$$

otherwise known as the Lebesgue constant.

Points $f_1, f_2, \ldots, f_N \in K$ are said to be Fekete points of degree n if they maximize $\mathrm{vdm}(x_1, \ldots, x_N)$ over K^N. Collecting Fekete points for degrees $n = 1, 2, \ldots$ we get a Fekete *array*, $F_1, F_2, \cdots$. We note that they need not be unique!

Fekete points have the basic properties that $\max_{x\in K} |\ell_i(x)| = 1$ and that the Lebesgue constant $\|\pi\| = \max_{x\in K} \sum_{i=1}^{N} |\ell_i(x)| \le N$. Consequently, for $p \in \mathcal{P}_n(K)$,

$$\|p\|_K \le N\|p\|_{F_n}. \tag{1}$$

Here, for $X \subset \mathbb{C}^d$, compact, and $f \in C(X)$,

$$\|f\|_X := \max_{z\in X} |f(z)|.$$

In words, the maximum norm of a polynomial of degree at most n, on *all* of K is at most N times its norm on F_n. A Norming Set is one for which this upper bound factor N may be replaced by a constant. Specifically, an array of subsets $X_n \subset K$, $n = 1, 2, \cdots$ is a *Norming Set* if there exists a constant C such that

$$\|p\|_K \le C\,\|p\|_{X_n},\ \forall p \in \mathcal{P}_n(K),\ n = 1, 2, \cdots.$$

Clearly, $\#(X_n) \ge N$ $(= \dim(\mathcal{P}_n(K)))$ and a Norming Set is said to be optimal if $\#(X_n) = O(N)$.

The first theorem in this regard is that of Ehlich and Zeller [7].

Theorem 1.1 (Ehlich-Zeller 1964)**.** *For any $a > 1$ the Chebyshev points of degree $\lceil an\rceil$ form an optimal Norming Set for $[-1, 1]$.*

The proof is simple, yet informative, based on the fact that the Chebyshev points are well-spaced with respect to the arc-cosine metric and uses an appropriate Markov-Bernstein inequality for the derivatives of polynomials. A rather general result, based on the so-called Dubiner distance is as follows.

Definition 1. Suppose that $K \subset \mathbb{R}^d$ is compact. Then the Dubiner distance between any two points $x, y \in K$ is defined as

$$d_K(x, y) := \sup_{n\ge1,\, p\in\mathcal{P}_n(K),\, \|p\|_K=1} \frac{1}{n} |\cos^{-1}(p(x)) - \cos^{-1}(p(y))|.$$

The Dubiner distance was introduced by Dubiner in [6] and extensively studied in [4] and [5]. In particular for $K = [-1, 1] \subset \mathbb{R}^1$,

$$d_K(x, y) = |\cos^{-1}(x) - \cos^{-1}(y)|$$

is but the arc-cosine metric.

Proposition 1.2. *(see [3] and [10, Prop. 1]) Suppose that $K \subset \mathbb{R}^d$ is compact and that $X_n \subset K$ is a subset with the property that there is some $\alpha < \pi/2$,*

$$\min_{y\in X_n} d_K(x, y) \le \frac{\alpha}{n},\ \forall x \in K.$$

Then, for all $p \in \mathcal{P}_n(K)$,

$$\|p\|_K \le \sec(\alpha)\|p\|_{X_n}.$$

Proof. Suppose that $x \in K$ is such that $|p(x)| = \|p\|_K$, which we may assume without loss to be $\|p\|_K = 1$. We may further assume, by normalizing by -1 if necessary, that $p(x) = 1$. By assumption there exists a point $y \in X_n$ such that $d_K(x, y) \le \alpha/n$. Hence

$$\begin{aligned} \frac{1}{n}\cos^{-1}(p(y)) &= \frac{1}{n}|\cos^{-1}(p(y))| \\ &= \frac{1}{n}|\cos^{-1}(p(x)) - \cos^{-1}(p(y))| \\ &\le d_K(x, y) \\ &\le \frac{\alpha}{n} \end{aligned}$$

from which it follows that

$$\cos^{-1}(p(y)) \le \alpha < \pi/2$$

and, in particular, $p(y) > 0$.

Consequently, as $\cos^{-1}$ is decreasing,

$$p(y) \ge \cos(\alpha)$$

and thus

$$\|p\|_K = 1 \le \frac{1}{\cos(\alpha)} p(y) \le \sec(\alpha)\|p\|_{X_n}. \quad \square$$

Remark. In the Ehlich-Zeller case, X_n is the set of Chebyshev points of degree $m := \lceil an \rceil$ (the zeros of $T_m(x)$). It is elementary to verify that for every $x \in K = [-1,1]$ there is a point $y \in X_n$ such that

$$d_K(x,y) \le \frac{\pi}{2m} \le \frac{\pi}{2a} n$$

i.e., Proposition 1.2 applies with $\alpha := \pi/(2a) < \pi/2$ and the Norming Constant $C = \sec(\pi/(2a))$. $\square$

Proposition 1.2 may also be used to prove an analogous result for the Fekete points for $K = [-1,1]$.

Proposition 1.3. *Suppose that $K = [-1,1]$ and that $a > 3/2$. Then the Fekete points of degree $m := \lceil an \rceil$, F_m, form a Norming Set with norming constant $C = \sec(3\pi/(4a))$.*

Proof. The proof will be a simple consequence of Sündermann's Lemma ([12, Lemma 1]) on the spacing of the Fekete points for the interval.

Lemma 1.4. *(Sündermann) Let $f_k = \cos(\theta_k)$, $1 \le k \le (m+1)$ denote the Fekete points of degree m for the interval $[-1,1]$, in decreasing order. Then*

$$\frac{(j-1)\pi}{m+1/2} \le \theta_j \le \frac{(j-1/2)\pi}{m+1/2}, \; j = 1, \cdots, (m+1).$$

Proof. As the Sündermann paper [12] is not easily accessible, we will reproduce his proof here. First note that for $\omega(x) := \prod_{k=1}^{m+1}(x - f_k)$, we may write the Lagrange polynomials as

$$\ell_k(x) = \frac{\omega(x)}{(x - f_k)\omega'(f_k)}, \; k = 1, \cdots, (m+1).$$

Then, from the facts that $f_1 = +1$, $f_{m+1} = -1$, and at the interior points $\max_{x\in[-1,1]} |\ell_k(x)| = 1$ and hence $\ell_k'(x_k) = 0$, $2 \le k \le m$, it follows easily that

$$(1 - x^2)\omega''(x) + n(n+1)\omega(x) = 0.$$

We note that it then follows that $\omega(x) = c(x^2 - 1)P_{m-1}'(x)$ for some constant c and where $P_m(x)$ is the classical Legendre polynomial of degree $(m-1)$.

For $u(\theta) := (\sin(\theta))^{-1/2}\omega(\cos(\theta))$ we consequently have

$$u''(\theta) + \left((m+1/2)^2 - \frac{3}{4\sin^2(\theta)}\right)u(\theta) = 0.$$

Now compare $u(\theta)$ with a solution of the differential equation

$$v''(\theta) + (m+1/2)^2 v(\theta) = 0.$$

Consider first $2 \le k \le (m-1)$ and the particular solution

$$v(\theta) = \sin((m+1/2)(\theta - \theta_k)).$$

By the Sturm Comparison Theorem (cf. [13, Thm. 1.82.1]) $v(\theta)$ has a zero in the open interval (θ_k, θ_{k+1}). But the zeros of v are just $(\theta - \theta_k) = j\pi/(m+1/2)$, $j = 0, \pm1, \pm2, \cdots$, i.e., for $\theta = \theta_k \pm j\pi/(m+1/2)$, $j = 0, 1, 2, \cdots$. Then $\theta \in (\theta_k, \theta_{k+1})$ implies that $j \ge 1$, i.e.,

$$\theta_k < \theta_k + j\pi/(m+1/2) < \theta_{k+1}$$

for some $j \ge 1$. Consequently

$$\theta_{k+1} - \theta_k > \frac{\pi}{m+1/2}, \; k = 2, \cdots, (m-1). \tag{2}$$

We claim that (2) also holds for $k = 1$ and $k = m$. To see this, note that $f_2 = \cos(\theta_2)$ is the largest zero of $P_m'(x)$. By [13, Thm. 6.21.1] it follows that this is smaller than the largest zero of $T_m'(x)$, i.e., $\theta_2 > \pi/m$. But as $f_1 = +1$, $\theta_1 = 0$, and hence

$$\theta_2 - \theta_1 = \theta_2 > \pi/m > \pi/(m+1/2).$$

The $k = m$ case follows by symmetry.

Summation of the inequalities (2) for $k = 1$ to $k = j-1$ yields $\theta_j \ge (j-1)\pi/(m+1/2)$ and by summation from $k = j$ through m we obtain $\theta_j \le (j-1/2)\pi/(m+1/2)$. $\square$

Continuing with the proof of the Proposition, the Sündermann Lemma implies that

$$\theta_{j+1} - \theta_j \le \frac{(j+1/2)\pi}{m+1/2} - \frac{(j-1)\pi}{m+1/2} = \frac{(3/2)\pi}{m+1/2}, \; j = 1, \cdots, m$$

from which it follows that for all $x \in [-1,1]$ there exists a Fekete point $f_k \in F_m$ of degree m such that

$$d_K(x, f_k) \le \frac{(3/4)\pi}{m+1/2} \le \frac{(3/4)\pi}{an+1/2} \le \frac{\alpha}{n}$$

for $\alpha := 3\pi/(4a) < \pi/2$ for $a > 3/2$. □

Remark. It is likely that the Proposition holds for any $a > 1$, but a proof would require a refinement of the Sündermann Lemma. □

It is also interesting to note that a very simple argument shows that Fekete points for degree $m = \lceil \log(n)n \rceil$, i.e., with a replaced by $\log(n)$, are always a near optimal Norming Set.

Proposition 1.5. *([2]) Suppose that $K \subset \mathbb{R}^d$ is a compact set for which there is an integer $s \le d$ such that $N_n(K) = O(n^s)$ (as is the case for compact subsets of algebraic varieties). Then the Fekete points F_m of degree $m = n\lceil \log(n) \rceil$ and $\#(X_n) = O((n\log(n))^s)$ form a Norming Set for K.*

Proof. First note that for $\deg(p) \le n$, $\deg(p^{\lceil \log(n) \rceil}) \le m$, and hence

$$\begin{aligned} \|p\|_K^{\lceil \log(n) \rceil} &= \|p^{\lceil \log(n) \rceil}\|_K \\ &\le \#(F_m)\|p^{\lceil \log(n) \rceil}\|_{F_m} \\ &= \#(F_m)\|p\|_{F_m}^{\lceil \log(n) \rceil} \end{aligned}$$

and hence

$$\|p\|_K \le (\#(F_m))^{1/\lceil \log(n) \rceil}\|p\|_{F_m}.$$

Now note that

$$(\#(X_n))^{1/\lceil \log(n) \rceil} = O((n\log(n))^{s/\log(n)})$$

where $(n\log(n))^{s/\log(n)} \to e^s$ as $n \to \infty$, and hence is bounded. □

2 The Unit Sphere

Marzo and Ortega-Cerdà [9] have shown, as a special case of a more general result, that Fekete points of degree $\lceil an \rceil$ form a Norming set for polynomials of degree at most n on the unit sphere.

Theorem 2.1 (Marzo and Ortega-Cerdà - 2010 [9]). *For any $a > 1$ the Fekete points of degree $m := \lceil an \rceil$ form a Norming Set for $K = S^{d-1} \subset \mathbb{R}^d$, the unit sphere.*

Proof. We note that in the case of $K = S^{d-1}$, as already shown by Dubiner [6] (cf. [4, 5]), the Dubiner distance is just geodesic distance on the sphere:

$$d_K(x,y) = \cos^{-1}(x \cdot y),\ \ x,y \in S^{d-1}.$$

Now, the key ingredients of their proof are:

1. The discrete equally-weighted measure based on the Fekete points is a bounded proxy for integrals of polynomials squared. Specifically, there is a constant $C > 0$ such that
$$\frac{1}{N_n}\sum_{k=1}^{N_n} P^2(f_k) \le C\int_{S^{d-1}} P^2(x)d\sigma(x),$$
for all $P \in \mathcal{P}_n(K)$, $n = 1, 2, \cdots$, where $d\sigma(x)$ is surface area measure on the sphere, normalized to be a probability measure.
2. For every point $A \in S^{d-1}$ and every degree n, there is a *peaking* polynomial $P_A(x) \in \mathcal{P}_n(K)$ such that $P_A(A) = 1$ and that
$$\int_{S^{d-1}} P_A^2(x)\,d\sigma(x) = O(N^{-1}).$$

Assuming these properties for the time being, their proof goes as follows.

Given $Q \in \mathcal{P}_n(S^{d-1})$, let $A \in S^{d-1}$ be such that

$$|Q(A)| = \|Q\|_{S^{d-1}}.$$

Further, let $P_A(x)$ be the peaking polynomial for $A \in S^{d-1}$ of degree $m := \lceil (a-1)n/2 \rceil$ postulated by Ingredient 2. It is important to note the specific degree of P_A. Then

$$R(x) = R_A(x) := Q(x)P_A^2(x)$$

is a polynomial of degree at most $\lceil an \rceil$ and has the property that $\|Q\|_{S^{d-1}} = |Q(A)| = |R(A)|$.

We let $\{f_1, f_2, \cdots, f_{N_{an}}\}$ denote a set of Fekete points for degree $\lceil an \rceil$ and $\ell_k(x)$ the associated Lagrange polynomials. Then

$$\begin{aligned} \|Q\|_{S^{d-1}} &= |R(A)| \\ &= \left|\sum_{k=1}^{N_{an}} R(f_k)\,\ell_k(A)\right| \\ &= \left|\sum_{k=1}^{N_{an}} Q(f_k)P_A^2(f_k)\,\ell_k(A)\right| \\ &\le \sum_{k=1}^{N_{an}} |Q(f_k)|P_A^2(f_k) \end{aligned}$$

as $\|\ell_k\|_K = 1$ for the Fekete points. Hence,

$$\begin{aligned}
|||Q||_{S^{d-1}} &\le \left\{\max_{1\le k\le N_{an}} |Q(f_k)|\right\} \sum_{k=1}^{N_{an}} P_A^2(f_k) \\
&= \left\{\max_{1\le k\le N_{an}} |Q(f_k)|\right\} N_{an} \left\{\frac{1}{N_{an}} \sum_{k=1}^{N_{an}} P_A^2(f_k)\right\} \\
&\le \left\{\max_{1\le k\le N_{an}} |Q(f_k)|\right\} N_{an} C \int_{S^{d-1}} P_A^2(x) d\sigma(x)
\end{aligned}$$

by Ingredient 1.

Consequently, by the integral property of the peaking polynomial P_A,

$$\begin{aligned}
|||Q||_{S^{d-1}} &\le C \left\{\max_{1\le k\le N_{an}} |Q(f_k)|\right\} \frac{N_{an}}{N_{(a-1)n}} \\
&\le C' \left\{\max_{1\le k\le N_{an}} |Q(f_k)|\right\}
\end{aligned}$$

for some constant C', using the fact that $N_{an}/N_{(a-1)n}$ is bounded. □

For completeness sake we will provide the details of their proofs of the two Ingredients above.

Proposition 2.2. *([8, Cor. 4.6]) There is a constant $C > 0$ such that for $n = 1, 2, \cdots$,*

$$\frac{1}{N_n} \sum_{k=1}^{N_n} P^2(f_k) \le C \int_{S^{d-1}} P^2(x) d\sigma(x),$$

for all $P \in \mathcal{P}_n(K)$, where $d\sigma(x)$ is surface area measure on the sphere, normalized to be a probability measure and $F_n := \{f_1, f_2, \cdots, f_{N_n}\}$ is a set of Fekete points for degree n.

Proof. We first note that Fekete points are well-spaced with respect to the Dubiner distance. Indeed, as shown by Dubiner [6],

$$d_K(f_i, f_j) \ge \frac{\pi}{2n},\ i \ne j. \tag{3}$$

The proof is quite simple – one just notes that

$$\begin{aligned}
d_K(f_i, f_j) &= \sup_{n\ge 1,\, p\in\mathcal{P}_n(K),\, \|p\|_K=1} \frac{1}{n} |\cos^{-1}(p(f_i)) - \cos^{-1}(p(f_j))| \\
&\ge \frac{1}{n} |\cos^{-1}(\ell_i(f_i)) - \cos^{-1}(\ell_i(f_j))| \\
&= \frac{1}{n} |\cos^{-1}(1) - \cos^{-1}(0)| \\
&= \frac{\pi}{2n}.
\end{aligned}$$

We will make use of the following notation:

- For $z \in \mathbb{R}^d$, $B_r(z) := \{x \in \mathbb{R}^d : |x - z| \le r\}$ will denote the *Euclidean* ball of radius r centred at z, and
- For $z \in S^{d-1}$, $\mathbb{B}_r(z) := \{x \in S^{d-1} : d_K(x, z) \le r\}$ will denote the *spherical* cap of radius r centred at z.

We note that

$$\mathrm{vol}_d(B_r(z)) = C_d r^d,\ \text{for some dimensional constant } C_d,\ \text{and} \tag{4}$$

$$\mathrm{vol}_{d-1}(\mathbb{B}_r(z)) \approx C'_{d-1} r^{d-1},\ z \in S^{d-1} \tag{5}$$

where here we mean that $\mathrm{vol}_{d-1}(\mathbb{B}_r(z))/r^{d-1}$ is bounded above and below by (positive) dimensional constants. We note also that $\mathrm{vol}_{d-1}(\mathbb{B}_r(z))$ is the same for any $z \in S^{d-1}$.

We make use of the following simple geometric facts.

Lemma 2.3. *Suppose that $x, y \in K = S^{d-1}$ and that $u \in \mathbb{R}^d$ has Euclidean norm $|u| = r > 0$. Then*

1. $d_K(x, y) \le \dfrac{\pi}{2} |x - y|$,
2. $\left| \dfrac{u}{|u|} - x \right| \le \dfrac{1}{\sqrt{r}} |u - x|$.

Proof. To see 1., note that this is equivalent to

$$\begin{aligned} \theta^2 &\le \frac{\pi^2}{4}2(1-\cos(\theta)),\ \cos(\theta)=x\cdot y\in[0,\pi] \\ \iff \theta^2 &\le \pi^2\sin^2(\theta/2) \\ \iff \sin(\theta/2) &\ge \frac{2}{\pi}\left(\frac{\theta}{2}\right),\ \theta/2\in[0,\pi/2], \end{aligned}$$

a well-known elementary inequality.

To see 2., just note that this is equivalent to

$$\begin{aligned} \left|\frac{u}{|u|}-x\right|^2 &\le \frac{1}{r}|u-x|^2 \\ \iff 2\left(1-\frac{u\cdot x}{|u|}\right) &\le \frac{1}{r}(|u|^2-2(u\cdot x)+1) \\ \iff 2r(1-\cos(\theta)) &\le r^2-2r\cos(\theta)+1,\ \cos(\theta)=(u\cdot x)/|u| \\ \iff 4r\sin^2(\theta/2) &\le (r^2-2r+1)+4r\sin^2(\theta/2) \\ &= (r-1)^2+4r\sin^2(\theta/2).\ \square \end{aligned}$$

Now, from the spacing (3) we may easily conclude that for every $0<c$ there is a constant $C=C(c)>0$ such that for every $0\neq u\in\mathbb{R}^d$ and $n=1,2,\cdots$

$$\#(F_n\cap B_{c/n}(u))\le C. \tag{6}$$

To see this, first note that by 2. of Lemma 2.3,

$$B_{c/n}(u)\cap S^{d-1}\subset B_{c'/n}(u/|u|)\cap S^{d-1}$$

where $c':=c/\sqrt{|u|}$ and that then, by 1.,

$$\left(B_{c/n}(u)\cap S^{d-1}\right)\subset\left(B_{c'/n}(u/|u|)\cap S^{d-1}\right)\subset\mathbb{B}_{c''/n}(u/|u|)$$

where $c'':=(\pi/2)c'$.

Suppose now that there are m distinct Fekete points $f_1,\cdots,f_m\in B_{c/n}(u)$. Necessarily then $f_1,\cdots,f_m\in\mathbb{B}_{c''/n}(x)$ where $x:=u/|u|\in S^{d-1}$.

Choose $a<1$ so that $ac<\pi/2$. Then, we have

$$\mathbb{B}_{ac/n}(f_j)\cap\mathbb{B}_{ac/n}(f_k)=\emptyset,\ j\neq k.$$

Also, there is constant $R_0=R_0(c)$ so that

$$\mathrm{vol}_{d-1}(\mathbb{B}_{c/n}(f_i)\cap\mathbb{B}_{ac/n}(f_j))\ge R_0\mathrm{vol}_{d-1}(\mathbb{B}_{ac/n}(f_j)),\ j=1,\cdots,m.$$

Hence

$$\begin{aligned} \mathrm{vol}_{d-1}(\mathbb{B}_{c/n}(x)) &\ge R_0\mathrm{vol}_{d-1}\left(\cup_{j=1}^m\mathbb{B}_{ac/n}(f_j)\right) \\ &\ge mC_0(ac/n)^{d-1}\ \text{(for some constant } C_0) \end{aligned}$$

and so

$$m\le\mathrm{vol}_{d-1}(\mathbb{B}_{c/n}(x))/(C_0(ac/n)^{d-1})\le C.$$

There is a further technical inequality that we will need. For $0<c<1$ we let

$$T_{c,n}:=\{x\in\mathbb{R}^d\ :\ ||x|-1|\le c/n\}$$

denote the tubular neighbourhood of the unit sphere S^{d-1}, of "radius" c/n. Then, given a polynomial $P\in\mathcal{P}_n(S^{d-1})$ it has a *harmonic* extension to all of $\mathbb{R}^d$. We denote this extension also by P.

Corollary 4.3 of [8] asserts (as a special case of a more general result) that there is a constant C such that

$$\int_{T_{c,n}}P^2(x)dx\le\frac{C}{n}\int_{S^{d-1}}P^2(x)d\sigma(x). \tag{7}$$

Their proof of this relies on the following lemma.

Lemma 2.4. *([8, Lemma 4.2]) For $r>0$ let $S_r^{d-1}\subset\mathbb{R}^d$ denote the sphere of radius r, centred at the origin. Then for $\rho>1$ and $P\in\mathcal{P}_n(S^{d-1})$ and any $|r-1|\le\rho/n$ there exists a constant C, depending only on ρ and d, such that*

$$\int_{S_r^{d-1}}P^2(x)d\sigma(x)\le C\int_{S^{d-1}}P^2(x)d\sigma(x).$$

Proof. Changing variables $x' = rx$, we have

$$\int_{S_r^{d-1}} P^2(x)d\sigma(x) = \int_{S^{d-1}} P^2(rx)d\sigma(x)$$

(as the measures are both normalized to be probability measures).

We claim that in fact, for any $r > 0$,

$$\int_{S^{d-1}} P^2(rx)d\sigma(x) \le \max\{1, r\}^{2\deg(P)} \int_{S^{d-1}} P^2(x)d\sigma(x)$$

from which the result follows easily. To see this, expand

$$P(x) = \sum_{k=0}^{n} a_k h_k(x)$$

where $h_k(x)$ is a harmonic, *homogeneous* polynomial of degree k, as is always possible to do. The $h_k(x)$ are mutually orthogonal and so

$$\begin{aligned}
\int_{S^{d-1}} P^2(rx)d\sigma(x) &= \int_{S^{d-1}} \left\{\sum_{k=0}^{n} a_k h_k(rx)\right\}^2 d\sigma(x) \\
&= \int_{S^{d-1}} \left\{\sum_{k=0}^{n} a_k r^k h_k(x)\right\}^2 d\sigma(x) \\
&= \sum_{k=0}^{n} a_k^2 r^{2k} \left\{\int_{S^{d-1}} h_k^2(x)d\sigma(x)\right\} \\
&\le \max\{1, r\}^{2n} \sum_{k=0}^{n} a_k^2 \left\{\int_{S^{d-1}} h_k^2(x)d\sigma(x)\right\} \\
&= \max\{1, r\}^{2n} \int_{S^{d-1}} P^2(x)d\sigma(x). \ \square
\end{aligned}$$

We now state and prove (7) as a lemma.

Lemma 2.5. *([8, Cor. 4.3]) There is a constant C such that for any harmonic polynomial $P(x)$ of degree at most n,*

$$\int_{T_{c,n}} P^2(x)dx \le \frac{C}{n} \int_{S^{d-1}} P^2(x)d\sigma(x).$$

Proof. First note that there is a dimensional constant C_d such that

$$\int_{T_{c,n}} P^2(x)dx = C_d \int_{r=1-c/n}^{r=1+c/n} \int_{S_r^{d-1}} r^{d-1} P^2(x)d\sigma(x)$$

where again $d\sigma(x)$ is normalized to be a probability measure., and hence by the preceeding Lemma,

$$\begin{aligned}
\int_{T_{c,n}} P^2(x)dx &= C_d \int_{r=1-c/n}^{r=1+c/n} \left\{\int_{S_r^{d-1}} r^{d-1} P^2(x)d\sigma(x)\right\} dr \\
&\le C \int_{r=1-c/n}^{r=1+c/n} \left\{\max\{1, r\}^{2n} \int_{S^{d-1}} P^2(x)d\sigma(x)\right\} dr \\
&\le C \frac{2c}{n} (1 + c/n)^{2n} \int_{S^{d-1}} P^2(x)d\sigma(x) \\
&\le C e^{2c} \frac{2c}{n} \int_{S^{d-1}} P^2(x)d\sigma(x). \ \square
\end{aligned}$$

We continue with the conclusion of the proof of Proposition 2.2. Indeed, by subharmonicity, there is a constant C such that for all harmonic polynomials $P(x)$ of degree at most n and and $z \in S^{d-1}$, we have

$$|P(z)|^2 \le C n^d \int_{\mathbb{B}(z,1/n)} P^2(x)\,dm(x)$$

where, as before, $B(z, 1/n)$ denotes the *Euclidean* ball of radius $1/n$ centred at z and $dm(x)$ denotes Lebesgue measure on $\mathbb{R}^d$. Hence

$$\begin{aligned}
\frac{1}{N_n}\sum_{k=1}^{N_n} P^2(f_k) &\le C\,\frac{1}{N_n}\sum_{k=1}^{N_n}\left\{n^d\int_{B(f_k,1/n)} P^2(x)dm(x)\right\}\\
&\le Cn^d\int_{C_{1,n}} P^2(x)\left\{\frac{1}{N_n}\sum_{k=1}^{N_n}\chi_{B(f_k,1/n)}(x)\right\}dm(x)\\
&= Cn^d\int_{C_{1,n}} P^2(x)\left\{\frac{1}{N_n}\sum_{k=1}^{N_n}\chi_{B(x,1/n)}(f_k)\right\}dm(x)\\
&\le Cn^d\int_{C_{1,n}} P^2(x)\left\{\frac{C}{N_n}\right\}dm(x)\ \text{(by (6))}\\
&\le C\,\frac{n^d}{N_n(K)}\int_{C_{1,n}} P^2(x)dm(x)\\
&\le C\,\frac{n^d}{N_n(K)}\frac{1}{n}\int_{S^{d-1}} P^2(x)d\sigma(x)\ \text{(by Lemma 2.5)}\\
&\le C\int_{S^{d-1}} P^2(x)d\sigma(x)
\end{aligned}$$

as $N_n(K) = O(n^{d-1})$. □

For Ingredient 2, we let for $x, y \in S^{d-1}$, $K_n(x, y)$ denote the reproducing kernel for polynomials of degree at most n with respect to the measure $d\sigma(x)$ on S^{d-1}. As is well known (see e.g. [11, p. 69])

$$K_n(x, x) \equiv N_n,\ x \in S^{d-1}.$$

Then, let

$$P_A(x) := \frac{1}{N_n}K_n(A, x).$$

We have $P_A(A) = N_n/N_n = 1$ and

$$\begin{aligned}
\int P_A(x)^2 d\sigma(x) &= \frac{1}{N_n^2}\int_{S^{d-1}} K_n(A, x)K_n(A, x)d\sigma(x)\\
&= \frac{1}{N_n^2}K_n(A, A) = \frac{1}{N_n}
\end{aligned}$$

as required. □

Concluding Remarks. We emphasize that the results of Marzo and Ortega-Cerdà are for the comparison of general L_p norms of polynomials with the corresponding discrete ℓ_p norms based on Fekete points. We have extracted the essentials of their proofs necessary for the L_∞ case, in which we are primarily concerned.

We conjecture that Fekete points of degree $\lceil an\rceil$, $a > 1$, are norming sets for general "sufficiently regular" compact sets $K \subset \mathbb{R}^d$. Indeed it would be sufficient to show that K has the analogous properties of Ingredients 1 and 2 above. The cases of K a ball or simplex will be discussed in a forthcoming paper.

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