Zeros of lacunary random polynomials

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Dedicated to Norm Levenberg on his 60th birthday

Abstract

We study the asymptotic distribution of zeros for the lacunary random polynomials. It is shown that the equidistribution of zeros near the unit circumference holds under more relaxed conditions on the random coefficients than in the case of Kac polynomials. Moreover, the zeros exhibit stronger convergence patterns towards the unit circumference.

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1 Asymptotic distribution of zeros

The distribution of zeros of the polynomials $P_n(z) = \sum_{k=0}^n A_k z^k$ with random coefficients is now a classical subject, see, e.g., Bharucha-Reid and Sambandham [4]. It is well known that the bulk of zeros for such polynomials are equidistributed near the unit circumference under mild conditions on the probability distribution of the coefficients. Let $\{Z_k\}_{k=1}^n$ be the zeros of a polynomial P_n of degree n, and define the *zero counting measure*

$$\tau_n = \frac{1}{n} \sum_{k=1}^n \delta_{Z_k}.$$

The fact of equidistribution for the zeros of random polynomials is expressed via the weak convergence of ν_n to the normalized arclength measure $\mu_{\mathbb{T}}$ on the unit circumference $\mathbb{T}$, where $d\mu_{\mathbb{T}}(e^{it}) := dt/(2\pi)$. Namely, we have that $\tau_n \xrightarrow{w} \mu_{\mathbb{T}}$ with probability 1 (abbreviated as a.s. or almost surely). Ibragimov and Zaporozhets [8] proved that if the coefficients are independent and identically distributed non-trivial random variables, then the condition $\mathbb{E}[\log^+ |A_0|] < \infty$ is necessary and sufficient for $\tau_n \xrightarrow{w} \mu_{\mathbb{T}}$ almost surely. Here, $\mathbb{E}[X]$ denotes the expectation of a random variable X, and X is called non-trivial if $\mathbb{P}(X = 0) < 1$. Further related results and history are found in the papers [9, 10], [11, 12], [13], [3], etc.

Let $\{r_k\}_{k=0}^\infty \subset \mathbb{N} \cup \{0\}$ be a strictly increasing deterministic sequence. We are interested in the asymptotic distribution for the zeros of lacunary polynomials

$$P_n(z) = \sum_{k=0}^n c_k z^k,$$

where $c_k = 0$ with probability one for all $k \notin \{r_j\}_{j=0}^\infty$, while the subsequence $c_{r_j} = A_j, j = 0, 1, 2, \dots,$ is given by non-trivial complex random variables $\{A_j\}_{j=0}^\infty$ that are often assumed to be i.i.d. In the case of bounded gaps, the equidistribution results remain essentially the same as for Kac polynomials, see [3] and references therein. But if the gaps grow sufficiently fast, then lacunary structure of P_n may prevent convergence $\tau_n \xrightarrow{w} \mu_{\mathbb{T}}$ along the whole sequence $n \in \mathbb{N}$, regardless of what is assumed about the non-trivial sequence of coefficients. For example, if $r_j = 2^j,\ j \in \mathbb{N}$, then $P_n(z) = P_{2^j}$ for all $n = 2^j, \dots, 2^{j+1} - 1$, so that

$$\tau_{2^{j+1}-1}(\mathbb{C}) \le \frac{2^j}{2^{j+1}-1} \to \frac{1}{2} \quad \text{as } j \to \infty,$$

with probability one. On the other hand, the only really interesting subsequence of these lacunary polynomials is given by

$$L_n(z) = \sum_{j=0}^{n} A_j z^{r_j}, \quad n \in \mathbb{N}. \tag{1.1}$$

We provide several results on the equidistribution of zeros for this sequence. Specifically, the zero counting measures for L_n satisfy

$$\nu_n := \frac{1}{r_n} \sum_{L_n(Z)=0} \delta_Z \xrightarrow{w} \mu_{\mathbb{T}} \quad \text{as } n \to \infty,$$

with probability one, under certain weak assumptions on $\{A_j\}_{j=0}^\infty$. In fact, these assumptions typically become more relaxed with faster growth of the sequence $\{r_k\}_{k=0}^\infty$. We mention that convergence questions on lacunary random polynomials were previously studied in [2].

Our first theorem is an analogue of the Ibragimov and Zaporozhets result [8] in the case of subexponential gaps.

Theorem 1.1. *Assume $\{A_n\}_{n=0}^\infty$ are non-trivial i.i.d. complex random variables such that $\mathbb{E}[\log^+ |A_n|] < \infty$. If for any $b > 0$ and $m(n) = [n - b \log n]$ we have*

$$\lim_{n\to\infty} \frac{r_{m(n)}}{r_n} = 1, \tag{1.2}$$

then $\nu_n \xrightarrow{w} \mu_{\mathbb{T}}$ almost surely.

If we have more detailed information about $\{r_n\}_{n=0}^\infty$, then we can relax the condition on the random coefficients as follows.

Theorem 1.2. *Let $a > 0$ and $p \ge 1$. Suppose $\{A_n\}_{n=0}^\infty$ are non-trivial i.i.d. complex random variables satisfying $\mathbb{E}[(\log^+ |A_n|)^{1/p}] < \infty$. If*

$$\lim_{n\to\infty} \frac{r_n}{a n^p} = 1,$$

then $\nu_n \xrightarrow{w} \mu_{\mathbb{T}}$ almost surely.

In the case of Hadamard gaps, i.e., when the sequence $\{r_n\}_{n=0}^{\infty}$ grows exponentially, one needs somewhat different assumptions on random coefficients. Non-triviality of random variables is no longer helpful, as seen from the following example. Let $r_n = 2^n$ and let A_n be i.i.d. Bernoulli random variables that take values 0 and 1 with equal probabilities. The second Borel-Cantelli Lemma implies that $A_n = 0$ infinitely often with probability one, cf. [7, p. 99]. Hence there is a subsequence such that

$$\nu_n(\mathbb{C}) \le \frac{2^{n-1}}{2^n} = \frac{1}{2},$$

with probability one. It is clear that $\nu_n \xrightarrow{w} \mu_{\mathbb{T}}$ as $n \to \infty$ cannot hold almost surely through the whole sequence $n \in \mathbb{N}$.

Note that we drop the independence assumption for the random coefficients below.

Theorem 1.3. *Assume that*

$$\liminf_{n\to\infty} r_n^{1/n} > 1.$$

If $\{A_n\}_{n=0}^{\infty} \subset \mathbb{C}$ is a sequence of identically distributed random variables satisfying $\mathbb{E}[\log^+ |\log |A_n||] < \infty$, then $\nu_n \xrightarrow{w} \mu_{\mathbb{T}}$ almost surely.

Theorem 1.3 may be generalized as follows to deal with faster growth of $\{r_n\}_{n=0}^{\infty}$.

Theorem 1.4. *Suppose that $\log r_n \ge h(n)$ for all $n \ge N \in \mathbb{N}$, where $h(x)$ is a strictly increasing function on $[N, \infty)$. Assume further that the inverse h^{-1} satisfies*

$$h^{-1}(x+y) \le c(h^{-1}(x) + h^{-1}(y))$$

for a fixed $c > 0$ and all $x, y \in [h(N), \infty)$. If $\{A_n\}_{n=0}^{\infty} \subset \mathbb{C}$ are identically distributed random variables such that $\mathbb{E}\left[h^{-1}\left(\log^+ |\log |A_n||\right)\right] < \infty$, then $\nu_n \xrightarrow{w} \mu_{\mathbb{T}}$ almost surely.

It is clear that iterated exponential growth of $\{r_n\}_{n=0}^{\infty}$ leads to the iterated logarithmic integral conditions on random coefficients, often considered in probability theory.

In a different direction, we state the following companion for Theorem 1.2.

Theorem 1.5. *Let $a > 0$ and $p \ge 1$. Suppose $\{A_n\}_{n=0}^{\infty}$ are identically distributed complex random variables satisfying $\mathbb{E}[|\log |A_n||^{1/p}] < \infty$. If $r_n \ge an^p$ for all large $n \in \mathbb{N}$, then $\nu_n \xrightarrow{w} \mu_{\mathbb{T}}$ almost surely.*

2 Discrepancy estimates and asymptotics

We considered a series of results on almost sure convergence $\nu_n \xrightarrow{w} \mu_{\mathbb{T}}$ that hold under various weak log-integrability conditions on random coefficients. Slightly more restrictive assumptions of finite fractional moments for A_n already suffice to study the deviation of ν_n from $\mu_{\mathbb{T}}$ in terms of discrepancy estimates that quantify the weak convergence of the zero counting measures of random polynomials to the normalized arclength on the unit circle.

We assume that the complex valued random variables $A_n,\ n = 0, 1, 2, \dots,$ are identically distributed (for the sake of convenience), but they may not necessarily be independent. It is standard to consider the discrepancy of ν_n and $\mu_{\mathbb{T}}$ in the annular sectors of the form

$$S_r(\alpha, \beta) = \{z \in \mathbb{C} : r < |z| < 1/r,\ \alpha \le \arg z < \beta\}, \quad 0 < r < 1.$$

We first state an immediate consequence of a known result from [12].

Theorem 2.1. *Suppose that $\{A_n\}_{n=0}^{\infty}$ are identically distributed complex random variables that satisfy $\mathbb{E}[|A_n|^t] < \infty$ for a fixed $t \in (0, 1]$, and $\mathbb{E}[\log |A_n|] > -\infty$. Then there is $C > 0$ such that for all large $n \in \mathbb{N}$ we have*

$$\mathbb{E}\left[\left|\nu_n(S_r(\alpha, \beta)) - \frac{\beta - \alpha}{2\pi}\right|\right] \le C\sqrt{\frac{\log r_n}{r_n}}. \tag{2.1}$$

Observe that the number of zeros of L_n in any set $S \subset \mathbb{C}$ denoted by $N_n(S)$ is equal to $n\nu_n(S)$, and the estimates for $\mathbb{E}[N_n(S)]$ readily follow from Theorem 2.1.

We now state an almost sure convergence result for discrepancy, restricting ourselves to the case of Hadamard gaps for simplicity.

Theorem 2.2. *Suppose that $\{A_n\}_{n=0}^{\infty}$ are identically distributed complex random variables such that $\mathbb{E}[|A_n|^t] < \infty$ for a fixed $t \in (0, 1]$, and $\mathbb{E}[\log |A_n|] > -\infty$. If*

$$\liminf_{n\to\infty} r_n^{1/n} = q > 1$$

then

$$\limsup_{n\to\infty} \left|\nu_n(S_r(\alpha, \beta)) - \frac{\beta - \alpha}{2\pi}\right|^{1/n} \le \frac{1}{\sqrt{q}} \quad a.s. \tag{2.2}$$

3 Proofs

We need several facts about limiting behavior of random coefficients. The first result is well known, and can be found in many papers, see [1], [8], [11], etc. It follows from the Borel-Cantelli Lemmas (see, e.g., [7, p. 96]) in a standard way.

Lemma 3.1. *If $\{A_n\}_{n=0}^{\infty}$ are non-trivial i.i.d. complex random variables that satisfy $\mathbb{E}[\log^+ |A_n|] < \infty$, then*

$$\limsup_{n\to\infty} |A_n|^{1/n} = 1 \qquad a.s. \tag{3.1}$$

and

$$\limsup_{n\to\infty} \left(\max_{0\le k\le n} |A_k|\right)^{1/n} = 1 \qquad a.s. \tag{3.2}$$

The next lemma shows that i.i.d. coefficients cannot be "too small too often." The exact form given below appeared in [13], but its roots are found in [5].

Lemma 3.2. *If* $\{A_k\}_{k=0}^\infty$ *are non-trivial i.i.d. complex random variables, then there is* $b > 0$ *such that*

$$\liminf_{n\to\infty} \left(\max_{n-b\log n<k\le n} |A_k| \right)^{1/n} \ge 1 \qquad a.s. \tag{3.3}$$

All of our results on the equidistribution of zeros are obtained by using the following deterministic proposition, which is a consequence of a result by Grothmann [6].

Proposition 3.3. *Let* $\{A_n\}_{n=0}^\infty \subset \mathbb{C}$ *and* $\{r_n\}_{n=0}^\infty \subset \mathbb{N}\cup\{0\}$ *be deterministic sequences, the latter one being strictly increasing. Suppose that*

$$\limsup_{n\to\infty} \left(\max_{0\le k\le n} |A_k| \right)^{1/r_n} \le 1. \tag{3.4}$$

If there is a sequence $m(n) \in \mathbb{N}$, *with* $m(n) \le n$, *such that*

$$\lim_{n\to\infty} \frac{r_{m(n)}}{r_n} = 1 \tag{3.5}$$

and

$$\liminf_{n\to\infty} \left(\max_{m(n)\le k\le n} |A_k| \right)^{1/r_n} \ge 1, \tag{3.6}$$

then the zero counting measures of L_n *defined by* (1.1) *satisfy* $\nu_n \stackrel{w}{\to} \mu_{\mathbb{T}}$ *as* $n \to \infty$.

Proof of Proposition 3.3. The proof is constructed as an application of Theorem 1 from [6]. Estimating

$$\max_{|z|=R} |L_n(z)| \le \sum_{k=0}^n |A_k| R^{r_k}, \quad R > 0, \tag{3.7}$$

we observe that (3.4) guarantees uniform convergence of L_n on compact sets $E \subset \mathbb{D}$ to an analytic function f in the unit disk. Since not all coefficients A_n are zero by (3.6), it follows that $f(z) = \sum_{n=0}^\infty A_n z^{r_n} \not\equiv 0$ in $\mathbb{D}$. Hurwitz's theorem now implies that

$$\lim_{n\to\infty} \nu_n(E) = 0 \tag{3.8}$$

for any compact set $E \subset \mathbb{D}$. Moreover, (3.7) and (3.4) give that

$$\limsup_{n\to\infty} \left(\max_{z\in\mathbb{T}} |L_n(z)| \right)^{1/r_n} \le \limsup_{n\to\infty} \left((n+1) \max_{0\le k\le n} |A_k| \right)^{1/r_n} \le 1. \tag{3.9}$$

Consider $R > 1$ and write

$$A_k = \frac{1}{2\pi i} \int_{|z|=R} \frac{L_n(z)\,dz}{z^{r_k+1}}, \quad k = 0, \dots, n.$$

It follows that

$$|A_k| \le \frac{\max\limits_{|z|=R} |L_n(z)|}{R^{r_k}}$$

and

$$\max_{|z|=R} |L_n(z)| \ge \max_{m(n)\le k\le n} (|A_k| R^{r_k}) \ge R^{r_{m(n)}} \max_{m(n)\le k\le n} |A_k|.$$

Applying (3.5) and (3.6) to the above estimate, we obtain that

$$\liminf_{n\to\infty} \left(\max_{|z|=R} |L_n(z)| \right)^{1/r_n} \ge \liminf_{n\to\infty} R^{r_{m(n)}/r_n} \liminf_{n\to\infty} \left(\max_{m(n)\le k\le n} |A_k| \right)^{1/r_n} \ge R.$$

The latter inequality combined with (3.8) and (3.9) show that all assumptions of Theorem 1 [6] are satisfied, and the desired convergence $\nu_n \xrightarrow{w} \mu_{\mathbb{T}}$ as $n \to \infty$ holds. $\square$

Proof of Theorem 1.1. Apply Proposition 3.3 with $m(n) = [n - b \log n]$, and use Lemmas 3.1 and 3.2. One only needs to note that (3.2) implies (3.4) because $r_n \ge n$, and that (3.3) implies (3.6) with probability one. $\square$

Proof of Theorem 1.2. We again use Proposition 3.3 with $m(n) = [n - b \log n]$, and Lemma 3.2. Equation (3.5) is verified immediately, and (3.3) implies (3.6) almost surely as before. It remains to show that (3.4) holds almost surely under current assumptions. For any fixed $\varepsilon > 0$, define events $\mathcal{E}_n = \{|A_n| \ge e^{\varepsilon r_n}\},\ n \in \mathbb{N}$. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathbb{P}(\mathcal{E}_n) &= \sum_{n=1}^{\infty} \mathbb{P}(\{\log^+ |A_n| \ge \varepsilon r_n\}) = \sum_{n=1}^{\infty} \mathbb{P}\left(\left\{ \left(\frac{1}{\varepsilon} \log^+ |A_0| \right)^{1/p} \ge r_n^{1/p} \right\} \right) \\
&\le C + \sum_{n=1}^{\infty} \mathbb{P}\left(\left\{ \left(\frac{2}{a\varepsilon} \log^+ |A_0| \right)^{1/p} \ge n \right\} \right) \\
&\le C + \left(\frac{2}{a\varepsilon} \right)^{1/p} \mathbb{E}\left[(\log^+ |A_0|)^{1/p} \right] < \infty,
\end{aligned}$$

where $C > 0$ is a constant. Hence $\mathbb{P}(\mathcal{E}_n$ occurs infinitely often$) = 0$ by the first Borel-Cantelli Lemma, so that the complementary event $\mathcal{E}_n^c$ must happen for all large n with probability one. This means $|A_n|^{1/r_n} \le e^{\varepsilon}$ for all sufficiently large $n \in \mathbb{N}$ almost surely, i.e., there is $N \in \mathbb{N}$ such that

$$\left(\max_{0\le k\le n} |A_k| \right)^{1/r_n} \le \max\left(\max_{0\le k\le N} |A_k|^{1/r_n}, e^{\varepsilon} \right) \quad \text{a.s.}$$

Letting $r_n \geq n \to \infty$, we obtain that

$$\limsup_{n\to\infty} \left(\max_{0\leq k\leq n} |A_k| \right)^{1/r_n} \leq e^{\varepsilon} \quad \text{a.s.}$$

Since $\varepsilon > 0$ is arbitrary, (3.4) now follows with probability one. $\square$

We prove the more general result of Theorem 1.4, and them deduce Theorem 1.3.

Proof of Theorem 1.4. In this case, we apply Proposition 3.3 with $m(n) = n$, so that (3.5) holds trivially. In order to verify that (3.4) and (3.6) hold almost surely, we first prove that for any sufficiently small $\varepsilon > 0$ the probability $\mathbb{P}\left(\left|\log |A_n|^{1/r_n}\right| \geq \varepsilon \ \text{ i. o.}\right) = 0$ by the first Borel-Cantelli Lemma. We estimate the corresponding series as follows:

$$\begin{aligned}
\sum_{n=N}^{\infty} \mathbb{P}(\{\left|\log |A_n|^{1/r_n}\right| \geq \varepsilon\}) &= \sum_{n=N}^{\infty} \mathbb{P}\left(\{\log|\log|A_0|| - \log\varepsilon \geq \log r_n\}\right) \\
&\leq \sum_{n=N}^{\infty} \mathbb{P}\left(\left\{\log^+|\log|A_0|| - \log\varepsilon \geq h(n)\right\}\right) \\
&= \sum_{n=N}^{\infty} \mathbb{P}\left(\left\{h^{-1}\left(\log^+|\log|A_0|| + \log\frac{1}{\varepsilon}\right) \geq n\right\}\right) \\
&\leq \sum_{n=N}^{\infty} \mathbb{P}\left(\left\{ch^{-1}\left(\log^+|\log|A_0||\right) + ch^{-1}\left(\log\frac{1}{\varepsilon}\right) \geq n\right\}\right) \\
&\leq \mathbb{E}\left[ch^{-1}\left(\log^+|\log|A_0||\right) + ch^{-1}\left(\log\frac{1}{\varepsilon}\right)\right] \\
&= c\,\mathbb{E}\left[h^{-1}\left(\log^+|\log|A_0||\right)\right] + ch^{-1}\left(\log\frac{1}{\varepsilon}\right) < \infty.
\end{aligned}$$

Consequently, for any $\varepsilon > 0$ the inequality $\left|\log |A_n|^{1/r_n}\right| < \varepsilon$ holds for all sufficiently large $n \in \mathbb{N}$ with probability one, and we obtain that

$$\lim_{n\to\infty} \log |A_n|^{1/r_n} = 0 \quad \text{a.s.}$$

Thus (3.6) holds almost surely with $m(n) = n$. Moreover, arguing as in the end of proof of Theorem 1.2, we obtain (3.4) with probability one. $\square$

Proof of Theorem 1.3. It is clear that the assumption on r_n is equivalent to $\log r_n \geq an$ for a fixed $a > 0$ and all large $n \in \mathbb{N}$. Hence we apply Theorem 1.4 with $h(x) = ax$ and complete the proof. $\square$

Proof of Theorem 1.5. Once again, Proposition 3.3 is used with $m(n) = n$, and (3.5) is clearly satisfied. Given any $\varepsilon > 0$ we prove that $\mathbb{P}\left(\left|\log |A_n|^{1/r_n}\right| \geq \varepsilon \ \text{ i. o.}\right) = 0$ by the first

Borel-Cantelli Lemma. Since $r_n \geq an^p$ for all $n \geq N \in \mathbb{N}$, we estimate

$$\begin{aligned}
\sum_{n=N}^{\infty} \mathbb{P}(\{\left|\log |A_n|^{1/r_n}\right| \geq \varepsilon\}) &= \sum_{n=N}^{\infty} \mathbb{P}\left(\{|\log |A_0|| \geq \varepsilon r_n\}\right) \\
&\leq \sum_{n=N}^{\infty} \mathbb{P}\left(\{|\log |A_0|| \geq \varepsilon a n^p\}\right) \\
&= \sum_{n=N}^{\infty} \mathbb{P}\left(\left\{\left(\frac{1}{a\varepsilon}|\log |A_0||\right)^{1/p} \geq n\right\}\right) \\
&\leq \left(\frac{1}{a\varepsilon}\right)^{1/p} \mathbb{E}\left[|\log |A_0||^{1/p}\right] < \infty,
\end{aligned}$$

It follows that the inequality $\left|\log |A_n|^{1/r_n}\right| < \varepsilon$ holds for all sufficiently large $n \in \mathbb{N}$ with probability one. Therefore, (3.6) holds almost surely with $m(n) = n$, because

$$\lim_{n\to\infty} \log |A_n|^{1/r_n} = 0 \quad \text{a.s.}$$

Another conclusion is that (3.4) holds with probability one, which comes via the same argument as in the end of proof of Theorem 1.2. $\square$

Proof of Theorem 2.1. Since the random coefficients A_n are identically distributed, the result follows immediately from Corollary 3.2 of [12]. $\square$

Proof of Theorem 2.2. We set for brevity

$$D_n := \left|\nu_n(S_r(\alpha,\beta)) - \frac{\beta-\alpha}{2\pi}\right|.$$

Consider any small $\varepsilon > 0$. Chebyshev's inequality and (2.1) give

$$\mathbb{P}\left(\left\{D_n \geq q^{(\varepsilon-1)n/2}\right\}\right) \leq q^{(1-\varepsilon)n/2}\mathbb{E}[D_n] \leq Cq^{-\varepsilon n/4}$$

for all $n \geq N \in \mathbb{N}$, where N is sufficiently large. Hence

$$\sum_{n=N}^{\infty} \mathbb{P}\left(\left\{D_n \geq q^{(\varepsilon-1)n/2}\right\}\right) < \infty,$$

and the first Borel-Cantelli Lemma implies $\mathbb{P}\left(\left\{D_n \geq q^{(\varepsilon-1)n/2} \text{ i.o.}\right\}\right) = 0$. This is equivalent to

$$D_n < q^{(\varepsilon-1)n/2} \quad \text{a.s.}$$

for all sufficiently large $n \in \mathbb{N}$. Finally, we obtain that

$$\limsup_{n\to\infty} D_n^{1/n} \leq q^{(\varepsilon-1)/2} \quad \text{a.s.},$$

and (2.2) follows as $\varepsilon > 0$ can be made arbitrarily small. $\square$

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