Optimal Polynomial Admissible Meshes on the Closure of $\mathscr{C}^{1,1}$ Bounded Domains

Workshop on Multivariate Approximation
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Outline

1. Introducing Polynomial Admissible Meshes
   - Defining **(Weakly) Admissible Meshes**, **Optimal Admissible Meshes**.
   - Main properties and motivations.
   - Relation with the Equilibrium Measure.
   - Building Admissible Meshes, the *state of the art*.
2. A new result for $\mathscr{C}^{1,1}$ bounded domains
   - Main Result.
   - Tools: **Bernstein Inequality** and regularity property of **oriented distance function**
   - Sketch of the proof.

First definitions (1)

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(Weakly) Admissible Meshes, (W)AM. [Calvi-Levenberg]

Let $K \subset \mathbb{R}^d$ (or $\mathbb{C}^d$) be a compact polynomial determining set. The sequence $\{A_n\}_{\mathbb{N}}$ of finite subsets of K is said to be an **Admissible Mesh** for K if there exist $C, s > 0$ such that

$$\mathrm{Card}\, A_n = \mathcal{O}(n^s)$$

$$\|p\|_K \leq C\|p\|_{A_n} \; \forall p \in \mathscr{P}^n(K)$$

If instead

$$\|p\|_K \leq C_n\|p\|_{A_n} \; \forall p \in \mathscr{P}^n(K)$$

$$limsup_n(C_n \,\mathrm{Card}\, A_n)^{1/n} = 1$$

then we say that A_n is a **Weakly Admissible Mesh**.

First definitions (1)

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A slightly enforced definition that is being used in the literature

Weakly Admissible Meshes, WAM

Let $K \subset \mathbb{R}^d$ (or $\mathbb{C}^d$) be a compact polynomial determining set. The sequence $\{A_n\}_{\mathbb{N}}$ of finite subsets of K is said to be an **Admissible Mesh** for K if there exist $s, q > 0$ such that

$$
\begin{aligned}
\text{Card}\, A_n &= \mathcal{O}(n^s) \\
C_n &= \mathcal{O}(n^q) \\
\|p\|_K &\leq C_n \|p\|_{A_n} \; \forall p \in \mathscr{P}^n(K).
\end{aligned}
$$

First definitions (2)

By the definitions both AMs and WAMs are determining for $\mathscr{P}^n(K)$, thus we have

$$\text{Card } A_n \geq \dim \mathscr{P}^n(K) = \binom{n+d}{n} = \mathcal{O}(n^d)$$

For this reason A. Kroó introduced

Optimal Admissible Mesh

The AM A_n w.r.t. $K \subset \mathbb{R}^d$ is said to be **optimal** if

$$\boxed{\text{Card } A_n = \mathcal{O}(n^d).}$$

DLS Approximation on a WAM - Calvi Levenberg (2008)

Let $K \subset \mathbb{R}^d$ (or $\mathbb{C}^d$) be compact and polynomial determining, A_n a WAM on it and $f \in \mathscr{C}^0(K)$, then one has

$$||f - \Lambda_{A_n} f||_K \leq \left(1 + C_n \left(||f||_K (1 + \sqrt{\mathrm{Card}(A_n)})\right)\right) d_n(f, K)$$

Where Λ_{A_n} is the discrete least squares (DLS) operator performed sampling f on A_n and $d_n(f, K)$ is the error of best polynomial approximation to f on K.

$\Downarrow$

Mild regularity of f and K $\Longrightarrow$ convergence of DLS operator.

Properties (1)

WAMs work nicely under some fundamental operations.

- Stability under affine mappings, unions and tensor products.
- “Weak” stability under polynomial mappings.
- Supersets of WAMs are WAMs.
- Good interpolation sets are WAMs.

Properties (2)

Discrete Extremal Sets - Bos, De Marchi, Sommariva and Vianello

Starting from a WAM one can extract by standard Numerical Linear Algebra

- AFP Approximate Fekete Points
- ALS Approximate Leja Sequences

such that

- Unisolvent sets.
- Slowly increasing Lebesgue constants.
- Same asymptotic (in measure theoretic sense) of true Fekete.

Properties (3)

$$A_n \dashrightarrow \mu_n := \frac{1}{\text{Card } A_n} \sum_{i=1}^{\text{Card } A_n} \delta_{x_i}$$

Asimptotically Bernstein Markov sequence of measures

If there exists $\{M_n\}_{\mathbb{N}}$ such that for any $p \in \mathscr{P}_n(\mathbb{C}^d)$ we have

$$\begin{aligned} \|p\|_K &\leq M_n \|p\|_{L^2_{\mu_n}} \\ \limsup_n (M_n)^{\frac{1}{n}} &= 1. \end{aligned}$$

Then $\{\mu_n\}_{\mathbb{N}}$ is an **as. BM** sequence of measures for K.

For measure arising from meshes we can choose $M_n := C_n \cdot \text{Card } A_n$.

A new motivation

Asimptotically B-M sequences of measures can be thought as a partial generalization of **Optimal Measures** introduced by Bloom Bos Levenberg and Waldron [1].

Proposition

Let μ_n be a as. BM sequence for the compact non-pluripolar set K, then

$$\limsup_n \left(G_n^{\mu_n}\right)^{\frac{1}{\alpha_n}} = \delta(K). \tag{1}$$

Where G_n^ν is the Gram matrix of $\left(\mathscr{P}_n^d, \langle \cdot, \cdot \rangle_{L^2_\nu}\right)$, $\alpha_n := \frac{d+1}{dNn}$ and

$$\boxed{\delta(K) := \lim_n \max_{\zeta \in K^N} |\text{VDM}(\zeta)|^{\frac{1}{\alpha_n}}}$$

is the **transfinite diameter** of K.

A new motivation

Let $\left\{q_j^{(n)}\right\}_{j=1,2,\dots,N}$ be an o.n.b. of $\mathscr{P}_n \cap L^2_{\mu_n}$, the **Bergman function** is

$$B_n^{\mu_n}(z) := \sum_{j=1}^N \left|q_j^{(\mu_n)}(z)\right|^2.$$

Thanks to (1) **Strong Bergman Asymptotic** applies.

Theorem.

Let A_n be a WAM for the compact non-pluripolar set $K \subset \mathbb{C}^d$ and μ_n as above then we have

$$\frac{B_n^{\mu_n}}{N}\mu_n \rightharpoonup^* \mu_K,$$

the pluripotential equilibrium measure of K.

New motivation for optimality

If A_n is an **optimal admissible mesh**, then $\frac{B_n^{\mu_n}}{N}$ is a bounded sequence of function, thus we can prove

Theorem [P.]

In the above hypothesis suppose that $\mu_n \rightharpoonup^* \mu$, then

$$D_\mu^-(\mu_K) \geq \liminf_n \frac{B_n^{\mu_n}}{N}.$$

Where D_μ^- is the lower Lebesgue Radon Nikodym derivative.

Unfortunately to prove

$$D_\mu(\mu_K) = \liminf_n \frac{B_n^{\mu_n}}{N}$$

we need further assumptions... (e.g. being $\frac{B_n^{\mu_n}}{N}$ decreasing.)

Problems

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Two questions naturally arise..

Question 1

How to build AMs or even WAMs for a given K?

Question 2

How to build Optimal AMs for a given K?

Solving (Q1)

One should choose/combine different results requiring K to have

particular shape and/or **smoothness**

- **Ehlich-Zeller:** Double degree Chebyshev points for the interval are AM of constant 2.
- **shape:** Use symmetries of K, polar coordinates, tensors, quadratic maps..
- **Calvi-Levenberg** used Multivariate Markov Inequality.
- **Kroó:** Star shaped bounded domains with smooth Minkowski functional + Bernstein Inequality .
- **P. and Vianello** Mapping and Perturbing WAMs and AMs.
- **Plesniak** improve for sub-analytic sets.
- **Kroó** result on Analytic Graph domains.
- **Bloom Bos Calvi and Levenberg** existence for L-regular sets.

Building AM by Markov Inequality

Markov Inequality (MI)

The set K is preserving a Markov Inequality of constant M_K and exponent r if

$$\|\,|\nabla p|\,\|_K \leq M_K n^r \|p\|_K. \ \forall p \in \mathscr{P}^n(\mathbb{R}^d)$$

MI holds under mild assumptions on K, typically $r = 2$.

Idea: take any equally spaced grid having step size $\mathcal{O}(n^{-r})$.

Calvi Levenberg (2008)

If $K \subset \mathbb{R}^d$ preserves a Markov Inequality of exponent r, then it has a AM with $\mathcal{O}(n^{rd})$ points.

Building AM by Bernstein Inequality and Star Shape

Using the classical Bernstein Inequality on segments and star-shaped property it has been proved

Kroó (2011)

Let $K \subset \mathbb{R}^d$ be compact and star-shaped with $\mathscr{C}^{1+\alpha}$ smooth Minkowski functional. Then K has an AM with $\mathcal{O}(n^{\frac{2d+\alpha-1}{\alpha+1}})$ points.

Mapping and Perturbing

Perturbation Result P. and Vianello

Let $K \subset \mathbb{C}^d$ be a polynomially convex and Markov compact set. If that there exists a sequence of compact sets $\{K_j\}_{\mathbb{N}}$ such that

- there exists $A_{n,j}$ (W)AM for K_j having constant $C_{n,j}$ and
- $d_{\mathscr{H}}(K, K_j) \leq \epsilon_j$ where $\limsup_j \epsilon_j C_{n,j} = 0 \; \forall n$.

Then K has a (W)AM.

Mapping Result P. and Vianello

(W)AMs are **"weakly" stable under smooth mapping**: for any holomorphic map $\varphi : Q \to K$ there exists $j_\varphi(n) = \mathcal{O}(\log n)$ such that $B_n := \varphi(A_{n \cdot j_\varphi(n)})$ is a WAM for K.

Nearly Optimal AM

AM having Card $A_n = \mathcal{O}((n \log n)^d)$ as the ones above are termed **nearly optimal.**

- W. Plesniak showed that Piazzon-Vianello results in particular apply to any *compact sub-analytic set* that hence has a nearly optimal AM.
- A. Kroó proved that *analytic graph domains* have a nearly optimal AM
- Bloom,Bos,Calvi and Levenberg [2] showed (non constructively) that any *L-regular* compact set has.

Solving (Q2)

Question 2

How to build Optimal AMs for a given K?

- Polytopes, Balls have Optimal AMs by 1dim techniques, symmetry or thanks to the particular shape and finite unions.
- The Kroó result applies to **star-shaped** $\mathscr{C}^2$ smooth sets.
- The Kroó result has been refined: if $d = 2$, then $\mathscr{C}^2$ smoothness can be replaced by uniform interior ball condition.

What about sets with a more general shape?

Main Result

Idea: Smoothness may completely replace Particular Shape.

$\downarrow$

P. 2013

Let Ω be a bounded $\mathscr{C}^{1,1}$ domain in $\mathbb{R}^d$, then there exists an optimal admissible mesh for $K := \overline{\Omega}$.

Bernstein Inequality

Let $p \in \mathscr{P}^n(\mathbb{R})$, then for any $a < b \in \mathbb{R}$ we have

$$\left|\frac{dp}{dx}(x)\right| \leq \frac{\mathbf{n}}{\sqrt{(x-a)(b-x)}}\|p\|_{[a,b]}. \tag{BI}$$

and thus $\left|\frac{dp}{dx}\left(\frac{a+b}{2}\right)\right| \leq \frac{2\mathbf{n}}{(b-a)}\|p\|_{[a,b]}$.

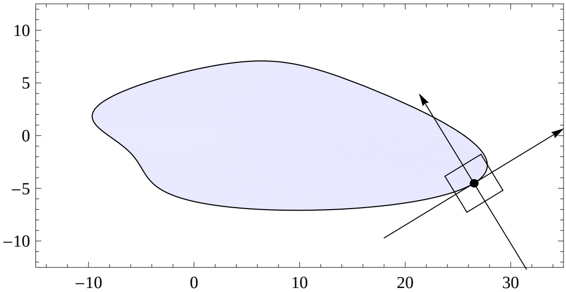
Markov Tangential Inequality

Let $p \in \mathscr{P}^n(\mathbb{R}^d)$, then for any $x_0 \in \mathbb{R}^d$, $r > 0$ and $v \in \cap \mathcal{T}_x \partial B(x_0, r)$, $|v| = 1$ we have

$$\left|\frac{dp}{dv}(x)\right| \leq \frac{\mathbf{n}}{r}\|p\|_{B(x_0,r]}. \tag{MTI}$$

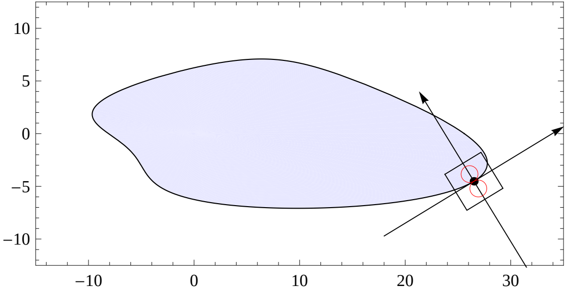
$\mathscr{C}^{1,1}$ domains

$\Omega \subset \mathbb{R}^d$ domain whose boundary is locally the graph of a $\mathscr{C}^{1,1}$ function of controlled norm. If Ω is bounded, then all the parameters involved in this definition can be chosen uniformly.



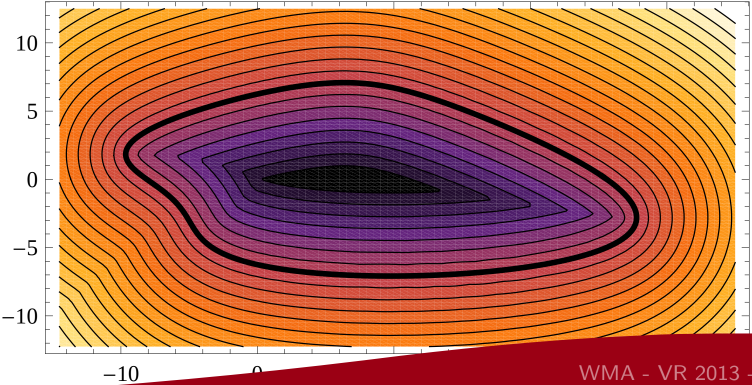
Geometric Characterization

Bounded $\mathscr{C}^{1,1}$ domains are characterized by the **uniform double sided ball condition**.



Oriented Distance Function

$$b_\Omega(x) := \inf_{y \in \overline{\Omega}} |x - y| - \inf_{z \in \complement\Omega} |x - z|$$



Tools V

Regularity Properties of $b_\Omega(\cdot)$

If Ω is a bounded $\mathscr{C}^{1,1}$ domain, then there exists $\bar{\delta}$ such that for any $0 < \delta < \bar{\delta}$ we have

- The metric projection on the boundary $x \mapsto \pi_{\partial\Omega}(x)$ is single valued on U_δ where U_δ is a δ-tubular neighborhood of $\partial\Omega$.
- $b_\Omega \in \mathscr{C}^{1,1}(U_\delta)$.
- $\nabla b_\Omega(x) = \frac{x - \pi_{\partial\Omega}(x)}{b_\Omega(x)} \neq 0$ in U_δ.
- $\nabla b_\Omega(x)$ defines the outer normal unit vector field w.r.t. Ω.

- Differentiability across the boundary.
- We can take $\bar{\delta}$ as the radius of the ball.
- Level sets of b_Ω are $\mathscr{C}^{1,1}$ manifolds.

Sketch of the Proof 1

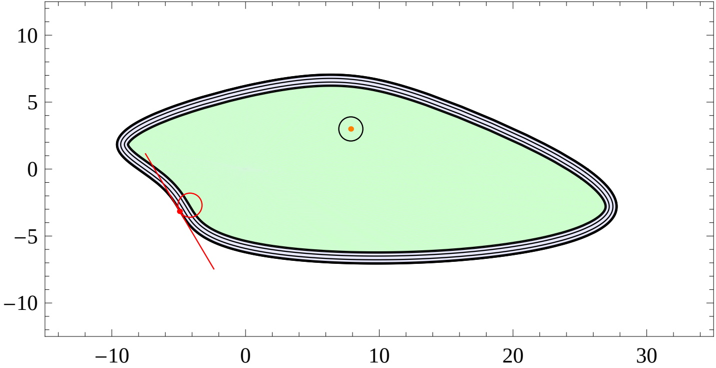
Bound normal derivatives of polynomials by a **modified Bernstein Inequality** along segments of metric projection. Thanks to boundary regularity.

$\Downarrow$

Find a **norming set** for $\overline{\Omega}$: by union of $m_n = \mathcal{O}(n)$ hypersurfaces which are **level sets of** b_Ω and $K_\delta := \{x \in \overline{\Omega} : |b_\Omega(x)| \geq \delta\}$

$$\|p\|_K \leq 2 \max \left\{ \|p\|_{K_\delta}, \|p\|_{\cup_{i=0}^{m_n} \Gamma^i} \right\}. \quad \forall p \in \mathscr{P}^n(\mathbb{R}^d)$$

construction



Sketch of the Proof 2

Bound any directional derivative of polynomials by a **modified Bernstein Inequality** holding in K_δ.

$$\Downarrow$$

Find a **weak norming set** for K_δ : w.r.t. $\overline{\Omega}$ by a grid mesh Z_n, of stepsize $\mathcal{O}(n^{-1})$ i.e.

$$\|p\|_{K_\delta} \leq \|p\|_{Z_n} + \frac{1}{\lambda}\|p\|_{\overline{\Omega}} \;\; \lambda > 2. \;\; \forall p \in \mathscr{P}^n(\mathbb{R}^d)$$

$$\text{Card } Z_n = \mathcal{O}(n^d)$$

Sketch of the Proof 3

Bound tangential derivatives of polynomials on Γ^is by the combination of

- Regularity of Γ^is $\Rightarrow$ Ball property.
- **Markov Tangential Inequality**

$$\Downarrow$$

Find **weak norming sets** for $\cup_{i=0}^{m}{}_n\Gamma^i$ w.r.t. $\overline{\Omega}$ by the union of geodesic meshes $Y_n^i \subset \Gamma^i$ having *geodesic fill distance* $\mathcal{O}(n^{-1})$

$$\|p\|_{\cup_{i=0}^{m_n}\Gamma^i} \leq \|p\|_{Y_n^i} + \frac{1}{\lambda}\|p\|_{\overline{\Omega}} \quad \lambda > 2. \;\; \forall p \in \mathscr{P}^n(\mathbb{R}^d)$$

$$\text{Card} \cup_{i=0}^{m_n}\Gamma^i = m_n\mathcal{O}(n^{d-1}) = \mathcal{O}(n^d).$$

Sketch of the Proof 4

Finally we set

$$\boxed{A_n := Z_n \cup \left(\cup_{i=0}^{m_n} \Gamma^i\right)}$$

and the inequalities above read as

$$\begin{array}{rcl} \|p\|_{\overline{\Omega}} & \leq & 2\|p\|_{A_n} + \dfrac{2}{\lambda}\|p\|_{\overline{\Omega}} \;\; \forall p \in \mathscr{P}^n(\mathbb{R}^d) \\ & \Downarrow & \\ \|p\|_{\overline{\Omega}} & \leq & \dfrac{2\lambda}{\lambda - 2}\|p\|_{A_n} \;\; \forall p \in \mathscr{P}^n(\mathbb{R}^d) \\ \text{Card}(A_n) & = & \mathcal{O}(n^d). \end{array}$$

We conclude by recalling the open problem

Conjecture Bloom Bos Calvi and Levenberg

If $K \subset \mathbb{C}^d$ is compact and L-regular then there exists $c := c(K)$ such that any array of degree $c(K)n$ Fekete points forms an Admissible mesh for K, that hence is Optimal.

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Thank you and...

Happy Birthday!!

more details....

Proof (1)

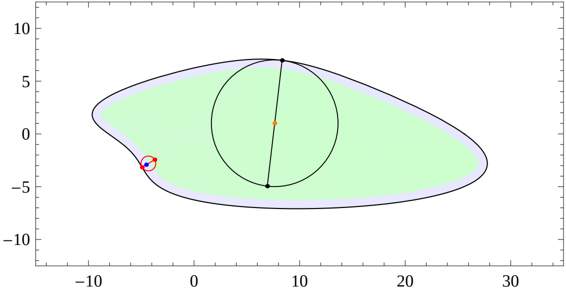
- **Step 1** we build a norming set.

We pick $\delta < r$ and work out a modification of **(BI)** of the form

$$|D_S p(x)| \leq n\varphi_\delta(b_\Omega(x))\|p\|_\Omega$$

Where S is a segment of metric projection,
φ_δ arises as follows..

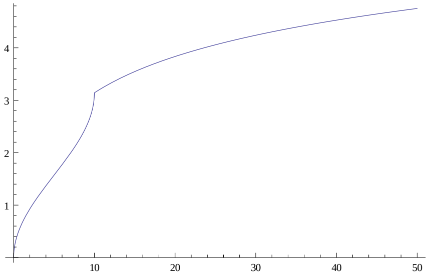
$$\varphi_\delta(\xi) := \begin{cases} \frac{1}{\sqrt{\xi(\delta-\xi)}}, & \text{if } \xi < \delta \\ \frac{1}{\xi}\quad, & \text{otherwise} \end{cases}. \tag{2}$$



Proof (3)

Then we define a function by integration along segments of metric projection

$$x \mapsto F_{n,\delta}(x) := n \int_0^{-b_\Omega(x)} \varphi_\delta(\xi) d\xi$$



Proof(4)

and consider equally spaced level sets

$$\Gamma^i_{n,\delta} := F_{n,\delta}(a^i) \;\; i = 0, 1, \dots m_n = \mathcal{O}(n),$$

where

$$\begin{aligned} a^i &= 0, \dots, \max_{\Omega} F_{n,\delta} \\ 1/2 &\geq a^{i+1} - a^i . \end{aligned}$$

Therefore we have

$$\begin{aligned} \|p\|_{\overline{\Omega}} &\leq \|p\|_{\cup_i \Gamma^i_{n,\delta}} + 1/2 \|p\|_{\overline{\Omega}} \Rightarrow \\ &\leq 2\|p\|_{\cup_i \Gamma^i_{n,\delta}} . \end{aligned}$$

Proof (5)

For some technical reasons we switch

$$\cup_{i=0}^{m_n}\Gamma_{n,\delta}^{i} = \cup_{i=0}^{\tilde{m}_n}\Gamma_{n,\delta}^{i} \Cup \cup_{i=\tilde{m}_n+1}^{m_n}\Gamma_{n,\delta}^{i}$$

where the second set is a subset of

$$\boxed{K_\delta := \{x \in \overline{\Omega} : |b_\Omega(x)| \geq \delta\}}$$

and then we can replace it by K_δ itself.

$$\boxed{\|p\|_{\overline{\Omega}} \leq 2\max\{\|p\|_{\cup_i^{\tilde{m}_n}\Gamma_{n,\delta}^{i}}, \|p\|_{K_\delta}\}.}$$

Proof (6)

- **Step 2:** finding a norming mesh Z_n for K_δ.

we use (BI) jointly with $B(x,\delta) \subset \overline{\Omega}$ $\forall x \in K_\delta$ to get

$$|\nabla p(x)| \le \frac{n}{\delta} \|p\|_{\overline{\Omega}}.$$

Thus we can build a suitable Z_n by a grid of step size $\frac{\delta}{4n} = \mathcal{O}(n^{-1})$ and hence cardinality

$$\text{Card } Z_n = \mathcal{O}(n^d)$$

obtaining

$$\|p\|_{K_\delta} \le \|p\|_{Z_n} + \frac{1}{4}\|p\|_{\overline{\Omega}}$$

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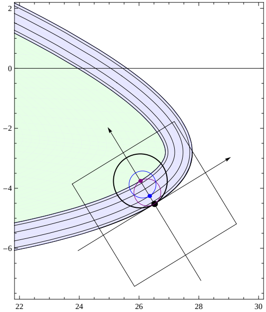
- **Step 3:** finding a norming mesh Y_n for $\cup_{i=0}^{\tilde{m}_n}\Gamma^i_{n,\delta}$
  - Any $\Gamma^i_{n,\delta}$ is a $\mathscr{C}^{1,1}$ manifold.
  - we can pick a tangent ball of radius $\delta/2$ lying in Ω.



thus..

Proof (8)

We can bound any tangential derivative by **MTI** applied to the ball to get

$$\left|\frac{dp}{dv}(x)\right| \leq \frac{n}{\delta/2}\|p\|_{\overline{\Omega}} \;\forall v \in \mathbb{S}^{d-1} \cap \mathcal{T}_x\Gamma^i_{n,\delta}$$

Therefore if we pick a mesh Y^i_n on each $\Gamma^i_{n,\delta}$ having controlled **geodesic fill distance** $h^i = \mathcal{O}(n^{-1})$ we get

$$\|p\|_{\cup_i^{\tilde{m}_n}\Gamma^i_{n,\delta}} \leq \|p\|_{\cup_i Y^i_n} + \frac{1}{4}\|p\|_{\overline{\Omega}}.$$

Proof (9)

The regularity of $\Gamma^i_{n,\delta}$'s ensures that we can produce suitable $Y^i_{n,\delta}$ using $\mathcal{O}(n^{d-1})$ points, since we have $m_n = \mathcal{O}(n)$ level set $\Gamma^i_{n,\delta}$ we have

$$\boxed{\mathrm{Card}\, Y_{n,\delta} := \mathrm{Card} \cup_i Y^i_{n,\delta} = \mathcal{O}(n^d)}$$

Proof (10)

- **Step 4:** joining all the inequalities.

Finally putting all together we get $\|p\|_{\overline{\Omega}} \leq 2 \left(\|p\|_{Y_{n,\delta} \cup Z_n} + \frac{1}{4} \|p\|_{\overline{\Omega}} \right)$ and thus

$$\|p\|_{\overline{\Omega}} \leq 4 \left(\|p\|_{Y_{n,\delta} \cup Z_n} \right) \quad \text{where} \tag{3}$$

$$\mathrm{Card}(Y_{n,\delta} \cup Z_n) = \mathcal{O}(n^d) \tag{4}$$

That is optimal.

[3]
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