# The Bernstein Markov Property and Applications to Pluripotential Theory

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Federico Piazzon

Department of Mathematics.
Doctoral School in Mathematical Sciences,
Applied Mathematics Area

Università degli Studi di Padova

# Outline

# Setting and Notation



- $K$ compact set in $\mathbb{C}$ or $\mathbb{C}^n$
- $\|f\|_K. = \max_K |f|.$
- $\mu$ positive finite Borel measure, $\operatorname{supp}\mu \subseteq K$. $\mu \in \mathcal{M}_+(K)$.
- $\mathscr{P}^m$ space of complex polynomials of degree at most $m$.
- $N_m := \dim \mathscr{P}^m = \binom{n+m}{m}$.
- $\mathscr{P}^m_\mu$ Hilbert space $\left(\mathscr{P}^m, \langle\cdot,\cdot\rangle_{L^2_\mu}\right)$.

Since $\mathscr{P}^m$ is a finite dimensional TVS all norms are comparable. In particular there exists $0 < C(\mu, K, m) < \infty$ such that

$$\frac{1}{\sqrt{\mu(K)}}\|p\|_{L^2_\mu} \leq \|p\|_K \leq C(\mu, K, m)\|p\|_{L^2_\mu} \quad \forall p \in \mathscr{P}^m(K).$$

The Bernstein Markov Property is a quantitative requirement on the asymptotic of the $m$–th root of the comparability constant $C(\mu, K, m)$.

### Bernstein Markov Property (BMP)

Let $K \subset \mathbb{C}$ be compact and $\mu \in \mathcal{M}_+(K)$ then the $(K, \mu)$ is said to enjoy the **Bernstein Markov Property** if exists a sequence $\{C_m\}_{m \in \mathbb{N}}$ such that

$$\begin{aligned} \|p\|_K \leq C_m \|p\|_{L^2_\mu} \;\; \forall p \in \mathscr{P}^m(K), \\ \limsup_m C_m^{1/m} \leq 1. \end{aligned} \tag{1}$$

# Definitions ..continued



Several variants have been introduced

- **Weighted BMP** Given a weight function $w : K \to [0, +\infty[$ one looks at $\|pw^m\|_K$ and $\|pw^m\|_{L^2_\mu}$ for $p \in \mathscr{P}^m$.
- **Strong BMP** If for any $w \in \mathcal{C}(K)$ $(K, \mu, w)$ has the WBMP.
- **Rational BMP** For a given compact set $P, K \cap P = \emptyset$ we set

$$\mathscr{R}^m(P) := \{p_m/q_m\ ,\ p_m, q_m \in \mathscr{P}^m(K),\ Z(q_m) \subset P\},$$

then we compare $\|r\|_K$ and $\|r\|_{L^2_\mu}$ for $r \in \mathscr{R}^m(P)$.

Weighted Rational and Strong Rational BMP...

# Related studies

- The first steps are made by Szego, Faber, Erdós and Turan.
- Classical weight on the real line.
- Leja $L^*$ condition.
- Systematic study for general measures in the plane erly 90's Stahl, Totik [10]. Regular measures.
- Determining measure: Widom and Ullman.

Here we follow the approach of Berman,Boucksom, Nymstrom [7], Bloom and Levenberg [3], which is more adapted to the svc context and pluripotential theory.

# Some Examples



1. $K = \overline{\Delta}, \mu = \delta_0$. This is not a BM couple.
   *It should be that $\mu$ defines at least a norm..*
2. $K := \overline{\Delta} \times \overline{\Delta}$, $\operatorname{supp} \mu = S(K)$ the Śilov boundary and $\mu := ds \times ds$. Instead is. Monomials are orthonormal..

$$\|p\|_K = |p(z_0)| \leq \sqrt{\sum_{|\alpha| \leq m} |c_\alpha|^2} \sqrt{\sum_{|\alpha| \leq m} |z_0|^{2\alpha}} = \sqrt{\frac{(m+2)(m+1)}{2}} \|p\|_{L^2_\mu}$$

   *$\mu$ should be thick on $S(K)$..*
3. It has been shown that there exists a BM measure for $\overline{\Delta}$ with discrete carrier in the interior of the disk. *In general we find out only sufficient conditions.*
4. The measure with the weight $w(z) = \exp(-|z|^2)$ makes $(\overline{\Delta}, ds, w)$ not a WBM triple.

$\mathscr{P}^m_\mu := (\mathscr{P}^m, \langle \cdot ; \cdot \rangle_{L^2_\mu(K)})$ is a reproducing kernel Hilbert space, being the kernel

$$K^\mu_m(z,\zeta) := \sum_{|\alpha|\leq m} q_\alpha(z,\mu)\bar{q}_\alpha(\zeta,\mu). \;\; \{q_\alpha\}_{|\alpha|\leq m} \text{ o.n.b.}$$

$$B^\mu_m(z) := K_m(z,z) = \langle K^\mu_m(z,\zeta); K^\mu_m(z,\zeta)\rangle_{L^2_{\mu(\zeta)}(K)}.$$

- Let $\delta_z \in L(\mathscr{P}^m_\mu, \mathbb{C})$ be the point-wise evaluation, for any $z \in K$ we have $\|\delta_z\| = \sqrt{B^\mu_m(z)}$.
- *The best possible constant in* (1) *is* $\sqrt{\|B^\mu_m\|_K}$.

# The dd$^c$ operator



For an open set $\Omega \subset \mathbb{C}^n$ and $u \in \mathcal{C}^2(\Omega)$ one defines first $d := (\partial + \bar{\partial})$ and $d^c := i(-\partial + \bar{\partial})$,

$$\mathrm{dd^c}\, u := 2i \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) dz_j \wedge d\bar{z}_k.$$

For any *plurisubharmonic* $u$ we can define by smoothing

dd$^c$ $u$ as a positive $(1,1)$ current

i.e. an element of the dual of the test forms of bidegree $(n-1, n-1)$ such that

$$\mathrm{dd^c}\, u \wedge \theta > 0 \quad \forall \theta \in SP^{(n-1,n-1)}(\Omega).$$

# Defining $(dd^c)^2$



Let $\Omega \subset \mathbb{C}^2$ be a domain

1. if $u \in \mathrm{PSH}$ and $\theta$ is a positive $(1,1)$ form

$$\langle dd^c u \wedge \theta, \psi \rangle := \langle dd^c u, \theta \wedge \psi \rangle \quad \forall \psi \in \mathcal{D}^{(n-2,n-2)}(\Omega) = C_c^\infty(\Omega).$$

2. if $u \in \mathcal{C}^2(\Omega)$ then we have $\forall \varphi \in \mathcal{C}_c^\infty(\Omega)$

$$\int_\Omega \varphi (dd^c u)^2 = \int_\Omega u \, dd^c \varphi \wedge dd^c u.$$

*...but the r.h.s. takes sense even for* $u \in \mathrm{PSH} \cap L^\infty_{loc}$...

# Extending to plurisubharmonic $u$



## Theorem [Chern Levine Nirenberg]

For any $K \subset\subset \Omega$ there exist $C > 0$ and compact set $L \subset \Omega \setminus K$ such that

$$\int_K (dd^c u)^n \leq C \|u\|_L^m \quad \forall u \in \mathcal{C}^2(\Omega).$$

Combining C.L.N. estimate and

## Proposition

Any $(p, p)$ positive current has measure coefficients [6].

Bedford and Taylor find out that

## Monge Ampere Operator

for $u \in \mathrm{PSH}(\Omega) \cap L^{\infty}_{\mathrm{loc}}$ one can iteratively define $(dd^c u)^n$ as a positive measure by

$$\langle (dd^c u)^{k+1}, \theta \rangle := \langle dd^c \theta \wedge (dd^c u)^k, u \rangle. \ \forall \theta \in \mathcal{D}^{n-k-1,n-k-1}(\Omega)$$

## Theorem [Bedford Taylor][2]

The operator $(dd^c)^n$ is continuous w.r.t. the weak $^*$ topology under point-wise converging decreasing sequence of functions.

This is a fully non linear partial diff operator that in the case $n = 1$ corresponds to the distributional Laplacian.

# Dirichlet problem



For "nice" compact set $K$ we can solve the Dirichlet problem

$$\begin{cases} (dd^c u)^n = 0 & \text{in } \Omega := \mathbb{C}^n \setminus K \\ u \equiv_{\text{q.e.}} 0 & \text{in } \partial K \\ u \in \mathcal{L} \text{ Lelong class.} \end{cases}$$

The unique solution $V_K^*$ is said **Extremal Function**.

# The extremal function



The solution is in the sense of Perron-Bremermann

$$V_K^*(z) := \limsup_{\zeta \to z} \left( \sup\{u(\zeta) \in \mathcal{L}\,,\ u|_K \leq 0\} \right)$$

If it is continuous the compact set is said $\mathcal{L}$**regular**.
By Siciak and Zaharyuta results we have

$$V_K = \log \Phi_K := \log^+ \sup\{|p|^{1/\deg p} : \|p\|_K \leq 1\}.$$

And **Bernstein-Walsh-Siciak Inequality** follows

$$|p(z)| \leq \|p\|_K \exp(\deg p V_K^*(z)).$$

The measure

$$\mu_K := (dd^c\, V_K^*)^n$$

is said the **equilibrium measure**.

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- **LS asymptotic.** If $(K,\mu)$ has the BMP then

$$\limsup_m d_\infty(f,\mathscr{P}^m)^{1/m} = \frac{1}{R} \text{ (i.e. } f \in hol(\{V_K < \log R\}) \text{ )}$$

$$\Downarrow$$

$$\limsup_m \|f - \mathscr{L}_m f\|_K^{1/m} = \frac{1}{R} \text{ (here } \mathscr{L}_m \text{ is the LS projection.)}$$

Moreover

$$\limsup_m \|f - \mathscr{L}_m f\|_{L^2_\mu}^{1/m} \leq \frac{1}{R} \Rightarrow f \in hol(\{V_K < \log R\}).$$

- *m*-**th roots asymptotic.** For regular compact set $K$ if then

$$(K,\mu) \quad \text{has} \quad \text{BMP}$$
$$\Updownarrow$$
$$\lim_m \frac{1}{2m} \log B_m^\mu \quad = \quad V_K \ \text{ loc. uniformly in } \mathbb{C}^n.$$

- **Free energy asymptotic.** If $(K,\mu)$ has the BMP then we have

$$\limsup_m \left( \int \dots \int \left| \underset{m}{\mathrm{VDM}}(z_1, \dots, z_{N_m}) \right|^2 d\mu(z_1)\dots d\mu(z_{N_m}) \right)^{\frac{n+1}{2nmN_m}} = \delta(K)$$

i.e. the l.h.s. is maximal among $\{\nu \in \mathcal{M}^+(K) : \nu(K) = \mu(K)\}$..
This is the main tool for proving

1. Strong Bergman Asymptotic. $\frac{B_m^\mu}{N_m}\mu \rightharpoonup^* \mu_K$.
2. Large Deviation Principle.

..and all these results go straightforward into the *weighted* setting.

# Mass Density Condition in $\mathbb{C}$



From the example we guess $\mu$ should be *thick* on $S(K)$.. We had $\lim_{r\to 0^+} \mu(B(z,r))/r = 1 \ \forall z \in S(K)$.

## Theorem [Stahl Totik]

Let $\mu$ be positive Borel measure with compact support $K = \operatorname{supp}\mu$ in $\mathbb{C}$, suppose that $K$ is a non-polar regular set w.r.t. the Dirichlet problem for the Laplace operator and there exists $t > 0$ such that

$$\lim_{r\to 0^+} \operatorname{cap}\left(\{z \in K : \mu(B(z,r)) > r^t\}\right) = \operatorname{cap}(K).$$

Then $(K, \mu)$ has the BMP.

# Capacity in $\mathbb{C}$



Here $\operatorname{cap}(K)$ is the logarithmic capacity of the set $K$,

$$\operatorname{cap}(K) := \max_{\nu \in \mathcal{M}_1(K)} \exp\left( \int \int \log |z - \zeta| d\nu(z) d\nu(\zeta) \right).$$

That is the (inverse of the exponential) of the minimum of the logarithmic energy functional: the variational formulation of the Dirichlet problem for the Laplacian in $\mathbb{C} \setminus K$.

# Mass Density Condition in $\mathbb{C}^n$



**Theorem [Bloom Levenberg]**

Let $\mu$ be positive Borel measure with compact support $K := \operatorname{supp}\mu \subset B(0,1)$ in $\mathbb{C}^n$, suppose that $K$ is a non-pluripolar $\mathcal{L}$-regular set and there exists $t > 0$ such that

$$\lim_{r\to 0^+} \operatorname{Cap}\Big(\{z \in K : \mu(B(z,r)) > r^t\}, B(0,1)\Big) = \operatorname{Cap}(K, B(0,1)). \tag{2}$$

Then $(K,\mu)$ has the BMP.

But here $\mathrm{Cap}(K, \Omega)$ is capacity in the **non-linear** pluripotential theory in $\mathbb{C}^n$ related to the Monge Ampere complex operator, namely the **relative capacity** $\mathrm{Cap}(K, \Omega)$ w.r.t. a hyperconvex sup-set $\Omega$ of $K$.

## Relative Capacity in $\mathbb{C}^n$

$$\mathrm{Cap}(K, \Omega) := \sup\left\{ \int_K (dd^c u)^n : u \in \mathrm{PSH}(\Omega), 0 \leq u \leq 1 \right\}$$

The proof of these results relays on the following facts

(A) The regularity assumption on the set: $V_K^*$ is continuous.

(B) Bernstein Walsh Siciak lemma.

$$|p(z)| \leq \|p\|_K \exp(\deg(p) V_K(z)). \tag{3}$$

(C) The following theorem

**Capacity Convergence [Bloom Levenberg]**

For any sequence of compact subsets of the compact non pluripolar $\mathcal{L}$–regular set $K$ the following facts are equivalent

(i) $\lim_j \mathrm{Cap}(K_j, B(0,1)) = \mathrm{Cap}(K, B(0,1))$.

(ii) $\lim_j V_{K_j} = V_K$ locally uniformly in $\mathbb{C}^n$.

# RBM mass density condition



**Motivation:** LDP for vector energy problems [4].

## Theorem [P.]

Let $K$ be a regular non polar compact set in the complex plane, $\Omega := \mathbb{C}_\infty \setminus \hat{K}$ and $P \subset \Omega$ a compactum. Let $\mu \in \mathcal{M}(K)$, $\operatorname{supp} \mu = K$ and suppose there exists a positive $t$ such that

$$\lim_{r \to 0^+} \operatorname{cap}\left(\{z : \mu(B(z,r)) \geq r^t\}\right) = \operatorname{cap}(K). \tag{4}$$

Then $\mu$ enjoys the Bernstein Markov property on $K$ for the rational functions with poles in $P$.

# idea of the proof

- We replace the Bernstein Walsh Siciak Inequality by

$$|r_m(z)| \leq \|r\|_K \exp\left( \sum_{z_j \in \mathrm{Poles}(r_m)} g_{\Omega_K}(z, z_j) \right) \quad \forall r_m \in \mathscr{R}_m(K, P).$$

  Here $g_{\Omega_K}(z, z_j)$ is the generalized Green function
- We recover a modified capacity convergence result.

## Proposition [P.]

Let $K \subset \mathbb{C}$ be a regular non polar compact set, let $\Omega_K$ be the unbounded component of $\mathbb{C} \setminus K$ and $P$ a compact subset of $\Omega_K$ such that $P \cap K = \emptyset$. Then there exist a domain $D$ such that $K \subset\subset D$ and $P \cap \overline{D} = \emptyset$, such that for any sequence $\{K_j\}$ of compact subsets of $K$ the following are equivalent (here $\Omega_{K_j}$ is defined similarly to $\Omega_K$).

$$\lim_j \mathrm{cap}(K_j) = \mathrm{cap}(K).$$

$$\lim_j g_{\Omega_j}(z, a) = g_{\Omega}(z, a) \text{ loc. unif. for } z \in D \text{ unif. for } a \in P.$$

# Unbounded sets in $\mathbb{C}$



In the case of a closed unbounded set $K$ and an admissible weight function $w : \mathbb{C} \to [0, +\infty[$ can we do something?

**Idea:**

- **Compactification**, we look to the real sphere.
- search for **Strong BMP**, leads to..
- **complexification**: $\mathcal{A} := \{z \in \mathbb{C}^3 : \sum z_j^2 = 1\}$ of the sphere
- use Pluripotential Theory for **Algebraic Submanifold**.

In such a setting the proof of an adapted formulation of sufficient mass density condition works provided an adapted version of the capacity convergence result.

# The adapted setting



There is a specific $\mathbb{C}$–linear change of (Rudin) coordinates[8] such that

$$\mathcal{A} \subset \{(z, w) \in \mathbb{C}^2 \times \mathbb{C} : |w|^2 \leq C(1 + |z|^2)\}.$$

Sadullaev [9] defined $V_K^*(\cdot, \mathcal{A})$ and gived sense to pluripotential theory on algebraic sets.
**Notation:** for $R >> 1$ we use the pseudo-ball

$$\Omega(r) := \{(z, w) \in \mathcal{A} : |z|^2 - R < r\}.$$

$\Omega := \Omega(-\sqrt{R^2 - 1})$.

# Capacity convergence on algebr. man.

## Theorem [P.]

Let $\mathcal{A} \subset \mathbb{C}^n$ be an algebraic variety of pure dimension $m < n$, $\mathcal{A}_{\text{reg}} \supset \Omega_0 \supset K$ where $K$ is a compact $\mathcal{L}$ regular nonpluripolar set. Let $\{K_j\}_{j\in\mathbb{N}}$ be a sequence of compact subsets of $K$, then the following are equivalent.

(i) $\lim_j \operatorname{Cap}(K_j, \Omega) = \operatorname{Cap}(K, \Omega)$.

(ii) $V^*_{K_j}(\cdot, \mathcal{A}) \to V^*_K(\cdot, \mathcal{A})$ locally uniformly on $\mathcal{A}$.

The proof is similar to the original one ...

**if**

we provide a modified version of the *Capacities Comparison Theorem* of Alexander and Taylor [1].

To do it ,first we re-defined the (modified) Tchebyshev constant as

$$m_\nu(K) := \inf\{\|p\|_K : p \in \mathscr{P}_{\mathbb{C}}(K), \deg p \le \nu, \|p\|_{|z|\le 1} \ge 1\},$$

$$T(K, \mathcal{A}) := \inf_{\nu>0} m_\nu^{1/\nu}(K).$$

# Capacities Comparison



## Theorem [P.]

Let $\mathcal{A}$ be a $m$-dimensional algebraic variety of $\mathbb{C}^n$, such that for an $R > 1$ $\mathcal{A}_{\text{reg}} \supset \Omega_0$, then for any $r < -\sqrt{R^2-1}$ there exist two positive constants $c_1, c_2$ such that for any compact $K \subset \Omega_r$

$$\exp\left[-\left(\frac{c_1}{\text{Cap}(K,\Omega)}\right)^{1/m}\right] \geq T(K,\mathcal{A}),$$

$$T(K,\mathcal{A}) \geq \exp\left(-\frac{c_2}{\text{Cap}(K,\Omega)}\right).$$

In particular for any $E \subseteq K$ we have

$$\|V_E(z,\mathcal{A})\|_\Omega \leq \frac{c_2}{\text{Cap}(E,\Omega)}.$$

# Element of the proof



Among other issue we need a further sharpening of Chern Levine Nirenberg Estimate for $u \leq 0$

CLN $\int_K (dd^c u)^n \leq C\|u\|_L^m \quad \forall u \in \mathcal{C}^2(B)$

AT $\int_K (dd^c u)^m \leq C(-u(0))\|u\|_B^{m-1} \quad u \in \mathrm{PSH}(B)$

New $\int_K (dd^c u)^m \leq C \int_{\Omega(z_0,r)} -u\,(dd^c \rho)^m \,\|u\|_\Omega^{m-1} \quad u \in \mathrm{PSH}(\Omega)$
$\Omega \subset \Omega(z_0, r) \subset \mathcal{A}$.

For the case of the relative extremal function the r.h.s. integral can be dominated by the same function for the projected set, i.e.

$$\int_{\Omega(z_0,r)} -U_{K,\Omega(z_0,r}\,(dd^c \rho)^m \leq C'(r)\left(-U_{\pi K, B_{\mathbb{C}^m}}(z_0)\right).$$

The main tool here is the Leelong Jensen Formula proven by Demailly [5].


Thank you for the attention.

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