Polynomial Admissible Meshes

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Definition

Let $K \subset \mathbb{C}^d$ (or $\mathbb{R}^d$) be a compact polynomial determining set. A sequence $\{A_n\}$ of its subsets is said to be a *weakly admissible mesh* (**WAM**) of constant C_n if

i) $\max_K |p| \leq C_n \max_{A_n} |p|$ for any polynomial p such that $\deg p \leq n$,

ii) $\mathrm{Card}(A_n)$ and C_n grow at most polynomially w.r.t. n.

If $\sup_n A_n =: C < \infty$ then A_n is said an *admissible mesh* (**AM**). If furthermore $\mathrm{Card}(A_n) = \mathcal{O}(n^d)$ then $\{A_n\}$ is termed *optimal*.

Relevant (univariate) examples

- **Chebyshev-Lobatto nodes.** For an interval $[a,b]$, $X_n(a,b) = \left\{\frac{b-a}{2}\xi_j + \frac{b+a}{2}\right\}$, where $\xi_j = \cos(j\pi/n)$, $0 \leq j \leq n$.
- **Chebyshev nodes.** $Z_n(a,b) = \left\{\frac{b-a}{2}\eta_j + \frac{b+a}{2}\right\}$, where $\eta_j = \cos\left(\frac{(2j+1)\pi}{2(n+1)}\right)$, $0 \leq j \leq n$.
- **Chebyshev-like subperiodic angular nodes.** $\Theta_n(\alpha,\beta) = \varphi_\omega(Z_{2n}(-1,1)) + \frac{\alpha+\beta}{2} \subset (\alpha,\beta)$, $\omega = \frac{\beta-\alpha}{2} \leq \pi$, where $\varphi_\omega(s) = 2\arcsin\left(\sin\left(\frac{\omega}{2}\right) s\right)$

The following inequalities hold with $c_n = \frac{2}{\pi}\log(n+1) + 1$.

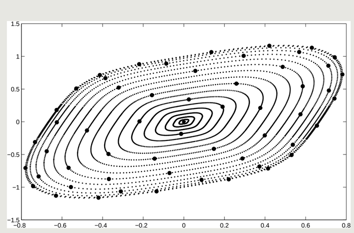
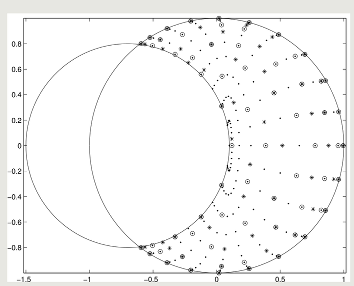
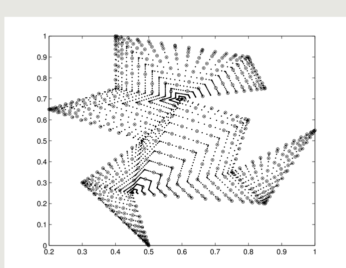
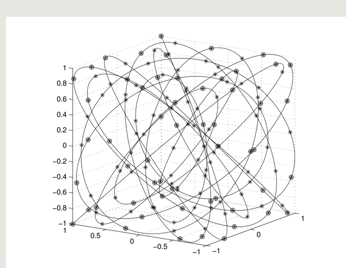
$$\|p\|_{[a,b]} \leq c_n \|p\|_{X_n} \;\forall p \in \mathbb{P}_n([a,b]).$$
$$\|p\|_{[a,b]} \leq c_n \|p\|_{Z_n} \;\forall p \in \mathbb{P}_n([a,b]).$$
$$\|t\|_{[\alpha,\beta]} \leq c_{2n} \|t\|_{\Theta_n} \;\forall t \in \mathbb{T}_n(S^1).$$

Basic Properties

- any *affine transformation* of a WAM is still a WAM, C_n being invariant;
- any sequence of unisolvent *interpolation sets* whose Lebesgue constant Λ_n grows at most polynomially with n is a WAM, with constant $C_n = \Lambda_n$;
- a *finite product* of WAMs is a WAM on the corresponding product of compacts, C_n being the product of the corresponding constants;
- a *finite union* of WAMs is a WAM on the corresponding union of compacts, C_n being the maximum of the corresponding constants.

Applications

- Polynomial least squares fitting. Theoretical and numerical bounds for the projection operator norm available.
- Discrete orthonormal polynomials computation and fitting, with algorithm for computing a stable basis.
- Extraction of unisolvent interpolation nodes with slow growth of Lebesgue constant.
- Collocation methods for pde.

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Building WAMs

The *bilinear transformation*

$$\sigma(s_1,s_2) = \frac{1}{4}\left((1-s_1)(1-s_2)\boldsymbol{v}_1 + (1+s_1)(1-s_2)\boldsymbol{v}_2 + (1+s_1)(1+s_2)\boldsymbol{v}_3 + (1-s_1)(1+s_2)\boldsymbol{v}_4\right.$$

maps the square $[-1,1]^2$ onto the *convex* quadrangle with vertices $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$; with a triangle, e.g. $\boldsymbol{v}_3 = \boldsymbol{v}_4$, as a special degenerate case.

Proposition 1 (Quadrangles and Triangles)**.** *The sequence* $A_n = \sigma(X_n(-1,1) \times X_n(-1,1))$ *is a WAM of the convex quadrangle* $\sigma([-1,1]^2)$, *with constant* $C_n = c_n^2 = \mathcal{O}(\log^2 n)$ *and* $\mathrm{card}(A_n) \leq (n+1)^2$.

Proposition 2 (Polygons)**.** *Let K be a polygon with ℓ sides. Then K has a WAM given by the union of the WAMs of $\ell-2$ triangles of a minimal triangulation, with constant* $C_n = c_n^2 = \mathcal{O}(\log^2 n)$ *and* $\mathrm{Card}(A_n) \sim (\ell-2)n^2$.

The blending transformation

$$\psi_{u,v}(s,\theta) = s\boldsymbol{u}(\theta) + (1-s)\boldsymbol{v}(\theta),$$

maps $[0,1]\times[\alpha,\beta]$ onto the region $K_{u,v}$ lying between $\boldsymbol{u}$ and $\boldsymbol{v}$, where $\boldsymbol{u}(\theta) = \boldsymbol{a}_1\cos(\theta) + \boldsymbol{b}_1\sin(\theta) + \boldsymbol{c}_1$, $\boldsymbol{v}(\theta) = \boldsymbol{a}_2\cos(\theta) + \boldsymbol{b}_2\sin(\theta) + \boldsymbol{c}_2$, $\theta \in [\alpha,\beta]$, $\boldsymbol{a}_i = (a_{i1}, a_{i2})$, $\boldsymbol{b}_i = (b_{i1}, b_{i2})$, $\boldsymbol{c}_i = (c_{i1}, c_{i2})$, $i = 1,2$, with $\boldsymbol{a}_i, \boldsymbol{b}_i$ not all zero and $[\alpha,\beta]$, $0 < \beta - \alpha \leq 2\pi$.

Proposition 3 (Circular sections)**.** *The sequence* $A_n = \psi_{u,v}(X_n(0,1) \times \Theta_n(\alpha,\beta))$ *is a WAM for* $K_{u,v}$ *with* $C_n = \mathcal{O}(\log^2 n)$ *and* $\mathrm{Card}(A_n) \leq (n+1)(2n+1)$.

Similar results are available in higher dimension e.g. for cylinders, cones, pyramids and solids of evolution.

Proposition 4 (Star-like UIBC bodies)**.** *Let* $K \subset \mathbb{R}^2$ *be the closure of a star-like domain satisfying the* Uniform Interior Ball Condition *(UIBC) of parameter* $\rho > 0$. *Then, for every fixed* $\alpha \in (0, 1/\sqrt{2})$, K *has an optimal admissible mesh* $\{A_n\}$ *such that* $C_n \equiv \frac{\sqrt{2}}{1-\alpha\sqrt{2}}$, $\mathrm{Card}\, A_n \sim n^2\, \frac{\mathrm{length}(\partial K)}{\alpha\rho}$.

Finally, WAMS have been shown to be stable under small perturbations and moreover *WAMs can be computed on sets that are images under smooth maps of sets where a WAM is given.*

Hyperinterpolation on the cube

$$\int_{[-1,1]^3} p(\boldsymbol{x}) \frac{d\boldsymbol{x}}{\sqrt{(1-x_1^2)(1-x_2^2)(1-x_3^2)}} = \pi^2 \int_0^\pi p(\boldsymbol{\ell}_n(\theta))\, d\theta\,, \quad \forall p \in \mathbb{P}_{2n}^3,$$

where we used the Lissajous curve

$$\boldsymbol{\ell}_n(\theta) = (\cos(\alpha_n\theta), \cos(\beta_n\theta), \cos(\gamma_n\theta)), \theta \in [0,\pi],$$

$$(\alpha_n, \beta_n, \gamma_n) = \begin{cases} \left(\frac{3}{4}n^2 + \frac{1}{2}n, \frac{3}{4}n^2 + n, \frac{3}{4}n^2 + \frac{3}{2}n + 1\right), n \text{ even}, \\ \left(\frac{3}{4}n^2 + \frac{1}{4}, \frac{3}{4}n^2 + \frac{3}{2}n - \frac{1}{4}, \frac{3}{4}n^2 + \frac{3}{2}n + \frac{3}{4}\right), n \text{ odd} \end{cases}.$$

Proposition 5 (WAM and onp on the cube)**.** *The sequence* $\{A_n\} = \{\boldsymbol{\ell}_n(s\pi/\nu)\}$, $s = 0, \dots, \nu = n\gamma_n + 1$ *is a WAM for the cube, with* $C_n = \mathcal{O}(\log^3 n)$ *and* $\mathrm{Card}\, A_n \sim \frac{3}{4}n^3$.

Compression and interpolation sets

Proposition 6. *Let* $\{A_n\}$ *be a WAM of a compact set* $K \subset \mathbb{R}^d$ *with* $\mathrm{Card}\, A_n > N_{2n} = \dim(\mathbb{P}_{2n})$ *and constant* C_n. *Then there exists (and can be numerically computed) a WAM* $A_n^* \subset A_n$ *with* $\mathrm{Card}\, A_n^* \leq N_{2n}$ *with constant* $C_n^* = C_n\sqrt{\mathrm{Card}\, A_n}$.

It is possible to extract unisolvent interpolation sets from a WAM by numerical linear algebra, namely *approximate Fekete points* (AFP) by QR and *Discrete Leja Sequences* (DLS) by LU factorization.

Proposition 7. *Let* $\mathcal{F}_n = \{\boldsymbol{\xi}_{i_1}, \dots, \boldsymbol{\xi}_{i_N}\}$ *the AFP or DLP extracted from a WAM of a compact set* $K \subset \mathbb{R}^d$ *(or* $K \subset \mathbb{C}^d$*). Then* $\lim_{n\to\infty} \frac{1}{N}\sum_{k=1}^N f(\boldsymbol{\xi}_{i_k}) = \int_K f(\boldsymbol{x})\, d\mu_K$ *for every* $f \in \mathscr{C}(K)$, *where* μ_K *is the pluripotential equilibrium measure of* K.

The norm of the interpolation operator $\mathcal{L}_{\mathcal{F}_n} : \mathcal{C}^0(K) \to \mathbb{P}_n(K)$ is bounded above by $\dim(\mathbb{P}_{2n})C_n$.