Optimal Admissible Meshes on Some Classes of Compact Subsets of $\mathbb{R}^d$

Federico Piazzon[1,]

Università degli Studi di Padova,
Dipartimento di Matematica, via Trieste 63 Padova, Italia.
fpiazzon@math.unipd.it

Abstract

We show that any compact subset of $\mathbb{R}^d$ which is the closure of a bounded star-shaped Lipschitz domain Ω, such that $\complement\Omega$ has positive reach in the sense of Federer, admits an optimal AM (admissible mesh), that is a sequence of polynomial norming sets with optimal cardinality. This extends a recent result of A. Kroó on $\mathscr{C}^2$ star-shaped domains.

Moreover, we prove constructively the existence of an optimal AM for any $K := \overline{\Omega} \subset \mathbb{R}^d$ where Ω is a bounded $\mathscr{C}^{1,1}$ domain. This is done by a particular multivariate sharp version of the Bernstein Inequality via the distance function.

Keywords: admissible meshes, multivariate polynomial approximation, positive reach, distance function.

1. Introduction

Let us denote by $\mathscr{P}^n(\mathbb{R}^d)$ the space of polynomials of d real variables having degree at most n. We recall that a compact set $K \subset \mathbb{R}^d$ is said to be *polynomial determining* if any polynomial vanishing on K is necessarily the null polynomial.

Let us consider a polynomial determining compact set $K \subset \mathbb{R}^d$ and let A be a subset of K. If there exists a positive constant C such that for any

Email address: fpiazzon@math.unipd.it (Federico Piazzon)

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polynomial $p \in \mathscr{P}^n(\mathbb{R}^d)$the following inequality holds

$$\|p\|_K \leq C_n \|p\|_{A_n},$$

then A_n is said to be a *norming set* for $\mathscr{P}^n(\mathbb{R}^d)$. Here and throughout the paper we use this notation: $\|f\|_X := \sup_{x\in X} |f(x)|$ for any bounded function on X.

Let $\{A_n\}_{\mathbb{N}}$ be a sequence of norming sets for $\{\mathscr{P}^n(\mathbb{R}^d)\}$ with constants $\{C_n\}$, and suppose that both C_n and $\text{Card}(A_n)$ grows at most polynomially with n (i.e., $\max\{C_n, \text{Card}(A_n)\} = O(n^s)$ for a suitable $s \in \mathbb{N}$), then $\{A_n\}_{\mathbb{N}}$ is said to be a *weakly admissible mesh (WAM)* for K; see[2] [14]. Observe that necessarily

$$\text{Card}(A_n) \geq N := \dim \mathscr{P}^n(\mathbb{R}^d) = \binom{n+d}{d} = O(n^d),$$

since a (W)AM is $\mathscr{P}^n(\mathbb{R}^d)$-determining, i.e., any polynomial in $\mathscr{P}^n(\mathbb{R}^d)$ vanishing on A_n is the zero polynomial.

If $C_n \leq C\ \forall n$, then $\{A_n\}_{\mathbb{N}}$ is said an *admissible mesh (AM)* for K (in the sequel, with a little abuse of notation, we term (weakly) admissible mesh not only the whole sequence but also its n-th element A_n). When $\text{Card}(A_n) = O(n^d)$, following Kroó [22], we speak of an *optimal* admissible mesh.

We recall that AMs are preserved by affine transformations and can be constructed incrementally by finite union and product. Moreover they are stable under small perturbations and smooth mappings [28, 30]. For a survey on WAMs properties and applications we refer to [10].

The study of AMs has several computational motivations. Indeed, it has been proved by Calvi and Levenberg that discrete least squares polynomial approximations based on (W)AMs are nearly optimal in the uniform norm, see [14, Thm. 1]. Moreover, discrete extremal sets extracted from (W)AMs (see for instance [10, 12]), are known to be good interpolation sets and to behave asymptotically like Fekete points, namely the corresponding sequences of uniform probability measures converge weakly* to the pluripotential equilibrium measure of the underlying compact set; see [5, 6] or the survey [24].

In principle, it is possible to construct an admissible mesh with $O(n^{rd})$ points on any real compact set satisfying a Markov Inequality [11] with ex-

[2]The original definition in [14] is actually a little weaker (sub-exponential growth), here we prefer to use the present one which is the most common in the literature.

ponent r. The mesh can be obtained by intersecting the compact set with a uniform grid having $O(n^{-r})$ by [14, Thm. 5].

Indeed, the hypothesis of [14, Thm. 5] are not too restrictive. For instance one has a Markov Inequality with exponent 2 for any compact set $K \subset \mathbb{R}^d$ satisfying a uniform cone condition [3], thus also for the closure of any bounded Lipschitz domain. However the Markov Inequality holds with an exponent possibly greater than 2 even for more general classes of sets; see [26] and [27] for details.

The cardinality growth order of AMs built by this procedure, however, causes severe computational drawbacks already for $d = 2$. This gives a strong practical motivation to construct low-cardinality admissible meshes, in particular optimal ones.

It has been proved in [7, Prop. 23] that for any compact polynomial determining $K \subset \mathbb{C}^d$ there exists an admissible mesh with $O((n \log n)^d)$ cardinality, unfortunately the method relies on the determinations of Fekete points, which are not known in general and whose construction is an extremely hard task.

In order to *build* meshes with nearly optimal cardinality growth order one can restrict his attention to sets with "easy" geometry as simplex, squares, balls and their images under any polynomial map (see for instance [8]) or can look at some specific geometric-analytic classes of sets; the present paper follows this line.

In [22] the author proves that any compact star-shaped set $K \subset \mathbb{R}^d$ with Minkowski Functional (see for instance [13][pg. 6]) having α-Lipschitz gradient has an admissible mesh $\{Y_n\}$ with

$$\operatorname{Card} Y_n = O(n^{\frac{2d+\alpha-1}{\alpha+1}}). \tag{1}$$

In particular he notices that this implies the existence of optimal AMs for the closure of any $\mathscr{C}^2$ star-shaped bounded domain.

In the meanwhile of writing this paper we received a new preprint (now published) by A. Kroó where the author improves his estimate (1) by a fine use of Minkowski Functional smoothness; [23, Theorem 3].

In [22] he also conjectured that *any* real convex body has an optimal admissible mesh. In this work we build such optimal admissible meshes on two relevant classes of compact sets.

The paper is organized as follows.

In **Section 2** we work on star-shaped compact sets in $\mathbb{R}^d$ with nearly minimal boundary regularity assumptions. We prove in Theorem 2.3 that if

$\Omega \subset \mathbb{R}^d$ is a bounded star-shaped Lipschitz domain such that $\complement\Omega$ has positive reach (see Definition Appendix A.1), then $K := \overline{\Omega}$ has an optimal admissible mesh.

In **Section 3** we address the same problem but we drop the star-shape assumption on K, it turns out that a little more boundary regularity is needed. In Theorem 3.6 we prove that if Ω is a bounded $\mathscr{C}^{1,1}$ domain of $\mathbb{R}^d$, then there exists an optimal admissible mesh for $K := \overline{\Omega}$.

To the author knowledge this is the first time for the construction of optimal admissible meshes on quite general sets (i.e, not polytopes, balls and their polynomial images).

In **the Appendices** we provide for the reader's convenience a quick review of some definitions and results from non-smooth and geometric analysis and geometric measure theory that are involved in the framework of this paper.

2. Optimal AMs for star-shaped sets having complement with positive reach

Let X be a compact subset of $\mathbb{R}^d$ and $Y \subset X$, the *fill distance* $h(Y)$ is often considered as mesh parameter, it is defined as

$$h(Y) := \sup_{x \in X} \inf_{y \in Y} |x - y|. \tag{2}$$

In this definition it is not important whether the segment $[x, y]$ lies in X or not. Instead we can look at the minimum length of paths joining x to y and supported in X. Hence we consider the following straightforward extension (being the same when X is convex) of the concept of fill distance given above.

Definition 2.1 (Geodesic Fill-Distance). Let Y be a subset of the set $X \subset \mathbb{R}^d$, then we set

$$\mathscr{A}_{x,y}(X) := \{\gamma \in \mathscr{C}([0,1], X) : \gamma(0) = x, \gamma(1) = y, \mathrm{Var}[\gamma] < \infty\}$$

and define

$$h_X(Y) := \sup_{x \in X} \inf_{y \in Y} \inf_{\gamma \in \mathscr{A}_{x,y}} \mathrm{Var}[\gamma], \tag{3}$$

the geodesic fill distance of Y over X.

Here and throughout the paper we denote by $\mathrm{Var}[\gamma]$ the total variation of the curve γ, that is

$$\mathrm{Var}[\gamma] := \sup_{N\in\mathbb{N}} \sup_{0=t_0<t_1\cdots<t_N=1} \sum_{i=1}^{N} |\gamma(t_i)-\gamma(t_{i-1})|.$$

Notice that if we make the further assumption of the local completeness of X, then there exists a length minimizer in $\mathscr{A}_{x,y}(X)$ provided that it is not the empty set. Thus if X has finite geodesic diameter, which will be the case of all instances considered in this paper, then we can replace $\inf_{\gamma\in\mathscr{A}_{x,y}} \mathrm{Var}[\gamma]$ by $\min_{\gamma\in\mathscr{A}_{x,y}} \mathrm{Var}[\gamma]$ in (3).

Now we want to build a mesh on the boundary of a bounded Lipschitz domain having a given geodesic fill distance but keeping as small as possible the cardinality of the mesh. Then we use such a "geodesic" mesh to build an optimal AM for the closure of the domain.

For the reader's convenience we recall here that a domain $\Omega \subset \mathbb{R}^d$ is termed a *Lipschitz domain* if there exist $0 < L < \infty$, $r > 0$ and an open neighborhood B of 0 in $\mathbb{R}^{d-1}$ such that for any $x \in \partial\Omega$ there exists $\varphi_x : B \to]-r, r[$ and $R_x \in SO_d$ such that $\varphi_x(0) = 0$, $\mathrm{Lip}(\varphi_x) \leq L$ and

$$R_x^{-1}\left(\Omega \cap (x + R_x(B\times]-r,r[)) - x\right) = \mathrm{epi}\,\varphi_x := \{(\xi,t) : \xi \in B, t \in]-R, \varphi_x(t)[\}.$$

The following result is a key element in our construction. Its proof relays on the fact that for a bounded Lipschitz domain the euclidean and geodesic distances restricted to the boundary are equivalent.

Proposition 2.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, then there exists $\bar{h} > 0$ such for any $0 < h < \overline{h}$ there exists $Y_h \subset X := \partial\Omega$ such that the following hold:*

(i) $\mathrm{Card}\, Y_h = O\left(h^{1-d}\right)$ *as $h \to 0$.*

(ii) $h_X(Y_h) \leq h$.

Proof. Here we denote by $B^s_\infty(x_0, r)$ the s dimensional ball of radius r centered at x_0 w.r.t. the norm $|x|_\infty := \max_{i\in\{1,2,\dots,s\}} |x_i|$, i.e. the coordinate cube centered at x_0 and having sides of length $2r$.

Since Ω is a Lipschitz domain using the above notation we can write

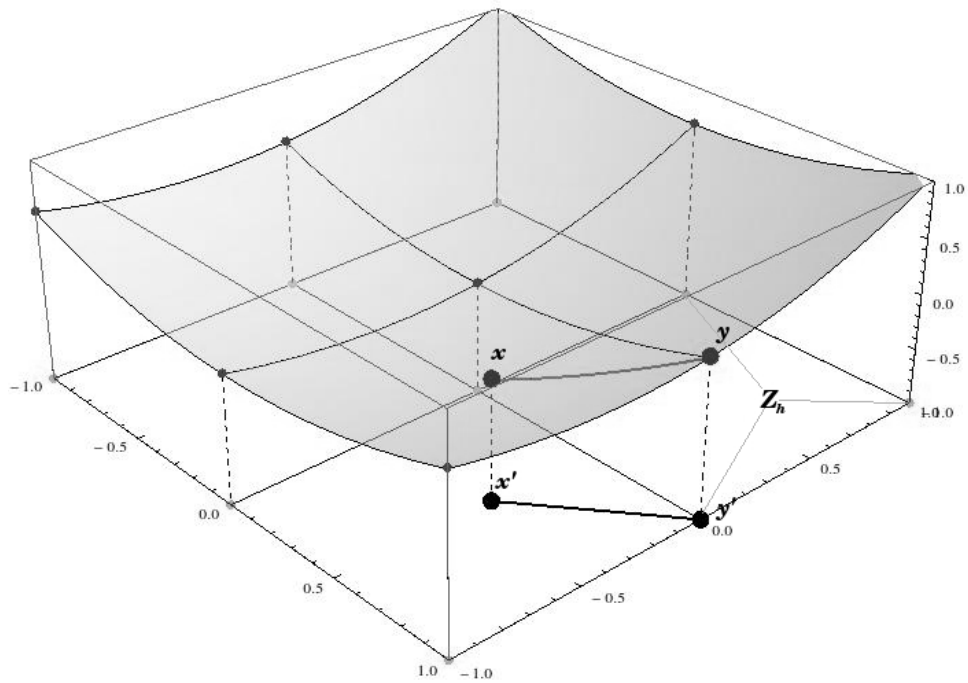

Figure 1: The *geodesic mesh* in the proof of Proposition 2.1 is built by lifting the grid mesh Z_h by the local parametrization of the boundary X. The curve γ_x connecting x to y is similarly produced by lifting the segment $[x', y']$.

$$\left(x + R_x B^d_\infty(0,r) \right) \cap \partial\Omega = R_x \,\mathrm{Graph}(\varphi_x).$$

Let us denote the *graph* function of φ_x by $g_x : B^{d-1}_\infty(0,r) \longrightarrow \mathbb{R}^d$, that is $B^{d-1}_\infty(0,r) \ni \boldsymbol{\xi} \mapsto \{\xi_1, \xi_2, \dots, \xi_{d-1}, \varphi_x(\boldsymbol{\xi})\} = g_x(\boldsymbol{\xi})$.

By compactness we can pick $x_1, x_2, \dots, x_{M(r)} \in \partial\Omega$ such that

$$\partial\Omega \subseteq \cup_{i=1}^{M(r)} X_i =: \cup_{i=1}^{M(r)} \left(\left(x_i + R_{x_i} B^d_\infty(0,r)\right) \cap \partial\Omega \right).$$

Let $\bar{h} := r\sqrt{1+L^2}$, take any $0 < h \le \bar{h}$ and let us consider the grid of step-size $\frac{h}{\sqrt{(1+L^2)}}$ in the $d-1$ dimensional cube

$$Z_h := \left(\left\{ -r + \frac{jh}{\sqrt{(1+L^2)}} \right\}_{j=0,1,\dots,\lceil \frac{2r\sqrt{1+L^2}}{h} \rceil} \right)^{d-1} \subset B^{d-1}_\infty(0,r),$$

where $\lceil\cdot\rceil$ is the ceil operator. Set

$$\begin{aligned} Y_h^i &:= x_i + R_{x_i}\left(g_{x_i}(Z_h)\right), \\ Y_h &:= \cup_{i=1}^{M(r)} Y_h^i. \end{aligned}$$

Now notice that

$$\begin{aligned} \operatorname{Card} Y_h &\leq \sum_{i=1}^{M(r)} \operatorname{Card} Y_h^i = M(r) \operatorname{Card} Z_h \\ &= M(r)\left(1 + \left\lceil \frac{2r\sqrt{1+L^2}}{h} \right\rceil\right)^{d-1} \\ &= O(h^{1-d}). \end{aligned}$$

In order to verify the *(ii)* for any $x \in \partial\Omega$ we explicitly find $y \in Y_h$ and build a curve γ_x connecting x to y whose variation gives an upper bound for the geodesic distance of x from Y_h. For the following construction we refer to the Figure 1.

Take any $x \in \partial\Omega$, then there exist (at least one) $i \in \{1, 2, \ldots, M(r)\}$ such that $x \in X_i$. Let us pick such an i.

Let us denote by proj_i the canonical projection on the first $d-1$ coordinates acting from $R_{x_i}^{-1}\left(\left(x_i + R_{x_i} B_\infty^d(0,r)\right) \cap \partial\Omega \; - \; x_i\right)$ onto $B_\infty^{d-1}(0,r)$.

Let $x' := \operatorname{proj}_i(x)$, by the very construction we can find $y' \in Z_h$ such that $|x'-y'| \leq \frac{h}{\sqrt{1+L^2}} =: h'$, moreover the whole segment $[x', y']$ lies in $B_\infty^{d-1}(0,r)$.

We consider the curve $\alpha_x : \xi \mapsto x' + \xi \frac{y'-x'}{|y'-x'|}, \xi \in [0, h']$ and we set $\gamma_x(\xi) := x_i + g_{x_i}(\alpha(\xi))$ the curve that joins x to $y := x_i + g_{x_i}(y') \in Y_h$ obtained by mapping the segment $[x', y']$ under g_{x_i}.

Now we use *Area Formula* [20] [18][Th. 1 pg. 96] to compute the length of the Lipschitz curve γ_x.

$$\operatorname{Var}[\gamma_x] = \int_0^{h'} \operatorname{Jac}[\gamma](t)dt = \tag{4}$$

$$= \int_0^{h'} \left[\sum_{i=1}^{d-1} \left(\frac{y_i' - x_i'}{|y'-x'|} \right)^2 + \cdots + \left(\nabla\varphi_x \left(x' + t\frac{y_i' - x_i'}{|y'-x'|} \right) \cdot \left(t \frac{y_i' - x_i'}{|y'-x'|}\right) \right)^2 \right]^{\frac{1}{2}} dt$$

$$= \int_0^{h'} \left[\left| \frac{y'-x'}{|y'-x'|} \right|^2 + L \left| \frac{y'-x'}{|y'-x'|} \right|^2 \right]^{\frac{1}{2}} dt \leq \sqrt{1+L^2} h' = h.$$

Here Jac is the Jacobian of a Lipschitz mapping, see [18][pg. 101].

We take the maximum over $x \in \partial\Omega$ using (3), notice that our γ_x by the construction is an element of $\mathscr{A}_{x,y}$,

$$h_{\partial\Omega}(Y_h) = \sup_{x\in X} \inf_{y\in Y_h} \inf_{\eta\in\mathscr{A}_{x,y}} \operatorname{Var}[\eta] \leq \sup_{x\in X} \operatorname{Var}[\gamma_x] \leq h.$$

□

Now we are ready to state and prove our main result of this section, where we build an optimal mesh for a star shaped Lipschitz bounded domain having complement of positive reach (see Appendix A) by the following technique.

First, we consider the hypersurfaces given by the images of the boundary of the domain under a one parameter family of homotheties, being the parameter chosen as Chebyshev points scaled to the suitable interval; see Figure 2. We prove that this family of hypersurfaces is a norming set for the given compact set.

The second key element is that on each such hypersurface we can use a Markov Tangential Inequality (see equation (7) below) for a particular tangent ball with exponent 1.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded star-shaped Lipschitz domain such that $\complement\Omega$ has positive reach (see Definition Appendix A.1), then $K := \overline{\Omega}$ has an optimal polynomial admissible mesh.*

Proof. We can suppose without loss of generality the center of the star to be 0 by stability of AM under euclidean isometries [10].

Let us consider $a_n^0, a_n^1, \ldots, a_n^{2n}$, the set of $2n+1$ Chebyshev points[3] for the standard interval $[-1,1]$ and set $b_n^i(r) := \frac{r}{2}(1+a_n^i)$ for any $r>0$ $i = 1, 2, \ldots 2n+1$. By a well known result ([17]) the set $G_n(r)$ of all $b_n^i(r)$'s (varying the index i) is an admissible mesh of degree n and constant $\sqrt{2}$ for the interval $[0,r]$:

$$\|p\|_{[0,r]} \leq \sqrt{2}\|p\|_{G_n(r)} \quad \forall p \in \mathscr{P}^n. \tag{5}$$

Let us take any $x \in X := \partial K$ and consider the set $\tilde{G}_n(x) := xG_n(1)$, notice that $\tilde{G}_n(x) \subset K$ because K is star-shaped.

One can set $Z_n := \cup_{x\in X}\tilde{G}_n(x)$, i.e., Z_n is the union of the images of X under the homotheties having parameters a_n^i's. See Figure 2.

[3]Here $a_n^j := \cos\frac{\pi(2n-j)}{2n}$.

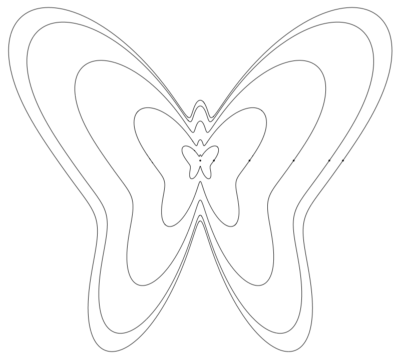

Figure 2: The geometry of Z_n.

Notice that the restriction of any polynomial of degree at most n in d variables to any segment is a univariate polynomial of degree at most n, then due to (5) we have

$$\|p\|_K \leq \sqrt{2}\|p\|_{Z_n} \ \ \forall p \in \mathscr{P}^n(\mathbb{R}^d). \tag{6}$$

Therefore we are reduced to finding an admissible polynomial mesh of degree n for Z_n, i.e., we can say that Z_n is a norming set for K (see (**??**) and below).

Let us consider any[4] Lipschitz curve $\gamma : [0, 1] \to X$, by Proposition Appendix A.1 for a.e. $s \in]0, 1[$ there exists $v \in \mathbb{S}^d$ such that

1. $B(\gamma(s) + rv, r) \subseteq K$ and
2. $\gamma'(s) \in \mathcal{T}_{\gamma(s)}\partial B(\gamma(s) + rv, r)$.

Since the boundary of the ball is a compact algebraic manifold, it admits *Markov Tangential Inequality* of degree 1 (see [9] and the references therein), moreover the constant of such an inequality is the inverse of the radius of the ball:

$$\left|\frac{\partial p}{\partial v}(x)\right| \leq \frac{|v|}{r} n\|p\|_{B(x_0,r)} \ \ \forall p \in \mathscr{P}^n(\mathbb{R}^d)\ , \forall v \in \mathcal{T}_x \partial B(x_0, r). \tag{7}$$

Hereafter $\mathcal{T}_p M$ is, as customary, the tangent space to M at $p \in M$.

[4]Notice that X is compact connected, nonempty and consists of an infinite number of points, obviously it contains an infinite number of Lipschitz curves.

Let us recall (see for instance [2][Lemma 1.1.4]) that any Lipschitz curve γ can be re-parametrized by arclength by the inversion of $t \mapsto \mathrm{Var}[\gamma|_{[0,t]}]$, obtaining a Lipschitz curve

$$\begin{aligned} \tilde{\gamma} : [0, \mathrm{Var}[\gamma]] &\rightarrow X \\ \mathrm{Var}[\tilde{\gamma}] &= \mathrm{Var}[\gamma] \\ \mathrm{Lip}[\tilde{\gamma}] &= 1 =_{\text{a.e.}} |\tilde{\gamma}'| \end{aligned}$$

Therefore (using Rademacher Theorem, see for instance [18][Th.2 pg 81]) for a.e. $s \in]0,1[$ we have

$$\left| \frac{\partial (p \circ \tilde{\gamma})}{\partial t}(t) \right| = |\nabla p(\tilde{\gamma}(t)) \cdot \tilde{\gamma}'(t)| \tag{8}$$

$$\leq \frac{|\tilde{\gamma}'(t)|n}{r} \|p\|_{B(\tilde{\gamma}(t)+rv,r)} \leq \frac{n}{r}\|p\|_K. \tag{9}$$

By Proposition 2.1 we can pick subsets $Y_{\frac{r}{2n}}$ on X such that $h_X\left(Y_{\frac{r}{2n}}\right) \leq \frac{r}{2n}$ and $\mathrm{Card}\, Y_{\frac{r}{2n}} = O(n^{d-1})$. For notational convenience we write Y_n in place of $Y_{\frac{r}{2n}}$.

Let us now pick any $x \in X$ and consider γ, an arc connecting a closest point y_n^i of Y_n to x and x itself such that $\mathrm{Var}[\gamma] \leq \frac{r}{2n}$, parametrized in the arclength.

By the Lebesgue Fundamental Theorem of Calculus for any $p \in \mathscr{P}^n(\mathbb{R}^d)$ one has

$$\begin{aligned} |p(x)| &\leq |p(y_n^i)| + \left| \int_0^{\mathrm{Var}[\gamma]} \frac{\partial (p \circ \gamma)}{\partial \xi}(\xi) d\xi \right| \\ &\leq |p(y_n^i)| + \int_0^{\mathrm{Var}[\gamma]} |\nabla p(\gamma(\xi)) \cdot \gamma'(\xi)|\, d\xi \\ &\leq |p(y_n^i)| + \int_0^{r/2n} \frac{n}{r}\|p\|_K d\xi \leq |p(y_n^i)| + \frac{1}{2}\|p\|_K \end{aligned}$$

where in the last line we used (9). Thus we have

$$\|p\|_X \leq \|p\|_{Y_n} + \frac{1}{2}\|p\|_K. \tag{10}$$

By the properties of rescaling, setting $b_n^i := b_n^i(1) = \frac{1+\cos(i\pi/n)}{2}$, we have also

$$\|p\|_{b_n^i X} \leq \|p\|_{b_n^i Y_n} + 1/2\|p\|_{b_n^i K} \leq \|p\|_{b_n^i Y_n} + \frac{1}{2}\|p\|_K,$$

for, consider the homothety $\Theta_n^i : \mathbb{R}^d \to \mathbb{R}^d$, where $\Theta_n^i(x) :?? = \frac{x}{b_n^i}$ and write the inequality (10) for each $q := p \circ \Theta_n^i$.

Therefore, taking the union over $i = 0, 1, 2n$ and using $x\tilde{G}_n = \cup_{i=0}^{m_n} b_n^i x$ and $Z_n = \cup_{x\in X} x\tilde{G}_n$, we have

$$\|p\|_{Z_n} = \|p\|_{\cup_{x\in X}(\cup_i b_n^i x)} \leq \|p\|_{\cup_i b_n^i Y_n} + \frac{1}{2}\|p\|_K.$$

Hence, setting $X_n := \cup_{i=0}^{2n} b_n^i Y_n$, we can write

$$\|p\|_{Z_n} \leq \|p\|_{X_n} + \frac{1}{2}\|p\|_K.$$

Now we can use (6) to get $\|p\|_K \leq \sqrt{2}\left(\|p\|_{X_n} + \frac{1}{2}\|p\|_K\right)$ and hence

$$\|p\|_K \leq \frac{2\sqrt{2}}{2-\sqrt{2}}\|p\|_{X_n} = 2(\sqrt{2}+1)\|p\|_{X_n}.$$

The set X_n is the disjoint union of $2n+1$ sets $b_n^i Y_n$,thus

$$\operatorname{Card} X_n = (2n+1)O(n^{d-1}) = O(n^d),$$

therefore X_n is an optimal admissible mesh of constant $2(\sqrt{2}+1)$. $\square$

This result should be compared to the recent preprint [23, Theorem 3]. The results achieved by this very new manuscript, even if they are set in a little more general context, still do not cover the case of a Lipschitz domain with complement having Positive Reach but not being $\mathscr{C}^{1,1-2/d}, d \geq 2$ globally smooth. The key element here is that inward pointing corners and cusps are allowed in our setting, while they are not in [23].

and to

Theorem 2.3 is formulated in a rather general way, here we provide two corollaries that specialize the same result.

It has been shown (see [1]) that $\mathscr{C}^{1,1}$ domains (see Appendix B.1) of $\mathbb{R}^d$ are characterized by the so called *uniform double sided ball condition*, that is, Ω is a $\mathscr{C}^{1,1}$ domain iff there exists $r > 0$ such that for any $x \in \partial\Omega$ there exist $v \in \mathbb{S}^{d-1}$ such that we have $B(x+rv, r) \subseteq \Omega$ and $B(x-rv, r) \subseteq \complement\overline{\Omega}$, this property in particular says that $\complement\Omega$ (and Ω itself) has positive reach Appendix A.1. Therefore the following is a straightforward corollary of our main result.

Corollary 2.4.1. *Let Ω be a bounded star-shaped $\mathscr{C}^{1,1}$ domain, then its closure has an optimal AM.*

It is worth recalling that such domains can also be characterized by the behavior of the oriented distance function of the boundary (i.e. $b_\Omega(x) := d(x,\Omega) - d(x,\complement\Omega)$). For any such $\mathscr{C}^{1,1}$ domain there exists a (double sided) tubular neighborhood of the boundary where the oriented distance function has the same regularity of the boundary, this condition characterizes $\mathscr{C}^{1,1}$ domains too. This framework is widely studied in [16] and [15].

In the planar case a similar result holds under slightly weaker assumptions.

Theorem 2.5 ([29]). *Let Ω be a bounded star-shaped domain in $\mathbb{R}^2$ satisfying a* Uniform Interior Ball Condition *(see Definition Appendix A.4), then $K := \overline{\Omega}$ has an optimal polynomial admissible mesh.*

A comparison of the statements of Theorem 2.3 and Theorem 2.5 reveals that actually in the second one we are dropping two assumptions, first the domain is no longer required to be Lipschitz, second we ask the weaker condition UIBC instead of complement of positive reach.

The first property is assumed to hold in the proof of the general case to make possible the construction of the geodesic mesh with a control on the asymptotics of the cardinality. In $\mathbb{R}^2$ the boundary of a bounded domain satisfying the UIBC is rectifiable [21] since, in particular, it satisfies the *cone property at the boundary.* Therefore, the geodesic mesh can be created by equally spaced (with respect to arc-length) points.

On the other hand the role of the second missing property is recovered by a deep fact in measure theory. If a set has the UIBC then then the set of points where the normal space (see Definition Appendix A.2) has dimension greater or equal to k has locally finite $d-k$ Hausdorff measure [25]. In our bi-dimensional (i.e., $d = 2$) case this result reads as follow: the normal space has dimension greater or equal to $k = 2$ on a subset having $0-$Hausdorff measure equal to 0, that is a finite set [19]. Moreover it can be proved that, apart from this small set, the *single valued* normal space is Lipschitz.

3. Optimal AM for $\mathscr{C}^{1,1}$ domains by distance function

As we mentioned above, in [22] the author conjectures that any real compact set admits an optimal AM, in this section we prove (in Theorem 3.6) that this holds at least for any real compact set K which is the the closure of a bounded $\mathscr{C}^{1,1}$ domain Ω, see Appendix B.1.

We denote by $d_{\complement\Omega}(\cdot)$ the distance function w.r.t. the complement of Ω, i.e.

$$d_{\complement\Omega}(x) := \inf_{y\in\complement\Omega} |y - x|, \tag{11}$$

and by $\mathrm{proj}_{\complement\Omega}(\cdot)$ the metric projection onto $\complement\Omega$, that is, the set of all (if any) minimizer of (11). We continue to use the same notation as in the previous section for the closure and the boundary of Ω, namely $X := \partial\Omega$ and $K := \overline{\Omega}$.

Let us give a sketch of the overall geometric construction before giving details.

First for a given $\mathscr{C}^{1,1}$ domain Ω we take $0 < \delta < 2r_\Omega$, where r_Ω is the maximum radius of the ball of the uniform interior ball condition satisfied by Ω. We stress that the double sided uniform ball condition fully characterizes $\mathscr{C}^{1,1}$ domains.

We can split $K := \overline{\Omega}$ as follows

$$\begin{aligned}
\overline{\Omega} &= K_\delta \cup \overline{\Omega^\delta} \text{ where} \\
K_\delta &:= \{x \in \Omega : d_{\complement\Omega}(x) \leq \delta\} \text{ and} \\
\Omega^\delta &= \Omega \setminus K_\delta.
\end{aligned}$$

To construct an AM of degree n on $\overline{\Omega}$ we work separately on K_δ and $\overline{\Omega^\delta}$ to obtain inequalities of the type

$$\begin{aligned}
\|p\|_{K_\delta} &\leq \|p\|_{Z_{n,\delta}} + \frac{1}{\lambda}\|p\|_K \text{ , } \lambda > 1 \text{ and} \\
\|p\|_{\Omega^\delta} &\leq 2\|p\|_{Y_{n,\delta}} + \frac{2}{\mu}\|p\|_K \text{ , } \mu > 1,
\end{aligned}$$

for $p \in \mathscr{P}^n$.

In the case of K_δ this is achieved by observing that $x \in K_\delta$ implies $\overline{B(x,\delta)} \subseteq \overline{\Omega}$ and therefore one can bound using the univariate Bernstein Inequality (see Theorem 3.1 below) any directional derivative of a given polynomial. The obtained inequality is a variant of a Markov Inequality with exponent 1 which is convenient and allow us to build a low cardinality mesh by a modification of the reasoning in [14].

The construction of an AM on $\overline{\Omega^\delta}$ is more complicated. The resulting mesh is given by points lining on some properly chosen level surfaces of $d_{\complement\Omega}$. The result is proved using the regularity property of the function $d_{\complement\Omega}$ in a small tubular neighborhood of X and the Markov Tangential Inequality for the sphere.

3.1. Bernstein-like Inequalities and polynomial estimates via the distance function.

For the reader's convenience we recall here the Bernstein Inequality.

Theorem 3.1 (Bernstein Inequality). *Let $p \in \mathscr{P}^n(\mathbb{R})$, then for any $a < b \in \mathbb{R}$ we have*

$$|p'(x)| \leq \frac{n}{\sqrt{(x-a)(b-x)}} \|p\|_{[a,b]}, \ x \in]a,b[. \tag{12}$$

Let us introduce the following notation illustrated in figure 3.

$$l(x) := \min_{y \in \mathrm{proj}_{\complement\Omega}(x)} \inf \left\{ \lambda > 0 : y + \lambda \frac{x-y}{|x-y|} \notin \Omega \right\} \quad x \in \Omega \tag{13}$$

$$l_\Omega := \inf_{x \in \Omega} l(x). \tag{14}$$

Remark 3.2. In the case when Ω is a $\mathscr{C}^{1,1}$ domain one has the estimate $l_\Omega \geq 2r$ where $r < \mathrm{Reach}(\partial\Omega)$ see Definition Appendix A.1 and thereafter.

The following consequence of *Bernstein Inequality* will play a central role in our construction.

Proposition 3.1. *Let Ω be a bounded domain in $\mathbb{R}^d$ and let us introduce the sequence of functions*

$$\varphi_n(x) := \begin{cases} \frac{n}{\sqrt{d_{\complement\Omega}(x)(l_\Omega - d_{\complement\Omega}(x))}}, & \text{if } d_{\complement\Omega}(x) < l_\Omega \\ \frac{n}{d_{\complement\Omega}(x)}, & \text{otherwise} \end{cases}. \tag{15}$$

For any $x \in \Omega$ let $v \in \{\frac{x-y}{|x-y|} : y \in \mathrm{proj}_{\complement\Omega}(x)\}$, then for any $p \in \mathscr{P}^n(\mathbb{R}^d)$ we have

$$|\partial_v p(x)| \leq \varphi_n(x) \|p\|_K. \tag{16}$$

If moreover we have $l_\Omega > 0$, let us pick any $0 < \delta < l_\Omega$ and define the sequence of functions

$$\varphi_{n,\delta}(x) := \begin{cases} \frac{n}{\sqrt{d_{\complement\Omega}(x)(\delta - d_{\complement\Omega}(x))}}, & \text{if } d_{\complement\Omega}(x) < \delta \\ \frac{n}{d_{\complement\Omega}(x)}, & \text{otherwise} \end{cases}. \tag{17}$$

Then the above polynomial estimate (16) *still holds when $\varphi_{n,\delta}$ is substituted by φ_n.*

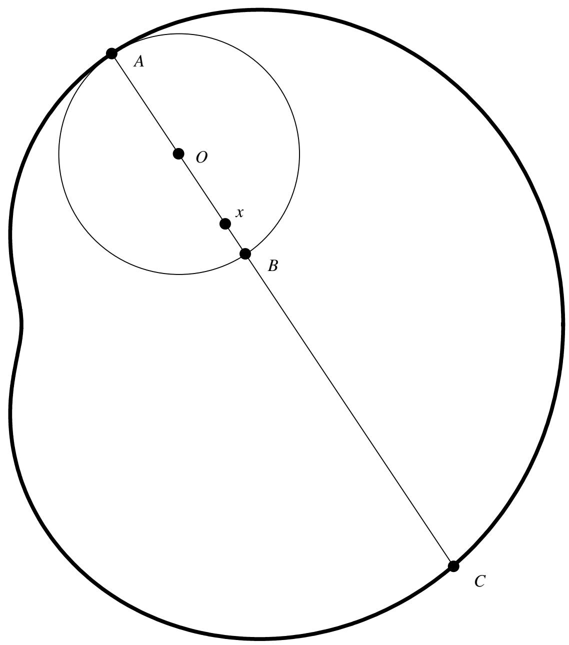

Figure 3: Here $A := \mathrm{proj}_{\complement\Omega}(x)$ and $l(x) = |A - C| \geq |A - B| = 2r$ is the length of the shortest segment inside Ω containing x and having direction $\frac{x-\mathrm{proj}_{\complement\Omega}(x)}{|x-\mathrm{proj}_{\complement\Omega}(x)|}$.

Proof. Pick $p \in \mathscr{P}^n(\mathbb{R}^d)$. Let us take $x \in \Omega$ such that $d_{\complement\Omega}(x) < l_\Omega$. We denoted by $S_v(x)$ the segment $x + [-d_{\complement\Omega}(x), l_\Omega - d_{\complement\Omega}(x)]v$, where v is as above and $x \in S_v(x)$ due to $d_{\complement\Omega}(x) < l_\Omega$. The restriction of p to this segment is an univariate polynomial $q(\xi) := p(x + v\xi)$ of degree not exceeding n, then we can use the Bernstein Inequality 3.1 to get

$$\left|\frac{\partial q}{\partial \xi}(\xi)\right| \leq \frac{n}{\sqrt{(\xi + d_{\complement\Omega}(x))(l_\Omega - d_{\complement\Omega}(x) - \xi)}}\|p\|_{S_v(x)},$$

evaluating at $\xi = 0$ we get

$$|\partial_v p(x)| \leq \frac{n\|p\|_{S_v(x)}}{\sqrt{d_{\complement\Omega}(x)(l_\Omega - d_{\complement\Omega}(x))}} \leq \frac{n\|p\|_K}{\sqrt{d_{\complement\Omega}(x)(l_\Omega - d_{\complement\Omega}(x))}}, \tag{18}$$

thus establishing the first case of (17).

Let x be such that $d_{\complement\Omega}(x) \geq l_\Omega$. Notice that $B(x, d_{\complement\Omega}(x)) \subseteq \Omega$ and hence $\forall \eta \in \mathbb{S}^{d-1}$ (the standard unit $d-1$ dimensional sphere) we can pick a segment in the direction of η having length $d_{\complement\Omega}(x)$ lying in K and having x as midpoint. The Bernstein Inequality gives

$$|\partial_v p(x)| \leq \max_{\eta\in\mathbb{S}^{d-1}} |\partial_\eta p(x)| \leq \frac{n}{d_{\complement\Omega}(x)}\|p\|_{B(x,d_{\complement\Omega}(x))} \leq \frac{n}{d_{\complement\Omega}(x)}\|p\|_K. \tag{19}$$

The last statement follows directly by the special choice of $\delta < l_\Omega$. The right hand side in (17) dominates (case by case) the r.h.s. in (15) when cases are chosen accordingly to (17). □

Actually the above proof proves also the following corollary, it suffices to take (17) and substitute $\frac{n}{d_{\complement\Omega}(x)}$ by $\frac{n}{\delta}$ in the second case.

Corollary 3.3.1. *Let Ω be an open bounded domain and δ a positive number such that $K_\delta := \{x \in \Omega : d_{\complement\Omega}(x) \geq \delta\} \neq \emptyset$. Then for any $v \in \mathbb{S}^{d-1}$ we have $\forall p \in \mathscr{P}^n(\mathbb{R}^d)$*

$$\|\partial_v p\|_{K_\delta} \leq \frac{n}{\delta}\|p\|_K. \tag{20}$$

A profitable technique in order to build an AM is to find a norming subset N of the given compact, then try to build an AM for N. We introduce the following in the spirit of [31].

Let us denote by $ds(\cdot)$ the standard length measure in $\mathbb{R}^d$.

Proposition 3.2. *Let Ω be a bounded domain in $\mathbb{R}^d$ such that $l_\Omega > 0$ and let $0 < \delta \leq l_\Omega$. Then*

(i) for any $x \in \Omega$ the map

$$\operatorname*{proj}_{\complement\Omega}(x) \ni y \mapsto \int_{[y,x]} \varphi_{n,\delta}(\xi)ds(\xi)$$

is constant, let $F_{n,\delta}(x)$ be its value.

(ii) We have

$$F_{n,\delta}(x) = \begin{cases} n\arccos(1 - \frac{2d_{\complement\Omega}(x)}{\delta}), & \textit{if } d_{\complement\Omega}(x) < \delta \\ n\left(\pi + \ln\frac{d_{\complement\Omega}(x)}{\delta}\right), & \textit{otherwise.}\end{cases} \tag{21}$$

In particular $F_{n,\delta}$ extends continuously to $\overline{\Omega}$.

(iii) $F_{n,\delta}$ *is constant on any level set of* $d_{\complement\Omega}(\cdot)$ *and* $\sup_{\Omega\setminus K_\delta} F_{n,\delta} = n\pi$.

Let us set $a^i_{n,\delta} := \frac{in\pi}{m_n}$ *where* $i = 0, 1, \dots m_n$ *and* m_n *is any positive integer greater than* $2n\pi$, *we denote by* $\Gamma^i_{n,\delta}$ *the* $a^i_{n,\delta}$*-level set of* $F_{n,\delta}$.

(iv) *We have*

$$\begin{aligned} \Gamma^i_{n,\delta} &= \{x \in K : d_{\complement\Omega}(x) = d^i_{n,\delta}\} \text{ , where} \\ d^i_{n,\delta} &:= \frac{\delta}{2}\left(1 - \cos\left(\frac{i\pi}{m_n}\right)\right). \end{aligned}$$

(v) *Let* $\Gamma_{n,\delta} := \cup_{i=0}^{m_n} \Gamma^i_{n,\delta}$, *then for any* $p \in \mathscr{P}^n(\mathbb{R}^d)$ *we have*

$$\|p\|_K \le \max\{2\|p\|_{\Gamma_{n,\delta}}, \|p\|_{K_\delta}\}. \tag{22}$$

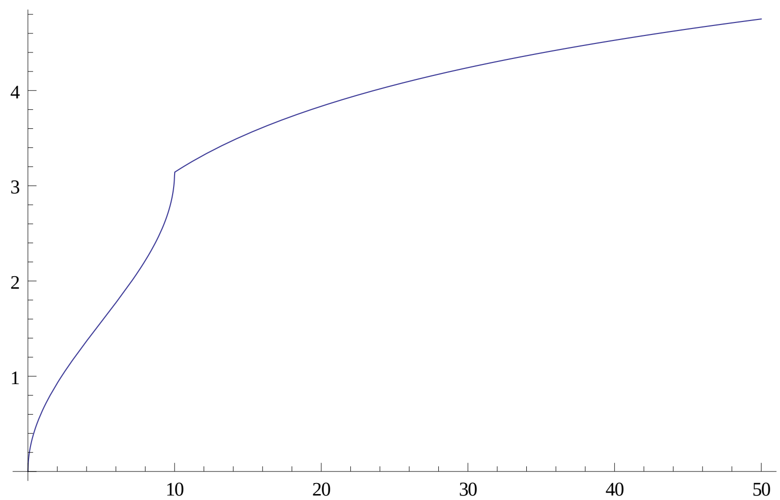

Figure 4: A plot of a section of $F_{n,\delta}$ along a segment of metric projection, where $\delta = 10$, $n = 1$. Abscissa here is the distance from the boundary.

PROOF. (i) The function $\varphi_{n,\delta}(\cdot)$ depends on its argument only by the distance function, $\varphi_{n,\delta}(x) =: g_{n,\delta}(d_{\complement\Omega}(x))$. The length of the segment $[y, x]$ is clearly constant when y varies in the set $\text{proj}_{\complement\Omega}(x)$.

Moreover for any $y, z \in \mathrm{proj}_{\complement\Omega}(x)$ let us denote by $R_{y,z}$ an euclidean isometry that maps $[y,x]$ onto $[z,x]$, one trivially has $d_{\complement\Omega}(\xi) = d_{\complement\Omega}(R_{y,z}\xi)$ for any $\xi \in [y,x]$. This is because $\mathrm{proj}_{\complement\Omega}(\xi) \ni y$ for any $\xi \in [x,y]$ by the Triangle Inequality and thus $d_{\complement\Omega}(\xi) = |\xi - y|$.

Thus we have

$$\begin{aligned} & \int_{[y,x]} \varphi_{n,\delta}(\xi) ds(\xi) = \int_{[y,x]} g_{n,\delta}(d_{\complement\Omega}(\xi)) ds(\xi) \\ = & \int_{[y,x]} g_{n,\delta}(d_{\complement\Omega}(R_{y,z}\xi)) ds(\xi) = \int_0^1 g_{n,\delta}\left(d_{\complement\Omega}\left(R_{y,z}\left(y + t\frac{x-y}{|x-y|}\right)\right)\right) dt \\ = & \int_0^1 g_{n,\delta}\left(d_{\complement\Omega}\left(z + t\frac{z-x}{|z-x|}\right)\right) dt = \int_{[z,x]} \varphi_{n,\delta}(\eta) ds(\eta). \end{aligned}$$

(ii) Let us parametrize the segment as $y + s\frac{x-y}{|x-y|}$, then we have

$$F_{n,\delta}(x) = \begin{cases} \int_0^{d_{\complement\Omega}(x)} \frac{n}{\sqrt{s(\delta - s)}} ds, & \text{if } d_{\complement\Omega}(x) < \delta \\ \int_0^{\delta} \frac{n}{\sqrt{s(\delta - s)}} ds + \int_\delta^{d_{\complement\Omega}(x)} \frac{n}{s} ds \quad , & \text{otherwise.} \end{cases} \tag{23}$$

The first integral can be solved by substitution: $s = \frac{\delta}{2}(1 - \cos\theta)$. The integration domain becomes $[0, \theta_x]$ where $\frac{\delta}{2}(1 - \cos(\theta_x)) = d_{\complement\Omega}(x)$, while the integral itself becomes $\int_0^{\theta_x} d\theta = \theta_x$, thus the first case in (21) is proven.

The second integral has an immediate primitive. $F_{n,\delta}$ depends on x only by the distance function, moreover we notice that

$$\lim_{s\to\delta^-} \arccos\left(1 - \frac{2s}{\delta}\right) = \pi = \lim_{s\to\delta^+}\left(\pi + \ln\frac{s}{\delta}\right),$$

hence $F_{n,\delta}$ is a continuous function of the distance function. Since $d_{\complement\Omega}$ is well known to be $1-$Lipschitz $F_{n,\delta}$ is continuous on Ω.

Since $d_{\complement\Omega}$ extends continuously to $\overline{\Omega}$, then $F_{n,\delta}$ does. Actually we must take $F_{n,\delta}|_{\partial\Omega} \equiv 0$.

(iii) We already used that $F_{n,\delta}$ depends on x only by the distance function and hence $F_{n,\delta}|_{d^{\leftarrow}_{\complement\Omega}(a)} = \text{constant}$[5], moreover the functions $\arccos\left(1 - \frac{2s}{\delta}\right)$ and

[5] We denote by $f^{\leftarrow}(a)$ the inverse image under $f : D \to \mathbb{R}$ of the number $a \in \mathrm{Range}[f]$, i.e., $\{x \in D : f(x) = a\}$ that, in general, is a set.

$\left(\pi + \ln \frac{s}{\delta}\right)$ are both increasing in $[0, \max_{x\in\overline{\Omega}} d_{\complement\Omega}(x)]$, see Figure 4, hence any level set of $F_{n,\delta}$ must coincide with a suitable level set of the distance function.

(iv) The conclusion follows immediately by inverting the equation

$$n \arccos\left(1 - \frac{2d^i_{n,\delta}}{\delta}\right) = a^i_{n,\delta}.$$

(v) Let $p \in \mathscr{P}^n(\mathbb{R}^d)$ be fixed, let us pick $x \in K$, then two possibilities can occur. In the first case $x \in K_\delta$. In this case we have $|p(x)| \leq \|p\|_{K_\delta}$. In the second we suppose $x \notin K_\delta$, let us consider $y \in \mathrm{proj}_{\complement\Omega}(x)$. The segment $[y, x]$ cuts $\Gamma^i_{n,\delta}$ for every i such that $d^i_{n,\delta} \leq d_{\complement\Omega}(x)$, moreover $[y, x] \cap \Gamma^i_{n,\delta} = \{y^i\}$, due to the monotonicity of $F_{n,\delta}$ along any segment where $d_{\complement\Omega}$ is monotone.

Let $i(x) := \max\{i : d^i_{n,\delta} \leq d_{\complement\Omega}(x)\}$ and let $y^{i(x)+1}$ be the unique intersection of $\Gamma^{i(x)+1}_{n,\delta}$ and the ray starting from x and having direction $\frac{x-y}{|x-y|}$.

Let $s(\cdot)$ be the arc length parametrization of the segment $[y^{i(x)}, y^{i(x)+1}]$ now we have

$$\begin{aligned}
|p(x)| &\leq |p(y^{i(x)})| + \int_0^{s^{-1}(x)} \left|\frac{\partial(p\circ s)}{\partial t}(t)\right| dt \\
&\leq |p(y^{i(x)})| + \int_0^{1} \left|\frac{\partial(p\circ s)}{\partial t}(t)\right| dt \\
&= |p(y^{i(x)})| + \int_0^{1} \|p\|_K \varphi_{n,\delta}(s(t))dt \\
&= |p(y^{i(x)})| + \int_{[y^{i(x)}, y^{i(x)+1}]} \|p\|_K \varphi_{n,\delta}(\xi)ds(\xi) \\
&\leq |p(y^{i(x)})| + \frac{\|p\|_K}{m_n} \int_{[y^0, y^{m_n}]} \varphi_{n,\delta}(\xi)ds(\xi) \\
&\leq \|p\|_{\Gamma^{i(x)}_{n,\delta}} + \frac{F_{n,\delta}(y^{m_n})}{m_n}\|p\|_K \leq \|p\|_{\Gamma^{i(x)}_{n,\delta}} + \frac{1}{2}\|p\|_K,
\end{aligned}$$

where we used (16) in the third line while the special choice of $a^i_{n,\delta}$ (and thus y^i) as equally spaced points in the image of $F_{n,\delta}$ and the choice of $m_n > 2n\pi$ has been used in the last two lines.

To conclude we take the maximum of the above estimates w.r.t. $x \in K$ thus letting i varying among $0, 1, \ldots, m_n - 1$ and considering both cases $x \in K_\delta$ and $x \notin K_\delta$. $\square$

Proposition 3.3. *Let Ω be a bounded $\mathscr{C}^{1,1}$ domain, $0 < r < \mathrm{Reach}(\partial\Omega)$ $0 < \delta \leq r$ and let $m_n > 2n\pi$, then*

(i) For any $i = 1, \ldots m_n$ $\Gamma^i_{n,\delta}$ is a $\mathscr{C}^{1,1}$ hypersurface.

(ii) For any $p \in \mathscr{P}^n(\mathbb{R}^d)$ any $x \in \Gamma^i_{n,\delta}$ and any $v \in \mathbb{S}^{d-1} \cap \mathcal{T}_x\Gamma^i_{n,\delta}$ where $i = 0, 1, \ldots, m_n$ we have

$$|\partial_v p(x)| \leq \begin{cases} \frac{n}{\delta}\|p\|_K & i = 0 \\ \frac{2n}{\delta}\|p\|_K & i = 1, 2, \ldots, m_n \end{cases}. \tag{24}$$

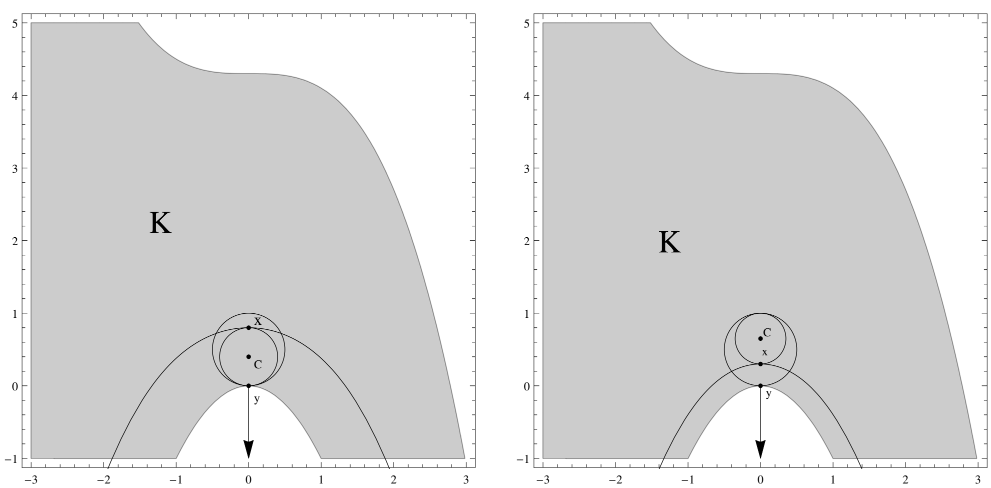

Figure 5: Different situations occurring in the proof of Proposition 3.3 (*ii*). On the left side the tangent ball at x is chosen outward and on the left side inward, this corresponds respectively to the first and the second case in (26). The arrow represent $\nabla b_\Omega(x)$.

PROOF. (i) Notice that we have, due to Appendix B.2,

$$0 < \min\{\mathrm{Reach}(\Omega), \mathrm{Reach}(\complement\Omega)\} = \mathrm{Reach}(\partial\Omega).$$

If $i > 0$ due to (B.1) and Theorem Appendix B.2. We have $\forall x \in \Gamma^i_{n,\delta}$

$$\nabla d_{\complement\Omega}(x) = -\nabla b_\Omega(x) = \frac{x - \mathrm{proj}_{\partial\Omega}(x)}{|x - \mathrm{proj}_{\partial\Omega}(x)|},$$

moreover this is a Lipschitz function when restricted to $\{|b_\Omega(x)| < \delta\}$ for any $0 < \delta < \min\{\text{Reach}(\Omega), \text{Reach}(\complement\Omega)\}$.

Also we have $b_\Omega|_{\overline{\Omega}} \equiv -d_{\complement\Omega}$.

We notice that $\nabla d_{\complement\Omega}(x) \neq 0$, therefore any level-set of $d_{\complement\Omega}$ contained in $\Omega \setminus K_\delta$ is a $\mathscr{C}^{1,1}$ $d-1$ dimensional manifold by the Implicit Function Theorem.

(ii) If $i = 0$ Theorem Appendix B.2 tells that for any x in $\Gamma^i_{n,\delta}$ we have $B_x := B(x + \delta\nabla b_\Omega(x), \delta) \subseteq \Omega$, (cfr. figure 5 point $C1$) moreover $\mathcal{T}_x\Gamma^i_{n,\delta} = \mathcal{T}_x\partial B_x$. Therefore we can apply the Markov Tangential Inequality to the ball B_x : for any polynomial $p \in \mathscr{P}^n$ and any $u \in \mathcal{T}_x\Gamma^i_{n,\delta} = \mathcal{T}_x\partial B_x$ we have

$$|\partial_u p(x)| \leq \frac{n}{\delta}\|p\|_{B_x} \leq \frac{n}{\delta}\|p\|_K. \tag{25}$$

Where the last inequality follows from $\overline{B}_x \subseteq K$.

Now we focus on $i > 0$. Let us take $x \in \Gamma^i_{n,\delta}$, then $y = \text{proj}_{\complement\Omega}(x) \Rightarrow \nabla b_\Omega(y) = \nabla b_\Omega(x)$ and hence we have $\mathcal{T}_x\Gamma^i_{n,\delta} = \mathcal{T}_y X, i = 0, 1, \dots, m_n$

Moreover we notice that

$$B^i_x := \begin{cases} B\left(y + \frac{d^i_{n,\delta}}{2}\nabla b_\Omega(x), \frac{d^i_{n,\delta}}{2}\right) \subset \Omega & d^i_{n,\delta} \geq \delta/2 \\ B\left(y + (d^i_{n,\delta} + \frac{2\delta - d^i_{n,\delta}}{2})\nabla b_\Omega(x), \frac{2\delta - d^i_{n,\delta}}{2}\right) \subset \Omega & d^i_{n,\delta} < \delta/2. \end{cases} \tag{26}$$

$$\mathcal{T}_x\Gamma^i_{n,\delta} = \mathcal{T}_x B^i_x. \tag{27}$$

This can be figured out by looking at in Figure 5 where the first occurrence is represented on the left and the second on the right.

Now we notice that the radius of B^i_x can be bounded below uniformly in i by $\delta/2$. Therefore The Markov Tangential Inequality for the ball gives us the following $\forall p \in \mathscr{P}^n$ and $\forall v \in \mathcal{T}_x\Gamma^i_{n,\delta}$, $|v| = 1$ we have

$$|\partial_v p(x)| \leq \frac{n}{\delta/2}\|p\|_{B^i_x}.$$

Now due to $\mathcal{T}_x\Gamma^i_{n,\delta} = \mathcal{T}_x B^i_x$ and $B^i_x \subset \Omega$ we have $\forall p \in \mathscr{P}^n, v \in \mathcal{T}_x\Gamma^i_{n,\delta}, |v| = 1, \forall i = 0, 1, m_n$

$$|\partial_v p(x)| \leq \frac{n}{\delta/2}\|p\|_K.$$

□

3.2. Proof of the main result

We developed all required tools to state and prove the main result of this paper, Theorem 3.6. The idea of its constructive proof is mixing the technique of Theorem 2.3 with an improvement of the one being used in [14][Th. 5]. More precisely the hypersurfaces Z_n of Theorem 2.3 here are replaced by the level sets $\Gamma^i_{n,\delta}$ which together with the set $K_\delta = \{x \in K : d_{\complement\Omega}(x) \geq \delta\}$ are shown to form a norming set for K.

Theorem 3.6. *Let Ω be a bounded $\mathscr{C}^{1,1}$ domain in $\mathbb{R}^d$, then there exists an optimal admissible mesh for $K := \overline{\Omega}$.*

PROOF. Notice that we have $0 < \min\{\text{Reach}(\Omega), \text{Reach}(\complement\Omega)\} = \text{Reach}\,\partial\Omega$ due to Appendix B.2 we fix $\delta \leq r < \text{Reach}\,\partial\Omega$

Let us recall the above notation

$$\begin{aligned}
K_\delta &:= \{x \in K : d_{\complement\Omega}(x) \geq \delta\},\\
\Gamma_{n,\delta} &:= \cup_i \Gamma^i_{n,\delta} \text{ where}\\
\Gamma^i_{n,\delta} &:= \{x \in K : d_{\complement\Omega}(x) = d^i_{n,\delta}\},\\
d^i_{n,\delta} &:= \frac{\delta}{2}\left(1 - \cos\left(\frac{i\pi}{m_n}\right)\right) \text{ , where we can take}\\
m_n &:= \lceil 2n\pi \rceil + 1.
\end{aligned}$$

Let $p \in \mathscr{P}^n(\mathbb{R}^d)$.

• **Claim 1.** *For any $\lambda > 1$ there exists $Z_{n,\delta,\lambda} \subset K_\delta$ such that*

$$\|p\|_{K_\delta} \leq \|p\|_{Z_{n,\delta,\lambda}} + \frac{1}{\lambda}\|p\|_K \text{ and} \tag{28}$$

$$\text{Card}\, Z_{n,\delta,\lambda} = O(n^d). \tag{29}$$

• **Proof of Claim 1.** Let us consider for any $\lambda > 1$ a mesh $Z_{n,\delta,\lambda}$ such that its fill distance

$$h(Z_{n,\delta,\lambda}) \leq \frac{\delta}{\lambda n + 1/2} =: h \text{ , see (2).}$$

Let us define $Z_{n,\delta,\lambda} \subset K_\delta$ as the intersection of K with a grid G with a step-size $\frac{h}{\sqrt{d}}$ on a suitable d dimensional cube containing K. It follows that $\text{Card}(Z_{n,\delta,\lambda}) = \left(\frac{\sqrt{d}}{h}\right)^d = O(n^d)$.

Now pick any $x \in K_\delta$ and find $y \in Z_{n,\delta,\lambda}$ such that $|x-y| \le h$ and define $v := \frac{x-y}{|x-y|}$ and notice that

$$\begin{aligned} & |p(x)| \\ \le \ & |p(y)| + \left| \int_0^{|x-y|} \partial_v p(x+sv) ds \right| \le \|p\|_{Z_{n,\delta,\lambda}} + |x-y| \|p\|_{[x,y]} \\ \le \ & \|p\|_{Z_{n,\delta,\lambda}} + \|\partial_v p\|_{B(K_\delta, h/2)}. \end{aligned}$$

Where we used $\min_{\xi \in [x,y]} \operatorname{dist}(\xi, K_\delta) \ge h/2$ due to the Triangle Inequality for the euclidean distance $\operatorname{dist}(\cdot, K_\delta)$ from K_δ.

By the observation $B(K_\delta, h/2) \subseteq K_{\delta - h/2}$ we can apply inequality (20) where δ is replaced by $\delta - h/2$.

$$|p(x)| \le |p(y)| + h \frac{n}{\delta - h/2} \|p\|_K$$

Taking maximum over $x \in K_\delta$ and using the particular choice $h := \frac{\delta}{\lambda n + 1/2}$ we are done.

• **Claim 2.** *For any $2 < \mu$ there exist finite sets $Y_{n,\delta}^i \subset \Gamma_{n,\delta}^i$, $i = 0, 1, ..m_n$, such that if we set $Y_{n,\delta} := \cup_i Y_{n,\delta}^i$ we get*

$$\begin{aligned} \|p\|_{\cup_i \Gamma_{n,\delta}^i} &\le \|p\|_{Y_{n,\delta}} + \frac{1}{\mu} \|p\|_K \text{ and} && (30) \\ \operatorname{Card} Y_{n,\delta} &= O(n^d). && (31) \end{aligned}$$

•**Proof of Claim 2.** Let us pick $Y_{n,\delta}^i \subset \Gamma_{n,\delta}^i$ such that

$$h_{\Gamma_{n,\delta}^i}(Y_{n,\delta}^i) \le \begin{cases} \frac{\delta}{\mu n} & i = 0 \\ \frac{\delta}{2\mu n} & i = 1, 2, \ldots, m_n \end{cases} \quad \text{(see Definition 3).} \qquad (32)$$

Now fix any $i \in \{0, 1, \ldots, m_n\}$, by (32) for any $x \in \Gamma_{n,\delta}^i$ there exist a point $y \in Y_{n,\delta}^i$ and a Lipschitz curve[6] γ lying in $\Gamma_{n,\delta}^i$, connecting x to y and such that $\operatorname{Var}[\gamma] \le h_{\Gamma_{n,\delta}^i}(Y_{n,\delta})$. Let us denote the arclength reparametrization of

[6]Notice that $\Gamma_{n,\delta}^i$ are compact $\mathscr{C}^{1,1}$ hypersurfaces, thus in particular they are locally complete with respect the geodesic distance. Therefore there exists a curve γ realizing the infimum in the definition of geodesic fill distance.

γ by $\tilde{\gamma}$, then we have

$$\begin{aligned}
|p(x)| &\leq |p(y)| + \int_0^{\mathrm{Var}[\gamma]} \frac{d(p\circ\tilde{\gamma})}{dt}(t)dt \\
&\leq \|p\|_{Y^i_{n,\delta}} + h_{\Gamma^i_{n,\delta}}(Y_{n,\delta}) \max_{\xi\in\Gamma_{n,\delta}, v\in\mathbb{S}^{d-1}\cap\mathcal{T}_\xi\Gamma^i_{n,\delta}} |\partial_v p(\xi)| \\
&\leq \|p\|_{Y^i_{n,\delta}} + \frac{1}{\mu}\|p\|_K.
\end{aligned}$$

Here, in the 3rd line, we used the inequality (24). Let us take the maximum w.r.t. x varying in $\Gamma^i_{n,\delta}$ and i varying over $\{0,1,\dots,m_n\}$, we obtain $\|p\|_{\Gamma_{n,\delta}} \leq \|p\|_{Y_{n,\delta}} + \frac{1}{\mu}\|p\|_K$.

We are left to prove that we can pick $Y^i_{n,\delta}$ such that $\mathrm{Card}(Y_{n,\delta}) = O(n^d)$.

When $i=0$ Proposition 2.1 ensures (X is a $\mathscr{C}^{1,1}$ hypersurface and a fortiori is Lipshitz) the existence of such an $Y^0_{n,\delta}$ with $h_{\Gamma^0_{n,\delta}}(Y^0_{n,\delta}) \leq \frac{\delta}{\mu n}$ and $\mathrm{Card}(Y^0_{n,\delta}) = O(n^{d-1})$. Let us study the case $i>0$.

Now let us notice that by *(v)* in Theorem Appendix B.2 one has $\mathrm{proj}_{\partial\Omega}|_{b_\Omega=\rho}$ is an injective function for any $0<\rho<\mathrm{Reach}(\partial\Omega)$. Since ∇b_Ω constant along metric projections we can also notice that $\nabla b_\Omega(x) = \nabla b_\Omega(\mathrm{proj}_{\partial\Omega}(x))$. Moreover by *(iii)* in Theorem Appendix B.2 if $x\in\Gamma^i_{n,\delta}$, $y=\mathrm{proj}_{\complement\Omega(x)}$ then

$$\begin{aligned}
y = \mathrm{proj}_{\complement\Omega}(x) &= x - |x - \mathrm{proj}_{\complement\Omega}(x)|\nabla b_\Omega(x) \\
&= x - d^i_{n,\delta}\nabla b_\Omega(x) = x - d^i_{n,\delta}\nabla b_\Omega(\mathrm{proj}_{\partial\Omega}(y)) \\
&= x - d^i_{n,\delta}\nabla b_\Omega(y).
\end{aligned}$$

Thus we can introduce the family of inverse maps $f_i := \left(\mathrm{proj}_{\complement\Omega}|_{\Gamma^i_{n,\delta}}\right)^{-1}$

$$\begin{aligned}
f_i : \Gamma^0_{n,\delta} &\longrightarrow \Gamma^i_{n,\delta} \\
x &\longmapsto x + d^i_{n,\delta}\nabla b_\Omega(x).
\end{aligned}$$

Notice that $\nabla b_\Omega|_{\partial\Omega}$ is a Lipschitz function, see Theorem Appendix B.2 *(iii)*. Let us denote L its Lipschitz constant.

Therefore $\{f_i\}_{i=1,2,\dots,m_n}$ is a family of equi-continuous functions of Lipschitz constant

$$\max_{i=1,2,\dots,m_n}(1+Ld^i_{n,\delta}) \leq (1+L\delta).$$

Now the *Area Formula* says that f_i (being $1+L\delta$ Lipschitz) maps a mesh of $\Gamma^0_{n,\delta}$ with geodesic fill distance $\frac{h}{1+\delta L}$ onto a mesh in $\Gamma^i_{n,\delta}$ having geodesic fill distance bounded by h. We already used this property and explained its application in more detail in the proof of Theorem 2.3, see (4) and thereafter.

Thanks to Proposition 2.1 we can pick the mesh $\tilde{Y}^i_{n,\delta} \subset \Gamma^0_{n,\delta}$ such that $h_{\Gamma^0_{n,\delta}}(\tilde{Y}^i_{n,\delta}) \leq \frac{\delta}{2\mu n(1+\delta L)}$ with the cardinality bound $\mathrm{Card}(\tilde{Y}^i_{n,\delta}) = O((\frac{n}{h})^{d-1})$ where we denote $\frac{\delta}{2\mu(1+\delta L)}$ by h. Let us set $Y^i_{n,\delta} := \{f_i(y), y \in \tilde{Y}^i_{n,\delta}\}$. Now we can notice that

$$\mathrm{Card}(Y_{n,\delta}) = \sum_{i=0}^{m_n} \mathrm{Card}\, Y^i_{n,\delta} = n^{d-1} + \sum_{i=1}^{m_n} O\left(\left(\frac{n}{h}\right)^{d-1}\right) = O(n^d).$$

• **Claim 3:** $A_{n,\delta} := Y_{n,\delta} \cup Z_{n,\delta,\lambda}$ *is an optimal admissible mesh for* K.

• **Proof of Claim 3.** By the special choice of $\delta < r \leq l_\Omega/2$ we can use jointly (22), (28) and(30) and we obtain

$$\|p\|_K \leq \max\{2\|p\|_{Y_{n,\delta}} + 2\frac{1}{\mu}\|p\|_K \ , \ \|p\|_{Z_{n,\delta,\lambda}} + \frac{1}{\lambda}\|p\|_K\}.$$

By the elementary properties of max we have

$$\|p\|_K \leq \max\{\frac{2\mu}{\mu-2}\, , \, \frac{1}{\lambda-1}\}\|p\|_{Y_{n,\delta}\cup Z_{n,\delta,\lambda}}. \tag{33}$$

Thus $Y_{n,\delta} \cup Z_{n,\delta,\lambda} =: A_{n,\delta}$ satisfies

$$\|p\|_K \leq C(\delta,\lambda,\mu)\|p\|_{A_{n,\delta}}\ \forall p \in \mathscr{P}^n(\mathbb{R}^d)\ \forall n \in \mathbb{N} \tag{34}$$

has the correct cardinality order of growth.

□

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Appendix A. Sets of positive reach

Here we provide very concisely some essential tools that we use in the proofs of the paper. Of course we do not even try to be exhaustive, since this is far from our aim.

We deal with Federer sets of positive reach, they were introduced in the outstanding article [19].

Definition Appendix A.1 (Reach of a Set). [19] Let $A \subset \mathbb{R}^d$ be any set, we denote by $\mathrm{proj}_A(x) = \{y \in A : |y - x| = d_A(x)\}$ the metric projection onto A, where we denoted by $d_A(x) := \inf_{y\in A} |x-y|$. Moreover let $Unp(A) := \{x \in \mathbb{R}^d : \exists! y \in A,\ \mathrm{proj}_A(x) = \{y\}\}$. Then we define

$$\mathrm{Reach}(A, a) := \sup_{r>0}\{r : B(a, r) \subseteq Unp(A)\} \text{ for any } a \in A, \tag{A.1}$$

$$\mathrm{Reach}(A) := \inf_{a\in A} \mathrm{Reach}(A, a). \tag{A.2}$$

The set A is said to be a set of positive reach if $\mathrm{Reach}(A) > 0$.

By this definition sets of reach $r > 0$ are precisely the subsets of $\mathbb{R}^d$ for which there exists a tubular neighborhood of radius r where the metric projection is unique and moreover this tubular neighborhood is maximal.

This class of sets was introduced by Federer in the study of *Steiner Polynomial* relative to a (very smooth) set, the polynomial that computed at $r > 0$ gives the d-dimensional measure of the r tubular neighborhood of the given set. The main interest on such a class of sets is that under this assumption (in place of high degree of smoothness) one can recover the coefficients of Steiner Polynomial as Radon measures, the Curvature Measures.

Sets with positive reach may be seen as a generalization of $\mathscr{C}^{1,1}$ bounded domains, in fact the latter can be characterized as domains such that the boundary has positive reach, a more restrictive condition. Moreover if Ω is a domain having positive reach it can be shown that the subset of $\partial\Omega$ where the distance function defines uniquely a normal vector field (as for $\mathscr{C}^{1,1}$ domains) is "big" in the right measure theoretic sense.

However from our point of view the most relevant feature of sets of positive reach is the one concerning the regularity properties of the distance function $d_A(\cdot)$. They can be found in [19][Section 4]. If A has positive reach then $d_A(\cdot)$ is differentiable at any point of $\mathbb{R}^d \setminus A$ having unique projection and we have $\nabla d_A(x) = \frac{x-\mathrm{proj}_A(x)}{d_A(x)}$ and this is a Lipschitz function in any set of the type $\{x : 0 < s \le d_A(x) \le r < \mathrm{Reach}(A)\}$.

In the sequel of the paper we need to use a little of tangential calculus on non-smooth structures, so we introduce the following.

Definition Appendix A.2 (Tangent and Normal). Let $A \subset \mathbb{R}^d$ be any set. Let $a \in A$ then we define respectively the tangent and the normal set to A at the point a as

$$\begin{aligned} \mathrm{Tan}(A,a) &:= \left\{ u \in \mathbb{R}^d : \forall \epsilon > 0\, \exists x \in A : \ |x-a| < \epsilon, \left| \frac{u}{|u|} - \frac{x-a}{|x-a|} \right| < \epsilon \right\} \\ \mathrm{Nor}(A,a) &:= \left\{ v \in \mathbb{R}^d : \langle v, u \rangle \leq 0 \ \forall u \in \mathrm{Tan}(A,a) \right\}. \end{aligned}$$

Here the idea is to take all possible sequences $x_n \in A$ approaching a and take the limit of $\frac{x_n - a}{|x_n - a|}$. For the normal set in the above definition the $\leq$ is preferred to the equality sign to allow to consider the non-smooth case and to work with more flexibility. The set $\mathrm{Nor}(A,a)$ actually is in general a cone given by the intersection of all half spaces *dual*[7] to a vector of $\mathrm{Tan}(A,a)$.

The notion of normal vector we introduced should be compared with other possible notions, the most relevant one is that of *proximal calculus.*

Definition Appendix A.3 (Proximal Normal). Let $A \subset \mathbb{R}^d$ and $x \in \partial A$. The vector $v \in \mathbb{S}^{d-1}$ is said to be a proximal normal to A at x (and we write $v \in N_A^P(x)$) iff there exists $r > 0$ such that

$$\left\langle v, \frac{y-x}{|y-x|} \right\rangle \leq \frac{1}{2r}|y-x|, \ \forall y \in \partial A. \tag{A.3}$$

Notice that the inequality A.3 implies that the boundary of A lies outside of $B(x + r\frac{v}{|v|}, r)$. If we focus on the boundary of a closed set the property of having non empty proximal normal set to the complement at each point of the boundary, i.e.

$$N^P_{\complement\Omega}(x) \neq \emptyset \ \forall x \in \partial\Omega$$

is known as *Uniform Interior Ball Condition (UIBC)* and it is usually stated in the following (equivalent) way

[7] Hereafter the word dual must be intended in the following sense [19], u is dual to $N \subset \mathbb{R}^d$ iff $\langle u, v \rangle \leq 0$ for any $v \in N$.

Definition Appendix A.4. Let $\Omega \subset \mathbb{R}^d$ be a domain, suppose that for any $x \in \partial\Omega$ there exists $y \in \Omega$ such that $B(y,r) \cap \complement\Omega = \emptyset$ and $x \in \partial B(y,r)$. Then Ω is said to admit the uniform Interior Ball Condition.

Such a condition (and some variants) appears in the literature also as External Sphere Condition (w.r.t. the complement of the set)in the context of the study of some properties of Minimum Time function in Optimal Control [25], while the previous nomenclature is more frequently used in the framework of regularity theory of PDE.

It is worthwile recalling that positive reach is a strictly stronger condition when compared to UIBC. Actually if a set A has positive reach, then it satisfies the UIBC at each point a of its boundary and *in any direction* of $\mathrm{Nor}(A,a)$.

We will use several times the following easy fact.

Proposition Appendix A.1. *Let $A \subset \mathbb{R}^d$, $\gamma : [0,1] \to \partial A$ a Lipschitz curve, $r > 0$ and let us suppose $\mathrm{Reach}(A) > r$. Then we have for a.e. $s \in]0,1[$ there exists $v \in \mathbb{S}^{d-1}$ such that*

(i) $B_s := B(\gamma(s) + rv, r) \subseteq A^c$,

(ii) $\gamma'(s) \in \mathcal{T}_{\gamma(s)} B_s$.

Proof. Let us consider the arclength re-parametrization $\tilde{\gamma}$ of γ that is a $1-$Lipschitz curve from $[0, \mathrm{Var}[\gamma]]$ to $\mathrm{supp}\,\gamma$. Notice that $\tilde{\gamma}$, being Lipschitz, is a.e. differentiable in $]0, \mathrm{Var}[\gamma][$, Let $\Sigma_{\tilde{\gamma}}$ be the set of singular points of $\tilde{\gamma}$ and let moreover t_0 be a point in $]0, \mathrm{Var}[\gamma][\setminus \Sigma_{\tilde{\gamma}}$.

First we claim that $\tilde{\gamma}'(t_0) \in \mathrm{Tan}(A, \tilde{\gamma}(t_0))$.

By differentiability of $\tilde{\gamma}$ at t_0 we have

$$\lim_{\substack{t \to t_0 \\ t \in [0, \mathrm{Var}[\gamma]] \setminus \Sigma_{\tilde{\gamma}}}} \frac{\tilde{\gamma}(t) - \tilde{\gamma}(t_0)}{t - t_0} = \tilde{\gamma}'(t_0). \tag{A.4}$$

Thus, recalling that $|\tilde{\gamma}'(t)| = 1 \neq 0$ in a neighborhood of t_0, we have

$$\lim_{\substack{t \to t_0 \\ t \in [0, \mathrm{Var}[\gamma]] \setminus \Sigma_{\tilde{\gamma}}}} \frac{\tilde{\gamma}(t) - \tilde{\gamma}(t_0)}{t - t_0} \frac{|t - t_0|}{|\tilde{\gamma}(t) - \tilde{\gamma}(t_0)|} = \frac{\tilde{\gamma}'(t_0)}{|\tilde{\gamma}'(t_0)|}.$$

Therefore we have

$$\lim_{\substack{t \to t_0 \\ t \in [t_0, \mathrm{Var}[\gamma]] \setminus \Sigma_{\tilde{\gamma}}}} \left| \frac{\tilde{\gamma}'(t_0)}{|\tilde{\gamma}'(t_0)|} - \frac{\tilde{\gamma}(t) - \tilde{\gamma}(t_0)}{|\tilde{\gamma}(t) - \tilde{\gamma}(t_0)|} \right| = 0.$$

Thus for any $\epsilon > 0$ we can build the point $x \in \operatorname{supp} \gamma$ of definition Appendix A.2 that realizes the vector $\tilde{\gamma}'(t_0)$ as a vector of $\mathrm{Tan}(A, a)$.

Moreover for a.e. s_0 in $]0, 1[$ the arc length $t_0 = t(s_0) := \mathrm{Var}[\gamma_{[0,s_0]}]$ is an element of $]0, \mathrm{Var}[\gamma][\setminus \Sigma_{\tilde{\gamma}}$ and $\frac{\gamma'}{|\gamma'|}(s_0) = \tilde{\gamma}'(t_0)$.

Now we recall [19] that since A has positive reach and $\gamma(s_0) \in \partial A$ then $\mathrm{Nor}(A, \gamma(s_0))$ is not $\{0\}$. Therefore $\exists v_0 \neq 0$ in $\mathbb{R}^d$ such that $\langle \gamma'(s_0), v_0 \rangle \leq 0$.

Now we can consider $\bar{\gamma}(s) := \gamma(1 - s)$ and $\bar{s}_0 := 1 - s_0$ and apply the same reasoning above to get

$$0 \leq \langle -\gamma'(s_0), v_0 \rangle = \langle \bar{\gamma}'(\bar{s}_0), v_0 \rangle \leq 0. \ \Rightarrow \gamma'(s_0) \in \langle v_0 \rangle^{\perp}.$$

Taking $v = \frac{v_0}{|v_0|}$ we are done. □

Appendix B. (Oriented) distance function and $\mathscr{C}^{1,1}$ domains

Now we switch to the case of a bounded $\mathscr{C}^{1,1}$ domain in $\mathbb{R}^d$. For the reader's convenience we clarify that here we are using the following definition, however several (essentially equivalent) variants are available.

Definition Appendix B.1. Let $\Omega \subset \mathbb{R}^d$ be a domain, then it is said to be a $\mathscr{C}^{1,1}$ domain iff the following holds.

There exist $r > 0$, $L > 0$ such that for any $x \in \partial\Omega$ there exist a coordinate rotation $R_x \in SO^d$ and $f_x \in \mathscr{C}^{1,1}\left(B^{d-1}(0, r),]-r, r[\right)$ (that is differentiable function having Lipschitz gradient) such that

$$\begin{aligned} f_x(0) &= 0 \\ \nabla f_x(0) &= 0 \\ \|f_x\|_{\mathscr{C}^{1,1}} &\leq L \\ x + R_x \,\mathrm{Graph}(f_x) &= \partial\Omega \cap (x + R_x B(x, r)), \end{aligned}$$

where$\|f_x\|_{\mathscr{C}^{1,1}} := \max\{\sup_D |f|, \sup_D |\nabla f|, \mathrm{Lip}(\nabla f)\}$.

In the spirit of [16] and [15] one may study regularity properties of a domain Ω comparing it to the smoothness of the distance function and the *Oriented Distance Function*

$$b_\Omega(\cdot) := d_\Omega(\cdot) - d_{\complement\Omega}.$$

We recollect all the properties we need of a $\mathscr{C}^{1,1}$ domain in $\mathbb{R}^d$ in the following theorem. Detailed proofs can be easily provided combining classical results that can be found in [4][Th. 5.1.9],[19],[1] and [15].

Theorem Appendix B.2. *Let $\Omega \subset \mathbb{R}^d$ be a $\mathscr{C}^{1,1}$ bounded domain. Then the following hold.*

(i) Both Ω and $\complement\Omega$ have positive reach,

$$\mathrm{Reach}(\partial\Omega) = \min\{\mathrm{Reach}(\Omega), \mathrm{Reach}(\complement\Omega)\}.$$

(ii) For any $0 < h < \mathrm{Reach}(\partial\Omega)$ $b_\Omega \in \mathscr{C}^1(U_h(\Omega))$ where $U_h(\Omega) := \{x \in \mathbb{R}^d : -h < b_\Omega(x) < h\}$.

(iii) For any $x \in U_h(\Omega)$, $0 < h < \mathrm{Reach}(\partial\Omega)$

$$\nabla b_\Omega(x) = -\frac{x - \mathrm{proj}_{\partial\Omega}(x)}{|x - \mathrm{proj}_{\partial\Omega}(x)|}, \tag{B.1}$$

where the right side is well defined also on $\partial\Omega$. Moreover ∇b_Ω is a Lipschitz function.

(iv) For any $x \in \partial\Omega$ we have $\mathrm{Tan}(x, \partial\Omega) = \mathcal{T}_x\partial\Omega$ and $\mathrm{Nor}(x, \Omega) = \langle \nabla b_\Omega(x) \rangle$.

(v) For all $x \in \partial\Omega$ an d for any $r < \mathrm{Reach}(\partial\Omega)$ we have

$$B(x - r\nabla b_\Omega(x), r) \subseteq \Omega \tag{B.2}$$
$$B(x + r\nabla b_\Omega(x), r) \subseteq \complement\Omega \tag{B.3}$$

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