Small perturbations of polynomial meshes *

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Abstract

We show that the property of being a (weakly) admissible mesh for multivariate polynomials is preserved by small perturbations on real and complex Markov compacts. Applications are given to smooth transformations of polynomial meshes and to polynomial interpolation.

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1 Introduction.

Global polynomial approximation is still a challenging topic in the multivariate setting. The geometry of the interpolation domain and of its discrete models play a key role, substantially not yet well understood.

A breakthrough has been recently given by the theory of *admissible meshes* by Calvi and Levenberg [12]. These are sequences of discrete subsets $\{\mathcal{A}_n\}$ of a compact set $K \subset \mathbb{R}^d$ (or $K \subset \mathbb{C}^d$), such that a *polynomial inequality* of the form

$$\|p\|_K \leq C\,\|p\|_{\mathcal{A}_n}\ ,\ \ \forall p \in \mathbb{P}_n^d \tag{1}$$

is satisfied, where $\mathbb{P}_n^d$ denotes the space of d-variate polynomials of degree not greater than n, and $\|f\|_X = \sup_{x\in X}|f(x)|$ for f bounded function on the compact X. These sets and inequalities are known also under different

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names in various contexts: (L^∞) norming sets and Marcinkiewicz-Zygmund inequalities (especially for the sphere), stability inequalities (even in more general functional settings); cf., e.g., [14, 20, 26].

A key feature of admissible meshes is that the cardinality of $\mathcal{A}_n$ grows at most polynomially with n,

$$\mathrm{card}(\mathcal{A}_n) = \mathcal{O}(n^s) \;, \quad s > 0 \tag{2}$$

In the case when $C = C(\mathcal{A}_n)$ is not constant but grows at most polynomially with n, namely

$$C = C(\mathcal{A}_n) = \mathcal{O}(n^\alpha) \;, \quad \alpha > 0 \tag{3}$$

they speak of a *weakly* admissible mesh. Observe that necessarily $\mathrm{card}(\mathcal{A}_n) \geq N = \dim(\mathbb{P}_n^d)$, since $\mathcal{A}_n$ is $\mathbb{P}_n^d$-determining.

In [12] it is shown that such meshes are near optimal for *least squares approximation*, and contain *Fekete-like interpolation* sets with a slowly increasing Lebesgue constant. Among their properties, it is worth to recall that (weakly) admissible meshes are preserved by affine mapping, and can be extended by finite union and product.

In some recent papers, the role of (weakly) admissible meshes in multivariate polynomial approximation has been deepened from both the theoretical and the computational points of view. It has been shown that discrete extremal sets of Fekete and Leja type can be extracted from such meshes working on the corresponding rectangular Vandermonde matrices, and using only basic procedures of numerical linear algebra, such as the QR and LU factorizations with pivoting; cf. [6, 7, 28] and references therein. Moreover, resorting to a recent deep result on the asymptotics of Fekete points (cf. [3]), in [6] it has been proved that such discrete extremal sets distribute asymptotically as the continuous Fekete points, i.e., the corresponding discrete measures converge weak-$*$ to the pluripotential equilibrium measure (cf. [15]).

In principle, following [12, Thm.5], it is always possible to construct an admissible mesh on a compact set which satisfies a Markov polynomial inequality (termed for brevity *Markov compacts*)

$$\|\nabla p\|_K \leq M n^r \|p\|_K \;, \quad \forall p \in \mathbb{P}_n^d \tag{4}$$

where $\|\nabla p\|_K = \max_{\boldsymbol{z} \in K} \|\nabla p(\boldsymbol{z})\|_\infty$, $\|\cdot\|_\infty$ denoting the max-norm of d-dimensional complex vectors. This can be done essentially by a uniform discretization of the compact set (or even only of its boundary in complex instances) with $\mathcal{O}(n^{-r})$ spacing, but the resulting mesh has then $\mathcal{O}(n^{rd})$ cardinality for real compacts and, in general, $\mathcal{O}(n^{2rd})$ for complex compacts. Since $r = 2$ for many compacts, for example real convex compacts (cf. [17]), the computational use of such admissible meshes becomes difficult or even impossible already for $d = 2, 3$ at moderate degrees.

On the other hand, weakly admissible meshes with approximately n^2 points and $C(\mathcal{A}_n) = \mathcal{O}(\log^2 n)$, and even (nonuniform) admissible meshes with $\mathcal{O}(n^2)$ points, can be constructed on some standard real bidimensional compacts like disks, triangles, quadrangles; cf. [8, 9]. Admissible and weakly admissible meshes with $\mathcal{O}(n^2)$ points can then be obtained on any convex or concave simple polygon (by polygon triangulation and finite union). These constructions are based on suitable algebraic or mixed algebraic-trigonometric transformations and one-dimensional Chebyshev-like points, and can be extended to higher dimension (balls, cylinders, tori, polyhedra), to obtain (weakly) admissible meshes with $\mathcal{O}(n^d)$ cardinality.

General results on the construction of admissible meshes in d-dimensional real compacts, have been recently proved by Kroó in [18]. In particular, it is shown that *"optimal" admissible meshes*, i.e., meshes with $\mathcal{O}(n^d)$ cardinality, always exist in d-dimensional polynomial graph domains (domains bounded by graphs of polynomial functions), in convex polytopes, and in star-like domains with C^2 boundary. It is also conjectured that any real convex body possesses an optimal admissible mesh. Moreover, admissible meshes with $\mathcal{O}(n^d \log^{k(d)} n)$ cardinality, $k(d) = \mathcal{O}(d^2)$, are constructed in d-dimensional analytic graph domains (domains bounded by graphs of analytic functions). The fact that near optimal admissible meshes can be obtained on Markov compacts by analytic transformations has been recently proved also in [22].

In this paper, we prove a general perturbation result: the property of being a (weakly) admissible mesh for multivariate polynomials is preserved by small perturbations on real and complex Markov compacts. This has a number of consequences. For example, it shows under which conditions small errors on the sampling points preserve unisolvence and the size of Lebesgue constants in multivariate polynomial interpolation. On the other hand, the result can be applied to the construction of (weakly) admissible meshes by smooth transformations, recovering also the case of analytic transformations studied in [22].

2 Some perturbation results.

We state and prove a general result on the perturbation of L^∞ finite norming sets for polynomials of a given degree on Markov compacts (say, a "perturbation principle"). We shall use the notion of *Hausdorff distance* of two d-dimensional compact sets, is defined as

$$\delta(K, H) = \inf \{\eta > 0 : \; K \subseteq H + B_\infty[0, \eta] \text{ and } H \subseteq K + B_\infty[0, \eta]\}$$

where $B_\infty[0, \eta]$ denotes the closed ball (in the max-norm) centered at 0 with radius η (usually termed *polydisk* in the complex case); cf., e.g., [27].

Remark 1 We generally work in $\mathbb{C}^d$ and consider polynomials with complex coefficients; when $K \subset \mathbb{R}^d$, "real" results can be obtained by considering polynomials with real coefficients.

Remark 2 Observe that $\mathcal{A}_n \subset K \subset \mathbb{R}^d$ is a (weakly) admissible mesh for polynomials with complex coefficients if and only if it is a (weakly) admissible mesh for polynomials with real coefficients. Concerning sufficiency, since for $\boldsymbol{x} \in K$ we have $|p(\boldsymbol{x})| = \sqrt{(Re\, p(\boldsymbol{x}))^2 + (Im\, p(\boldsymbol{x}))^2}$, we simply obtain that inequality (1) holds with $\sqrt{2}C$ replacing C.

Theorem 1 *Let $K \subset \mathbb{C}^d$ be a Markov compact with constant M and exponent r, cf. (4). Assume that there exists a compact K_n, $n \in \mathbb{N}$, such that the polynomial inequality*

$$\|p\|_{K_n} \leq C_n \|p\|_{\mathcal{N}_n} \ , \quad \forall p \in \mathbb{P}_n^d \tag{5}$$

is satisfied for a suitable finite subset $\mathcal{N}_n \subset K_n$, and $\delta(K, K_n) \leq e_n$ in the Hausdorff metric δ, with

$$e_n = e_n(\theta) = \frac{\theta}{(1 + C_n)Mn^r} \tag{6}$$

for a fixed $\theta \in (0, t^*/d)$, where $t^* = 0.703\ldots$ solves the equation

$$t \exp{(t/2)} = 1 \ . \tag{7}$$

Consider a small perturbation of $\mathcal{N}_n$, say $\widetilde{\mathcal{N}}_n \subset K$, constructed by choosing a point $\tilde{\boldsymbol{\xi}} \in B_\infty[\boldsymbol{\xi}, e_n] \cap K$ for every $\boldsymbol{\xi} \in \mathcal{N}_n$.

Then, the following polynomial inequality holds

$$\|p\|_K \leq \frac{C_n}{1 - d\theta \exp{(d\theta/2)}} \|p\|_{\widetilde{\mathcal{N}}_n} \ , \quad \forall p \in \mathbb{P}_n^d \ . \tag{8}$$

Proof. Fix $p \in \mathbb{P}_n^d$ and $\boldsymbol{z} \in K$, take $\boldsymbol{z}_n \in K_n$ such that $\|\boldsymbol{z} - \boldsymbol{z}_n\|_\infty \leq e_n$. By the mean-value inequality we can write

$$|p(\boldsymbol{z})| \leq |p(\boldsymbol{z}_n)| + |p(\boldsymbol{z}) - p(\boldsymbol{z}_n)| \leq |p(\boldsymbol{z}_n)| + \max_{\boldsymbol{s} \in [\boldsymbol{z}, \boldsymbol{z}_n]} \|\nabla p(\boldsymbol{s})\|_\infty \|\boldsymbol{z} - \boldsymbol{z}_n\|_1$$

$$\leq |p(\boldsymbol{z}_n)| + \|\nabla p(\boldsymbol{s}_n)\|_\infty \, d\|\boldsymbol{z} - \boldsymbol{z}_n\|_\infty \leq |p(\boldsymbol{z}_n)| + d\|\nabla p(\boldsymbol{s}_n)\|_\infty \, e_n$$

for a suitable $\boldsymbol{s}_n$ in the segment $[\boldsymbol{z}, \boldsymbol{z}_n]$. Observe that $\mathrm{dist}_\infty(\boldsymbol{s}_n, K) \leq e_n$. Using the fact that K is a Markov compact, by the estimate $|q(\boldsymbol{s})| \leq \exp{(dMn^r\varepsilon)}\|q\|_K$, valid for every $q \in \mathbb{P}_n^d$ and for every $\boldsymbol{s} \in \mathbb{C}^d$ such that $\mathrm{dist}_\infty(\boldsymbol{s}, K) \leq \varepsilon$ (cf. [12, Lemma 6]), applied to components of $\nabla p = (\partial_1 p, \cdots, \partial_d p)$, we get

$$\|\nabla p(\boldsymbol{s}_n)\|_\infty \leq \exp{(dMn^r e_n)} \max_{1 \leq i \leq d} \|\partial_i p\|_K = \exp{(dMn^r e_n)}\|\nabla p\|_K \ ,$$

and hence

$$|p(\boldsymbol{z})| \le |p(\boldsymbol{z}_n)| + \exp(dMn^r e_n)\|\nabla p\|_K \, de_n \le |p(\boldsymbol{z}_n)| + \sigma_n \|p\|_K \tag{9}$$

where

$$\sigma_n = dMn^r e_n \exp(dMn^r e_n) = \frac{d\theta \exp(d\theta/(1+C_n))}{(1+C_n)} \, . \tag{10}$$

Now, by (5) we have $|p(\boldsymbol{z}_n)| \le C_n \|p\|_{\mathcal{N}_n}$ and thus, taking the maximum in the left-hand side of (9),

$$\|p\|_K \le C_n \|p\|_{\mathcal{N}_n} + \sigma_n \|p\|_K \, . \tag{11}$$

The next step is to bound $\|p\|_{\mathcal{N}_n}$ in a similar fashion. Fix $\boldsymbol{\xi} \in \mathcal{N}_n$, and take $\tilde{\boldsymbol{\xi}} \in \widetilde{\mathcal{N}}_n$ such that $\|\boldsymbol{\xi} - \tilde{\boldsymbol{\xi}}\|_\infty \le e_n$. Exploiting the Markov inequality as above, we arrive to the estimate

$$|p(\boldsymbol{\xi})| \le |p(\tilde{\boldsymbol{\xi}})| + \sigma_n \|p\|_K \le \|p\|_{\widetilde{\mathcal{N}}_n} + \sigma_n \|p\|_K \, .$$

Taking the maximum in the left-hand side and inserting the resulting bound for $\|p\|_{\mathcal{N}_n}$ into (11) we get finally

$$\|p\|_K \le C_n \|p\|_{\widetilde{\mathcal{N}}_n} + \beta_n \|p\|_K \tag{12}$$

where (observe that necessarily $C_n \ge 1$)

$$\beta_n = (1+C_n)\sigma_n \le d\theta \exp(d\theta/2) \tag{13}$$

which in view of (12) gives (8). $\square$

2.1 Weakly admissible meshes and interpolation sets.

A first relevant consequence of Theorem 1 concerns perturbation of (weakly) admissible meshes, cf. (1)-(3). This can be stated by the following:

Corollary 1 *Let $K \subset \mathbb{C}^d$ be a Markov compact, and let $\mathcal{A}_n$ be a (weakly) admissible mesh for K. Fix $\theta \in (0, t^*/d)$ and consider a small perturbation of $\mathcal{A}_n$, say $\widetilde{\mathcal{A}}_n \subset K$, constructed by choosing a point $\tilde{\boldsymbol{\xi}} \in B_\infty[\boldsymbol{\xi}, \varepsilon_n] \cap K$ for every $\boldsymbol{\xi} \in \mathcal{A}_n$, where*

$$\varepsilon_n = \frac{\theta}{Mn^r(1+C(\mathcal{A}_n))} \tag{14}$$

Then $\widetilde{\mathcal{A}}_n$ itself is a (weakly) admissible mesh for K, such that $C(\widetilde{\mathcal{A}}_n) \le C(\mathcal{A}_n)/(1 - d\theta \exp(d\theta/2))$ and $\mathrm{card}(\widetilde{\mathcal{A}}_n) \le \mathrm{card}(\mathcal{A}_n)$.

The proof of this Corollary is immediate, simply by taking $K_n \equiv K$, $\mathcal{N}_n \equiv \mathcal{A}_n$ and thus $C_n \equiv C(\mathcal{A}_n)$, $\widetilde{\mathcal{N}}_n \equiv \tilde{\mathcal{A}}_n$. The cardinality inequality follows immediately by construction of $\widetilde{\mathcal{A}}_n$. When the mesh is admissible, Corollary 1 says that we get still an admissible mesh under perturbations not exceeding $\varepsilon_n = \mathcal{O}(n^{-r})$.

A special instance is that of subsets $\mathcal{A}_n$ unisolvent for *polynomial interpolation*, $\mathrm{card}(\mathcal{A}_n) = \dim(\mathbb{P}_n^d)$ and nonzero Vandermonde determinant, such that the *Lebesgue constant* $\Lambda_n = C(\mathcal{A}_n)$ grows (at most) polynomially with n. Here Corollary 1 ensures that we get still a unisolvent interpolation set, with a Lebesgue constant proportional to the original one, under perturbations not exceeding $\varepsilon_n = \mathcal{O}((n^r\Lambda_n)^{-1})$. Indeed, the characteristic polynomial inequality of a weakly admissible set

$$\|p\|_K \le C(\widetilde{\mathcal{A}}_n)\,\|p\|_{\widetilde{\mathcal{A}}_n}\ ,\ \ \forall p \in \mathbb{P}_n^d$$

ensures that $\widetilde{\mathcal{A}}_n$ is *unisolvent* for polynomial interpolation of degree n, since $card(\widetilde{\mathcal{A}}_n) \le card(\mathcal{A}_n)$ (and thus, necessarily, $card(\widetilde{\mathcal{A}}_n) = card(\mathcal{A}_n)$). Notice that the latter result also implies that, for any pair $\xi, \eta \in \mathcal{A}_n$, the intersection $B[\boldsymbol{\xi}, \varepsilon_n] \cap B[\boldsymbol{\eta}, \varepsilon_n] \cap K$ must be empty (otherwise, we could choose a unisolvent interpolation set $\widetilde{\mathcal{A}}_n$ with $card(\widetilde{\mathcal{A}}_n) < \dim(\mathbb{P}_n^d)$). In the case of a convex compact, for example, this means that the separation distance of the interpolation set $\mathcal{A}_n$ (the minimal distance between pairs of points) is greater than $2\varepsilon_n$.

Moreover, denoting by $L_n f$ the polynomial that interpolates a continuous function f at $\widetilde{\mathcal{A}}_n$, the chain of inequalities

$$\|L_n f\|_K \le C(\widetilde{\mathcal{A}}_n)\,\|L_n f\|_{\widetilde{\mathcal{A}}_n} = C(\widetilde{\mathcal{A}}_n)\,\|f\|_{\widetilde{\mathcal{A}}_n} \le C(\widetilde{\mathcal{A}}_n)\,\|f\|_K$$

ensures for general Markov compacts that the Lebesgue constant of the interpolation set $\widetilde{\mathcal{A}}_n$, say $\widetilde{\Lambda}_n$, is such that $\widetilde{\Lambda}_n \le C(\widetilde{\mathcal{A}}_n)$.

Consider for example the case of *Fekete points* of a (Markov) compact, that are points that maximize the modulus of the Vandermonde determinant. Here $\Lambda_n \le N$, and the considerations above ensure that we get unisolvent interpolation points with a Lebesgue constant proportional to N, under perturbations not exceeding $\varepsilon_n = \mathcal{O}((n^r N)^{-1})$. This is clearly an underestimate of the size of the possible perturbations. In the only two explicitly known instances of Fekete points, the interval ($n+1$ Gauss-Lobatto points) where $r = 2$ and the complex circle ($2n+1$ equispaced points) where $r = 1$, we have $\Lambda_n = \mathcal{O}(\log n)$, and thus $\varepsilon_n = \mathcal{O}((n^2 \log n)^{-1})$ and $\varepsilon_n = \mathcal{O}((n \log n)^{-1})$, respectively.

The perturbation estimate for Gauss-Lobatto points is shared by many near optimal interpolation sets on an interval $[a, b]$, which have $\Lambda_n = \mathcal{O}(\log n)$, such as for example its Chebyshev-Lobatto points. In this case there is the explicit estimate $\Lambda_n \le 1 + \frac{2}{\pi}\log(n+1)$, cf. [11]. Moreover, a Markov inequality with $r = 2$ and $M = 2/(b-a)$ holds. Corollary 1 ensures that any perturbation of the Chebyshev-Lobatto points of $[a, b]$ not exceeding

$$\varepsilon_n = \frac{\theta(b-a)}{4n^2(1 + \frac{1}{\pi}\log(n+1))}$$

with $\theta < t^* = 0.703\ldots$, is a unisolvent interpolation set, with Lebesgue constant not greater than $(1 + \frac{2}{\pi} \log(n+1))/(1 - \theta \exp(\theta/2))$. This result is on the same line of the so-called *mock-Chebyshev* subset interpolation proposed in [10], where numerical evidence is given that interpolation at the points of a sufficiently dense uniform grid that are closest to the Chebyshev-Lobatto points, defeats the Runge phenomenon.

We stress that Corollary 1 has a nontrivial practical meaning:

- *weakly admissible meshes (in particular, good interpolation sets) are "stable" under small perturbations on Markov compacts.*

That is, if we make small errors on the sampling locations of a weakly admissible mesh, then stability and convergence of polynomial least-squares approximation (cf. [12, Thm.1]), and in particular of polynomial interpolation, are preserved. This is not only qualitative, since we have at hand an explicit (over)estimate of such errors by (14), that allows to control the sampling when (estimates of) some parameters are at hand (namely, the Markov constant and exponent, and the mesh constant).

To give a multivariate example of the stability property stated above, in Figure 1 we show the (numerically evaluated) Lebesgue constants corresponding to small random perturbations of the Padua interpolation points in $[-1,1]^2$, for degree $n = 5, \ldots, 30$. These points are the only explicitly known instance of near optimal total-degree interpolation sets for $d > 1$, cf. [5]. We have $\Lambda_n = \mathcal{O}(\log^2 n)$, $r = 2$ (and $M = 1$, cf. [12, 17]), thus $\varepsilon_n = \mathcal{O}((n^2 \log^2 n)^{-1})$. In the example, for any given Padua point, say $\boldsymbol{\xi}$, we take $\tilde{\boldsymbol{\xi}} = \boldsymbol{\xi} + (u_1, u_2)$, where u_1 and u_2 are uniformly random distributed in $[-\alpha/(n^2 \log^2 n), \alpha/(n^2 \log^2 n)]$, $\alpha = 0, 1, 5, 10$ (with the constraint that $\tilde{\boldsymbol{\xi}} \in [-1,1]^2$). For $\alpha = 1$ the Lebesgue constants are quite close to those of the unperturbed case, whereas they begin to oscillate by increasing α, but with good values even for $\alpha = 10$.

2.2 Smooth transformations of (weakly) admissible meshes.

In this Section we prove two Corollaries of Theorem 1 on smooth geometric transformations of (weakly) admissible meshes, and we give two applications concerning complex parametric curves and planar domains with smooth boundary. Indeed, also transformations of polynomial meshes can be treated in the framework of the "perturbation principle" on Markov compacts. We begin by recalling some definitions from polynomial approximation theory in $\mathbb{R}^d$ and in $\mathbb{C}^d$.

A fat compact set $Q \subset \mathbb{R}^d$ (i.e., $Q = \overline{\mathrm{int} Q}$) is termed a *Jackson compact* if it admits a *Jackson inequality*, namely for each $k \in \mathbb{N}$ there exist a positive integer m_k and a positive constant c_k such that

$$n^k \, \mathrm{dist}_Q(f, \mathbb{P}_n^d) \leq c_k \sum_{|\boldsymbol{i}| \leq m_k} \|D^{\boldsymbol{i}} f\|_Q \ , \ n > k \ , \ \ \forall f \in C^{m_k}(Q) \qquad (15)$$

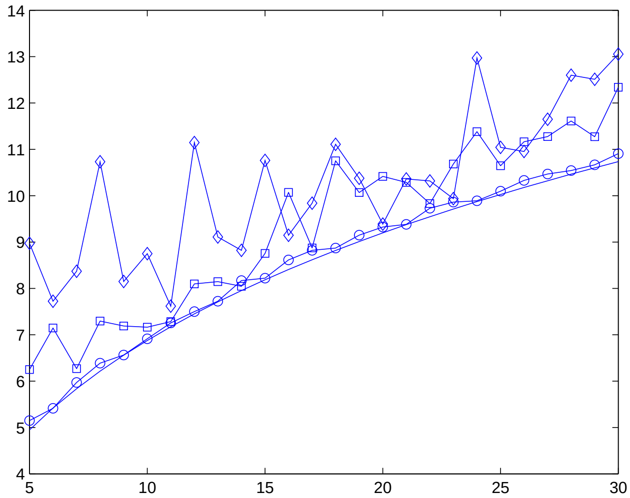

Figure 1: Lebesgue constants of randomly perturbed Padua points in $[-1,1]^2$ for $n = 5, \ldots, 30$, with perturbation radius $\alpha/(n^2 \log^2 n)$: Padua points ($\alpha = 0$, lowest solid line), $\alpha = 1$ ($\circ$), $\alpha = 5$ ($\Box$), $\alpha = 10$ ($\diamond$).

where $\mathrm{dist}_Q(f, \mathbb{P}_n^d) = \inf\{\|f - p\|_Q \,,\, p \in \mathbb{P}_n^d\}$. Examples of Jackson compacts are d-dimensional cubes (with $m_k = k + 1$, cf. [21]) and euclidean balls (with $m_k = k$, cf. [25]); see [2, 23] for some recent results on the multivariate Jackson inequality.

Given a compact set $Q \subset \mathbb{C}^d$, its *polynomial convex hull* is

$$\hat{Q} = \{\boldsymbol{z} \in \mathbb{C}^d : |p(\boldsymbol{z})| \leq \|p\|_Q \,,\ \forall p \in \mathbb{P}_n^d\} \tag{16}$$

and Q is termed *polynomially convex* if $\hat{Q} = Q$. In one complex variable, this is equivalent to the fact that Q has a connected complement ($\hat{Q}$ being the union of Q with the bounded components of its complement). On the other hand, any compact $Q \subset \mathbb{R}^d$ is polynomially convex. We refer the reader e.g. to [15, 19] for a discussion on this concept in the context of pluripotential theory and multivariate polynomial approximation. Finally, we specify that by *analytic* function on a compact set we mean a function that is holomorphic in an open neighborhood of the set.

Corollary 2 *Let $K \subset \mathbb{C}^d$ be a Markov compact, cf. (4). Let $Q \subset \mathbb{C}^d$ be a compact such that $K = \boldsymbol{\phi}(Q)$, and let $\mathcal{A}_n$ be a (weakly) admissible mesh for Q, cf. (1)-(3). Assume that*

(A) *$\boldsymbol{\phi}$ is analytic on $\hat{Q}$ (the polynomial convex hull of Q, cf. (16)).*

Then, there exist a logarithmic sequence of natural numbers, say $j(n) =$

$\mathcal{O}(\log n)$, *such that* $\mathcal{A}'_n = \boldsymbol{\phi}(\mathcal{A}_{nj(n)})$ *is a (weakly) admissible mesh for* K, *with* $C(\mathcal{A}'_n) = \mathcal{O}(C(\mathcal{A}_{nj(n)}))$ *and* $\mathrm{card}(\mathcal{A}'_n) \le \mathrm{card}(\mathcal{A}_{nj(n)})$.

Corollary 3 *Let* $K \subset \mathbb{C}^d$ *be a Markov compact, cf. (4). Let* $Q \subset \mathbb{R}^d$ *be a compact such that* $K = \boldsymbol{\phi}(Q)$, *and let* $\mathcal{A}_n$ *be a (weakly) admissible mesh for* Q, *cf. (1)-(3). Assume that*

(B) *Q is a Jackson compact and* $\boldsymbol{\phi} \in C^{m_k}(Q)$ *for some* $k > r + 2\alpha$ *(cf. (15)), with* $\alpha = 0$ *if* $C(\mathcal{A}_n) = o(n^q)$ *for every* $q > 0$ *(in particular, when* $\mathcal{A}_n$ *is an admissible mesh).*

Then, there exist a sublinear sequence of natural numbers, say $j(n) = \mathcal{O}(n^{\frac{r+2\alpha}{k}})$, *such that the same conclusions of Corollary 2 hold.*

Proof of the Corollaries. Let $\boldsymbol{\pi}_j$ be the best uniform vector polynomial approximation to $\boldsymbol{\phi}$ on Q of degree not greater than j, and $\varepsilon_j = \max_{\boldsymbol{w} \in Q} \|\boldsymbol{\pi}_j(\boldsymbol{w}) - \boldsymbol{\phi}(\boldsymbol{w})\|_\infty$ the corresponding error. Observe that $\boldsymbol{\pi}_j(\mathcal{A}_{nj})$ is a (weakly) admissible mesh for $\boldsymbol{\pi}_j(Q)$, with constant $C(\mathcal{A}_{nj}) = \mathcal{O}((nj)^\alpha)$.

By the Uniform Bernstein-Walsh-Siciak Theorem for analytic functions of several complex variables on polynomially convex compacts [23, Lemma 3], applied componentwise to $\boldsymbol{\phi}$, under assumption (A) of Corollary 2 we have that $\varepsilon_j = \mathcal{O}(a^j)$ for a suitable $a \in (0, 1)$.

On the other hand, under assumption (B) of Corollary 3 we have that $\varepsilon_j = \mathcal{O}(j^{-k})$, applying componentwise to $\boldsymbol{\phi}$ the multivariate Jackson inequality (15).

Fix $\theta \in (0, t^*/d)$, and define

$$j(n) = \min\{j : (1 + C(\mathcal{A}_{nj}))\varepsilon_j \le \theta/(Mn^r)\} \tag{17}$$

(observe that $C(\mathcal{A}_{nj})\varepsilon_j \to 0$ as $j \to \infty$). The assumptions of Theorem 1 are then all satisfied, with $K_n = \boldsymbol{\pi}_{j(n)}(Q)$, $\mathcal{N}_n = \boldsymbol{\pi}_{j(n)}(\mathcal{A}_{nj(n)})$ and $C_n = C(\mathcal{A}_{nj(n)})$, $\widetilde{\mathcal{N}}_n = A'_n = \phi(\mathcal{A}_{nj(n)})$, since $\delta(K, K_n) \le \varepsilon_{j(n)} \le e_n(\theta)$.

Now, define $m(n) = \lceil b \log n \rceil$ with $b > (r + \alpha)/|\log a|$ for Corollary 2, and $m(n) = \left\lceil n^{\frac{r+2\alpha}{k}} \right\rceil$ for Corollary 3: in both cases, $n^r C_{m(n)}\, \varepsilon_{m(n)} \to 0$ as $n \to \infty$, thus $m(n)$ satisfies the inequality in (17) and $j(n) \le m(n)$ for n sufficiently large. We conclude by (8). The assertion on cardinalities follows from the fact that $\boldsymbol{\phi}$ is not injective, in general. $\square$

Corollary 2 is essentially Theorem 1 in [22], where in addition it is proved that $C(\mathcal{A}'_n) \sim C(\mathcal{A}_{nj(n)})$ as $n \to \infty$. Indeed, this is true also here, if we take $j(n) = m(n)$ for n sufficiently large, observing that σ_n in (10) can be substituted by $\hat{\sigma}_n = dMn^r\varepsilon_{j(n)} \exp(dMn^r\varepsilon_{j(n)})$, which becomes an infinitesimal sequence.

The two Corollaries above allow the contruction of low cardinality admissible meshes by geometric transformations. This is on the line of what is done in [18], concerning a class of compacts, the so-called "graph domains". We state the following Proposition, whose proof is immediate. In the analytic case we may speak of a *near optimal* admissible mesh.

Proposition 1 *Let the assumptions of Corollary 2 or 3 be satisfied, and assume in addition that Q possesses an optimal admissible mesh (i.e., an admissible mesh with $\mathcal{O}(n^d)$ cardinality). Then $K = \boldsymbol{\phi}(Q)$ possesses an admissible mesh with cardinality $\mathcal{O}(n^d \log^d n)$, or $\mathcal{O}\left(n^{d(1+(r+2\alpha)/k)}\right)$, respectively.*

2.2.1 Complex parametric curves.

To make a first example, we can consider the case of a complex parametric curve, $K = \phi([a, b]) \subset \mathbb{C}$, admitting a smooth parametrization $\phi \in C^k([a, b])$, $k \geq 1$. Being a one-dimensional connected compact, K satisfies a Markov inequality with exponent at most 2, in view of a well-known result of Pommerenke [24]. Then, being at least C^1 the curve has in any case an admissible mesh with $\mathcal{O}(n^2)$ cardinality, which is the image of a suitable set of equally spaced parameters, as it has been shown in [4, Prop.17] following the construction of [12, Thm.5].

On the other hand, further regularity allows to construct admissible meshes with a lower cardinality. The assumptions of Proposition 1 are satified, when $k > 2$, or when ϕ is even analytic. In fact, $Q = [a, b]$ is a Jackson compact, and is also polynomially convex (i.e., it coincides with its polynomial convex hull) like all real compacts. Moreover, it has optimal admissible meshes (for real polynomials, but see Remark 2), for example its $2n + 1$ Chebyshev-Lobatto points, as it has been shown in [9] using a classical polynomial inequality by Ehlich and Zeller. The ϕ-image of $2nj(n) + 1$ Chebyshev-Lobatto points of $[a, b]$ is then an admissible mesh on the curve (see Corollaries 2 and 3 for the definition of $j(n)$).

Summarizing, as already proved in [22], we get that

- *any analytic parametric curve in $\mathbb{C}$ possesses a near optimal admissible mesh with $\mathcal{O}(n \log n)$ cardinality.*

Moreover, we can also assert that

- *any C^k parametric curve in $\mathbb{C}$, $k > 2$, possesses an admissible mesh with $\mathcal{O}\left(n^{1+2/k}\right)$ cardinality.*

Observe that the curve doesn't need to be geometrically regular (tangent defined everywhere), simple or closed. The results extend immediately by finite union to any piecewise smooth (analytic) parametric curve, and by definition to the polynomial convex hull of the curve (cf. (16)).

In order to show a numerical example, in Figure 2 we display the (numerically evaluated) Lebesgue constants of approximate Fekete points extracted from meshes with different cardinality on the complex unit circle, compared with the Lebesgue constant of the true Fekete points (that in this case are known to be $n+1$ equally spaced points). Such approximate Fekete points have been computed by the algorithm developed in [6, 28], specialized to the one-dimensional complex case. Notice that starting from Chebyshev-Lobatto meshes of increasing $\mathcal{O}(n \log n)$ cardinality we obtain soon good quality interpolation sets, with a very low computational cost (essentially that of a QR factorization of a rectangular $\mathcal{O}(n \log n) \times (n+1)$ Vandermonde matrix, cf. [28]).

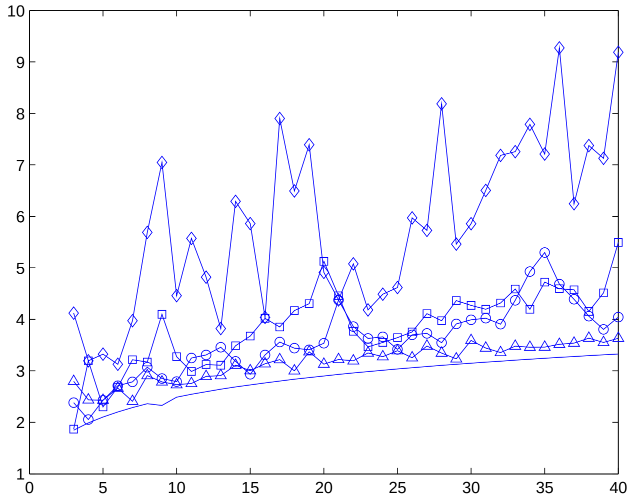

Figure 2: Lebesgue constants of approximate Fekete points extracted from different meshes on the complex unit circle $\phi(\theta) = exp(i\theta)$, $0 \leq \theta \leq 2\pi$, for $n = 3, \ldots, 40$: true Fekete points (lowest solid line), ϕ-image of n^2 equally spaced points ($\triangle$), ϕ-image of $\lceil bn \log n \rceil$ Chebyshev-Lobatto points in $[0, 2\pi]$, $b = 1$ ($\diamond$), $b = 2$ ($\Box$), $b = 3$ ($\circ$).

2.2.2 Planar domains with smooth boundary.

As a second example, we treat a class of compact domains in two real variables. Consider $K = \overline{\Omega}$, the closure of a simply connected bounded open set $\Omega \subset \mathbb{R}^2$, such that $\partial\Omega$ is a regular closed parametric curve of class C^k, $k \geq 1$. Being a Lipschitz domain (the boundary is locally the graph of a Lipschitz continuous function), K satisfies a uniform interior cone condition [13] and thus a Markov inequality with exponent 2. By [12, Thm.5], we can always construct an admissible mesh on K with $\mathcal{O}(n^4)$ cardinality.

On the other hand, if we consider Ω as a subset of the complex plane, it is known that every conformal map from Ω to the interior of the complex unit disk D, can be extended to a C^{k-1} map $\overline{\Omega} \to \overline{D}$, and the same is true for its inverse by the Painlevé Lemma (cf. [16, Thm.5.2.4]). This means that, starting from a conformal map as above (that always exists with holomorphic inverse by the Riemann Mapping Theorem), and taking the real and imaginary parts of its inverse extended to the boundary, we have at hand a C^{k-1} transformation $\boldsymbol{\phi} : (\overline{D} \subset \mathbb{R}^2) \to K$. It is worth recalling that recently some advances have been made in the numerical construction of smooth conformal mappings for domains with (piecewise) smooth boundary, cf. [1].

The assumptions of Proposition 1 concerning Corollary 3 are satisfied, when $k > 3$. Indeed, the closed unit disk is a Jackson compact in $\mathbb{R}^2$ with $m_k = k$ (as every d-dimensional euclidean ball, cf. [25]), and, as it has been shown in [9], it has an optimal admissible mesh with $4n^2$ points (and mesh constant $C = 2$). Then we can assert that

- *any planar simply connected compact domain with C^k boundary, $k > 3$, possesses an admissible mesh with $\mathcal{O}\left(n^{2+4/(k-1)}\right)$ cardinality.*

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