# Polynomial Admissible Meshes

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## Definition

Let $K \subset \mathbb{C}^d$ (or $\mathbb{R}^d$) be a compact polynomial determining set. A sequence $\{A_n\}$ of its subsets is said to be a *weakly admissible mesh* (**WAM**) of constant $C_n$ if

i) $\max_K |p| \leq C_n \max_{A_n} |p|$ for any polynomial $p$ such that $\deg p \leq n$,

ii) $\mathrm{Card}(A_n)$ and $C_n$ grow at most polynomially w.r.t. $n$.

If $\sup_n A_n =: C < \infty$ then $A_n$ is said an *admissible mesh* (**AM**). If furthermore $\mathrm{Card}(A_n) = \mathcal{O}(n^d)$ then $\{A_n\}$ is termed *optimal*.

## Relevant (univariate) examples

- **Chebyshev-Lobatto nodes.** For an interval $[a,b]$, $X_n(a,b) = \left\{\frac{b-a}{2}\xi_j + \frac{b+a}{2}\right\}$, where $\xi_j = \cos(j\pi/n)$, $0 \leq j \leq n$.
- **Chebyshev nodes.** $Z_n(a,b) = \left\{\frac{b-a}{2}\eta_j + \frac{b+a}{2}\right\}$, where $\eta_j = \cos\left(\frac{(2j+1)\pi}{2(n+1)}\right)$, $0 \leq j \leq n$.
- **Chebyshev-like subperiodic angular nodes.** $\Theta_n(\alpha,\beta) = \varphi_\omega(Z_{2n}(-1,1)) + \frac{\alpha+\beta}{2} \subset (\alpha,\beta)$, $\omega = \frac{\beta-\alpha}{2} \leq \pi$, where $\varphi_\omega(s) = 2\arcsin\left(\sin\left(\frac{\omega}{2}\right)s\right)$

The following inequalities hold with $c_n = \frac{2}{\pi}\log(n+1)+1$.

$$\|p\|_{[a,b]} \leq c_n \|p\|_{X_n} \ \forall p \in \mathbb{P}^1_n.$$
$$\|p\|_{[a,b]} \leq c_n \|p\|_{Z_n} \ \forall p \in \mathbb{P}^1_n.$$
$$\|t\|_{[\alpha,\beta]} \leq c_{2n} \|t\|_{\Theta_n} \ \forall t \in \mathbb{T}^1_n.$$

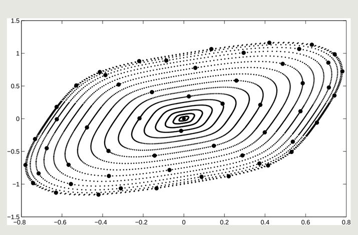
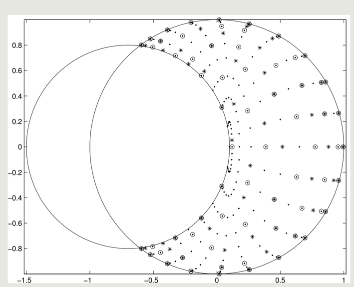
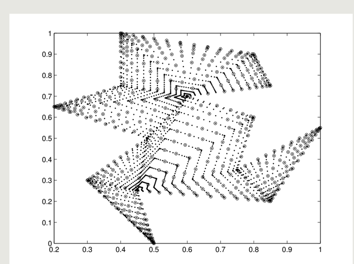
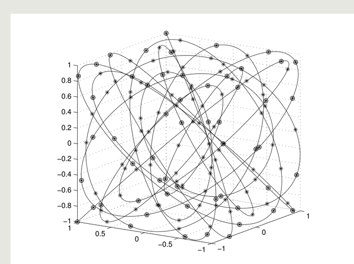
## Basic Properties

- Any *affine transformation* of a WAM is still a WAM, $C_n$ being invariant;
- Any sequence of unisolvent *interpolation sets* whose Lebesgue constant $\Lambda_n$ grows at most polynomially with $n$ is a WAM, with constant $C_n = \Lambda_n$;
- A *finite product* of WAMs is a WAM on the corresponding product of compacts, $C_n$ being the product of the corresponding constants;
- A *finite union* of WAMs is a WAM on the corresponding union of compacts, $C_n$ being the maximum of the corresponding constants.

## Applications

- Polynomial least squares fitting. Theoretical and numerical bounds for the projection operator norm available.
- Discrete orthonormal polynomials computation and fitting, with algorithm for computing a stable basis.
- Extraction of unisolvent interpolation nodes with slow growth of Lebesgue constant.
- Collocation methods for PDEs.

## convex set, lune, polygon, Lissajous WAM

## Constructing WAMs

The *bilinear transformation*

$$\sigma(s_1,s_2) = \frac{1}{4}\left((1-s_1)(1-s_2)\boldsymbol{v}_1 + (1+s_1)(1-s_2)\boldsymbol{v}_2 + (1+s_1)(1+s_2)\boldsymbol{v}_3 + (1-s_1)(1+s_2)\boldsymbol{v}_4\right)$$

maps the square $[-1,1]^2$ onto the *convex* quadrangle with vertices $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$; with a triangle, e.g. $\boldsymbol{v}_3 = \boldsymbol{v}_4$, as a special degenerate case.

**Proposition 1** (Quadrangles and Triangles)**.** *The sequence $A_n = \sigma(X_n(-1,1) \times X_n(-1,1))$ is a WAM of the convex quadrangle $\sigma([-1,1]^2)$, with constant $C_n = c_n^2 = \mathcal{O}(\log^2 n)$ and $\mathrm{card}(A_n) \leq (n+1)^2$.*

**Proposition 2** (Polygons)**.** *Let $K$ be a polygon with $\ell$ sides. Then $K$ has a WAM given by the union of the WAMs of $\ell-2$ triangles of a minimal triangulation, with constant $C_n = c_n^2 = \mathcal{O}(\log^2 n)$ and $\mathrm{Card}(A_n) \sim (\ell-2)n^2$.*

The blending transformation

$$\psi_{u,v}(s,\theta) = s\boldsymbol{u}(\theta) + (1-s)\boldsymbol{v}(\theta),$$

maps $[0,1] \times [\alpha,\beta]$ onto the region $K_{u,v}$ lying between $\boldsymbol{u}$ and $\boldsymbol{v}$, where $\boldsymbol{u}(\theta) = \boldsymbol{a}_1\cos(\theta) + \boldsymbol{b}_1\sin(\theta) + \boldsymbol{c}_1$, $\boldsymbol{v}(\theta) = \boldsymbol{a}_2\cos(\theta) + \boldsymbol{b}_2\sin(\theta) + \boldsymbol{c}_2$, $\theta \in [\alpha,\beta]$, $\boldsymbol{a}_i = (a_{i1}, a_{i2})$, $\boldsymbol{b}_i = (b_{i1}, b_{i2})$, $\boldsymbol{c}_i = (c_{i1}, c_{i2})$, $i = 1,2$, with $\boldsymbol{a}_i, \boldsymbol{b}_i$ not all zero and $[\alpha,\beta]$, $0 < \beta - \alpha \leq 2\pi$.

**Proposition 3** (Circular sections)**.** *The sequence $A_n = \psi_{u,v}(X_n(0,1) \times \Theta_n(\alpha,\beta))$ is a WAM for $K_{u,v}$ with $C_n = \mathcal{O}(\log^2 n)$ and $\mathrm{Card}(A_n) \leq (n+1)(2n+1)$.*

Similar results are available in higher dimension e.g. for cylinders, cones, pyramids and solids of rotation.

**Proposition 4** (Star-like UIBC bodies)**.** *Let $K \subset \mathbb{R}^2$ be the closure of a star-like domain satisfying the* Uniform Interior Ball Condition *(UIBC) of parameter $\rho > 0$. Then, for every fixed $\alpha \in (0, 1/\sqrt{2})$, $K$ has an optimal admissible mesh $\{A_n\}$ such that $C_n \equiv \frac{\sqrt{2}}{1-\alpha\sqrt{2}}$,* $\mathrm{Card}\, A_n \sim n^2 \frac{\mathrm{length}(\partial K)}{\alpha\rho}$.

Finally, WAMS have been shown to be stable under small perturbations and moreover *WAMs can be computed on sets that are images under smooth maps of sets where a WAM is given.*

## Lissajous WAMS in the cube

$$\int_{[-1,1]^3} p(\boldsymbol{x}) \frac{d\boldsymbol{x}}{\sqrt{(1-x_1^2)(1-x_2^2)(1-x_3^2)}} = \pi^2 \int_0^\pi p(\boldsymbol{\ell}_n(\theta))\, d\theta, \ \forall p \in \mathbb{P}^3_{2n},$$

where we used the Lissajous curve

$$\boldsymbol{\ell}_n(\theta) = (\cos(\alpha_n\theta), \cos(\beta_n\theta), \cos(\gamma_n\theta)), \theta \in [0,\pi],$$

$$(\alpha_n, \beta_n, \gamma_n) = \begin{cases} \left(\frac{3}{4}n^2 + \frac{1}{2}n, \frac{3}{4}n^2 + n, \frac{3}{4}n^2 + \frac{3}{2}n + 1\right), n \text{ even}, \\ \left(\frac{3}{4}n^2 + \frac{1}{4}, \frac{3}{4}n^2 + \frac{3}{2}n - \frac{1}{4}, \frac{3}{4}n^2 + \frac{3}{2}n + \frac{3}{4}\right), n \text{ odd} \end{cases}.$$

**Proposition 5** (Lissajous WAMs by hyperinterpolation)**.** *The sequence $\{A_n\} = \{\boldsymbol{\ell}_n(s\pi/\nu)\}$, $s = 0, \ldots, \nu = n\gamma_n + 1$ is a WAM for the cube, with $C_n = \mathcal{O}(\log^3 n)$ and $\mathrm{Card}\, A_n \sim \frac{3}{4}n^3$.*

## Compression and interpolation sets

**Proposition 6.** *Let $\{A_n\}$ be a WAM of a compact set $K \subset \mathbb{R}^d$ with $\mathrm{Card}\, A_n > N_{2n} = \dim(\mathbb{P}_{2n})$ and constant $C_n$. Then there exists (and can be numerically computed) a WAM $A_n^* \subset A_n$ with $\mathrm{Card}\, A_n^* \leq N_{2n}$ with constant $C_n^* = C_n\sqrt{\mathrm{Card}\, A_n}$.*

It is possible to extract unisolvent interpolation sets from a WAM by numerical linear algebra, namely *approximate Fekete points* (AFP) by QR and *Discrete Leja Sequences* (DLS) by LU factorization.

**Proposition 7.** *Let $\mathcal{F}_n = \{\boldsymbol{\xi}_{i_1}, \ldots, \boldsymbol{\xi}_{i_N}\}$ the AFP or DLP extracted from a WAM of a compact set $K \subset \mathbb{R}^d$ (or $K \subset \mathbb{C}^d$). Then $\lim_{n\to\infty} \frac{1}{N}\sum_{k=1}^N f(\boldsymbol{\xi}_{i_k}) = \int_K f(\boldsymbol{x})\, d\mu_K$ for every $f \in \mathscr{C}(K)$, where $\mu_K$ is the pluripotential equilibrium measure of $K$.*

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