Why should people in Approximation Theory care about (pluri-)Potential Theory?

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Outline

$\boxed{\text{Interpolation in the complex plane}}$

Classical Interpolation Problem

Q: Let $E \subset \mathbb{C}$ be a compact set, $f : E \to \mathbb{C}$ be a continuous function. Can we uniformly approximate f by polynomials on E?

A: YES, provided E has infinitely many points and $f \in hol(\hat{E})$ (Runge Theorem).

Q: Can we do it by interpolation?

Q: How is the result depending on nodes?

Lagrange interpolation I

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Consider the monomial basis $\{z^j\}_{j=1,2,\ldots,k}$ of $\mathscr{P}^k$ and $k+1$ distinct points $\boldsymbol{z} = \{z_0, \ldots, z_k\}$ in E, then the interpolation problem

$$p(z_j) = f(z_j),\ j = 0, \ldots, k \ \ \deg(p) = k$$

can be written in matrix form as

$$[z_i^j]_{i,j} c = (f(z_0), \ldots, f(z_k))^t,$$

the matrix $\mathrm{VDM}(\boldsymbol{z}) := [z_i^j]_{i,j}$ is the Vandermonde matrix (has non-zero determinant).
Using the Lagrange Basis we can write

$$\ell_{i,k}(z) := \frac{\det \mathrm{VDM}(z_0, \ldots, z_{i-1}, z, z_{i+1}, \ldots, z_k)}{\det \mathrm{VDM}(z_0, \ldots, z_k)}$$

$$p(z) := \sum_{i=0}^{k} f(z_i)\ell_{i,k}(z).$$

Lagrange interpolation II

The norm of the interpolation operator
$I_k : (\mathscr{C}(E), \|\cdot\|_E) \to \left(\mathscr{P}^k, \|\cdot\|_E\right)$ is the Lebesgue constant

$$\Lambda_k := \max_{z \in E} \sum_{m=0}^{k} \left|\ell_{m,k}(z)\right|.$$

Task: holding the Lebesgue constant growth rate; it will provide a very good interpolant in terms of uniform norm distance from f.

- Lebesgue nodes are nearly impossible to compute, Fekete are just very very hard. . .

Fekete Points I

Fekete Points definition

Let $E \subset \mathbb{C}$ be a compact set and $\boldsymbol{z}_k := (z_0, \ldots, z_k) \in E^{k+1}$. If we have $\det \mathrm{VDM}_k(\boldsymbol{z}_k) = \max_{\zeta \in E^{k+1}} \det \mathrm{VDM}_k(\zeta)$, then $\boldsymbol{z}_k$ is said to be a Fekete array of order k, its elements are said Fekete points.

The relevance of Fekete points is easy to see: one has

$$\Lambda_k(\boldsymbol{F}_k) = \max_{z \in E} \sum_{m=0}^{k} \left| \frac{\det \mathrm{VDM}(z_0, \ldots, z_{i-1}, z, z_{i+1}, \ldots, z_k)}{\det \mathrm{VDM}(z_0, \ldots, z_k)} \right| \leq \sum_{j=0}^{k} 1 = k + 1$$

for any Fekete array $\boldsymbol{F}_k$; moreover notice that in general $\boldsymbol{F}_k$ is not unique.

Fekete points on the unit disc

Let $\mathbb{D}$ be the unit circle and $E := \overline{\mathbb{D}}$. Take any set of distinct points $\boldsymbol{z} = \{z_0, \dots, z_k\}$ and consider the Vandermonde determinant $V(\boldsymbol{z}) := \mathrm{VDM}_k(\boldsymbol{z})$. We have

$$\|V_{:,j}(\boldsymbol{z})\|_2 = \left\| \left(z_0^j, z_1^j \dots, z_k^j \right) \right\|_2 = \sqrt{k+1}, \text{ for any } j = 0, 1, \dots k.$$

Therefore Hadamard Inequality for determinants implies $|\mathrm{VDM}_k(\boldsymbol{z})| \leq \prod_{j=0}^{k} \|V_{:,j}(\boldsymbol{z})\|_2 = (k+1)^{\frac{k+1}{2}}$. This upper bound is achieved by k-th roots of unity, therefore $\{\frac{2i\pi j}{k+1}\}_{j=0,\dots,k}$ is a Fekete set for E.

- for any set of points in E we have

$$\left|\det \mathrm{VDM}_k(z_0, \ldots, z_k)\right| = \prod_{0 \le i < j \le k} |z_i - z_j|.$$

- For any Fekete array $\boldsymbol{F}_k$ for E the sequence of kth diameters $d_k(E) := \left|\det \mathrm{VDM}_k(\boldsymbol{F}_k)\right|^{\frac{1}{\binom{k+1}{2}}}$ is decreasing.

Transfinite diameter definition

$$d(E) := \lim_k d_k(E).$$

Logarithmic potential theory

We recall that a $\mathscr{C}^2(\Omega)$ function u is harmonic if $\Delta u = 0$, while an usc function v is subharmonic if for any open relatively compact subset $\Omega_1 \subset \Omega$ and any harmonic function u on Ω such that $v \leq u$ on $\partial\Omega_1$ it follows that $v \leq u$ on Ω_1.

- By Green's Identity it follows that $\Delta \log |z| = \delta_0$, i.e., $\log |\cdot|$ is the *fundamental solution* for the Laplacian.
- For any given positive finite Borel measure we introduce its logarithmic potential

$$U^\mu(z) := -\mu * \log |\cdot|(z) = \int \log \frac{1}{|z-\zeta|} d\mu(\zeta).$$

- By definition it follows that (up to a normalizing constant) $-\Delta U^\mu = \mu$.
- U^μ is Harmonic on $\mathbb{C} \setminus \operatorname{supp} \mu$ and globally superharmonic.

Logarithmic Energy

Logarithmic Energy Minimization Problem

Let E be a compact subset of $\mathbb{C}$, minimize

$$I[\mu] := \int\int \log \frac{1}{|z-\zeta|} d\mu(\zeta) d\mu(z) = \int U^{\mu}(z) d\mu(z), \qquad \text{(LEM)}$$

among $\mu \in \mathcal{M}_1(E)$, the set of Borel probability measures on E endowed with the weak* topology.

We notice that if we consider the electrostatic interaction between $k+1$ unitary charges in the plane, the force acting on the i-th particle is $\sum_{j \neq i} \frac{(x_i - x_j, y_i - y_j)}{(|x_i - x_j|^2 + |y_i - y_j|)^{1/2}} = -\nabla U^{\mu}$, where μ denotes the uniform probability measure associated to the charges distribution.

We will see that he limit problem is LEM...

Solving LEM

- $I[\cdot]$ is a lower semi-continuous functional
- $\mathcal{M}(E)$ is a locally compact space.

We can use the Direct Methods of Calculus of Variation. Two situations may occur.

1. Either $I[\mu] = +\infty$ for all $\mu \in \mathcal{M}_1(E)$, the set is *too small* (polar) for log potential theory.
2. Either there exists a unique minimizer that is the equilibrium measure of E and is usually denoted by μ_E.

In the latter case one has

$$U^{\mu_E}(z) = -\log c(E) - g_E(z), \text{ where}$$

$$g_E := G_{\mathbb{C}_\infty \setminus E} \text{ Green function with logarithmic pole at } \infty \text{ for } \mathbb{C} \setminus E.$$

- The number $c(E)$, called *logarithmic capacity* of E, is defined as
$$c(E) := \inf_{\mu \in \mathcal{M}_1(E)} I[\mu],$$
- $U^{\mu_E}(z) = -\log c(E)$ quasi everywhere on E, that is outside from a set of zero logarithmic capacity.

Sets of zero logarithmic capacity are polar, they are (locally) the $\{-\infty\}$ set of a subharmonic function.

Fundamental Theorem of Logarithmic Potential Theory

Let $E \subset \mathbb{C}$ be a compact non polar set, we have

$$c(E) = d(E).$$

Therefore, for any sequence of (asymptotically) Fekete arrays $\{\boldsymbol{F}_k\}$, setting $\mu_k := \frac{1}{k+1}\sum_{j=0}^{k} \delta_{F_k^j}$, we have

$$\mu_k \rightharpoonup^* \mu_E.$$

Moreover, locally uniformly on $\mathbb{C} \setminus E$, we have

$$\lim_k -U^{\mu_k}(z) := \lim_k \frac{1}{k+1} \log \prod_{j=0}^{k+1} |z - F_k^j| = g_E(z) - \log c(E) = -U^{\mu_E}.$$

but not all that glitters is gold...

- In the sense of the Theorem μ_E is *the asymptotic* of any asymptotically Fekete array, in particular of *very good interpolation points*, but...
- this is only a necessary conditions, actually

> There are arrays tending to μ_E with exponentially growing Lebesgue constant.

Polynomials and Green function

Bernstein Walsh results

Let E be a compact polynomially determining non polar set, then we have

$$g_E(z) = \overline{\lim}_{\zeta \to z}\left(\left\{\frac{1}{\deg p}\log^+|p(\zeta)|, \|p\|_E \leq 1\right\}\right).$$

Moreover

$$|p(z)| \leq \|p\|_E \exp(\deg p\, g_E(z)) \quad \forall p \in \mathscr{P}(\mathbb{C}).$$

(Bernstein Walsh Ineq.)

If $f : E \to \mathbb{C}$ is any continuous function, we set $d_k(f, E) := \inf\{\|f - p\|_E : p \in \mathscr{P}^k\}$, then for any real number $R > 1$ the following are equivalent

1. $\lim_k d_k(f, E)^{1/k} < 1/R$
2. f is the restriction to E of $\tilde{f} \in hol(D_R)$, where $D_R := \{g_E < \log R\}$.

$\boxed{L^2 \text{ theory in the plane}}$

We show that some analogues of results for Fekete points (that are L^∞ maximizers, in some sense) hold for particular measures in a L^2 fashion.

BM definition

Let $E \subset \mathbb{C}$ be a compact set and μ be a Borel measure such that $\operatorname{supp}\mu \subseteq E$, assume that

$$\overline{\lim}_k \left(\frac{\|p_k\|_E}{\|p_k\|_{L^2_\mu}} \right)^{1/\deg(p_k)} \leq 1,$$

for any sequence of non zero polynomials $\{p_k\}$. Then we say that (E, μ) has the Bernstein Markov property, BMP for short, or equivalently μ is a Bernstein Markov measure on E.

Bernstein Markov measures II

- Example: arclength on unit circle.
- We have plenty of them. Sufficient condition is $\exists t > 0$ such that
$$\lim_{r \to 0^+} c\left(\{z \in E : \mu(B(z,r)) \geq r^t\}\right) = c(\operatorname{supp} \mu).$$
- Necessary conditions are not known…

- Upper bound on diagonal of *reproducing kernel* of $\left(\mathscr{P}^k, \langle\cdot,\cdot\rangle_{L^2_\mu}\right)$ gives good behaviour of uniform polynomial approximation by L^2_μ projection $f \to \mathcal{L}^\mu_k[f] := \sum_{j=0}^k \langle f, q_j\rangle q_j(z)$. Setting $K^\mu_k(z,\zeta) := \sum_{j=0}^k q_j(z)\bar{q}_j(\zeta)$, $B^\mu_k(z) := K^\mu_k(z,z)$ for a orthonormal basis $\{q_j\}$ we have

$$\|\mathcal{L}^\mu_k[f]\|_E \leq \left(\sum_{j=0}^k |\langle f, q_j\rangle|^2\right)^{1/2} \left\|\left(\sum_{j=0}^k |q_j(z)|^2\right)^{1/2}\right\|_E \leq \|f\|_{L^2_\mu} \sqrt{\|B^\mu_k(z)\|_E} \leq \|f\|_E \sqrt{\mu(E)\|B^\mu_k(z)\|_E}.$$

- L^2 version of Bernstein Walsh lemma: it reads precisely as the uniform one.
- Asymptotic of random polynomials and matrices...
- Approximation of potential theoretic objects.

Rephrasing the asymptotic property

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Up to constants depending on k and tending to 1 we have

$$
\begin{aligned}
&|\det \mathrm{VDM}(F_k)|^{\frac{2}{\binom{k+1}{2}}}\\
\simeq&\left(\int \dots \int |\det \mathrm{VDM}(\zeta_0,\dots,\zeta_k)|^2 d\mu_k(\zeta_0)\dots d\mu_k(\zeta_k)\right)^{\frac{1}{\binom{k+1}{2}}} =: Z_k(\mu)\\
\simeq&(\det G_k^{\mu_k})^{\frac{1}{\binom{k+1}{2}}}, \text{ where } G_k^{\mu_k} = [\langle z^i \bar{z}^j\rangle_{L^2_{\mu_k}}]_{i,j}.
\end{aligned}
$$

BM measure are asymptotically Fekete

Let E be a compact non polar set and μ a Borel probability measure such that $\operatorname{supp}\mu \subset E$ and satisfying the Bernstein Markov property. We have

$$\lim_k Z_k(\mu) = \lim_k \left[(k(k+1))! \det G_k^{\mu}\right]^{\frac{1}{k(k+1)}} = d(E).$$

Recovering g_E and μ_E

Theorem

Let E be a compact non polar set and μ a Borel probability measure such that $\operatorname{supp}\mu \subset E$ and satisfying the Bernstein Markov property. We have

i) $\lim_k \frac{1}{2k(k+1)} \log B_k^\mu(z) = g_E(z)$ point-wise, locally uniformly if E is regular.

ii) $\lim_k \frac{B_k^\mu}{k(k+1)}\mu = \mu_E$ in the weak* sense.

$\boxed{\text{Several Complex Variables Case}}$

Several of the previous results have a scv counterpart, but a suitable translation is needed.

some difficulties and solutions

- No classical proof of limit of Vandermonde determinants existence.
- No correspondence between Vandermonde determinants and product of distances.
- The log (discrete) energy and Laplace operator are no more related to Fekete points.
- Therefore subharmonic functions are not the correct space to look at.

So we change our setting.

- subharmonic functions $\rightsquigarrow$ plurisubharmonic (psh) functions.
- Laplace operator $\rightsquigarrow$ complex Monge Ampere operator.
- Harmonic functions $\rightsquigarrow$ maximal psh functions.
- Green function $\rightsquigarrow$ extremal psh function.
- Equilibrium measure $\rightsquigarrow$ pluripotential equilibrium measure.

PSH functions

$u : \mathbb{C}^n \to [-\infty, +\infty[$ is plurisubharmonic, psh for short, if is upper semicontinuous and subharmonic along each complex line.

We use operators $d := \partial + \bar{\partial}$ and $d^c := i(\bar{\partial} - \partial)$, where

$$\partial u := \sum_{j=1}^{n} \frac{\partial u}{\partial z_j} dz_j \ , \ \ \bar{\partial} u := \sum_{j=1}^{n} \frac{\partial u}{\partial \bar{z}_j} d\bar{z}_j \ \forall u \in \mathscr{C}^2.$$

The operator $\mathrm{dd^c} = 2i\partial\bar{\partial}$ is strictly related to plurisubharmonic functions:

for any $u \in \mathscr{C}^2$, $u \in$ psh if and only if $\mathrm{dd^c}\, u$ is a positive $(1,1)$ form. If u is just psh one can define $\mathrm{dd^c}\, u$ as a $(n-1, n-1)$ positive *current*.

Monge Ampere operator

For any $u \in \mathscr{C}^2$ we set

$$(dd^c u)^n := dd^c u \wedge \cdots \wedge dd^c u = \det[\frac{\partial^2}{\partial z_i \partial \bar{z}_j} u(z)]_{i,j} dV_n,$$

where dV_n is the standard volume form on $\mathbb{C}^n$.

Bedford and Taylor showed that $(dd^c \cdot)^n$ extends to a *fully non linear* differential operator on psh $\cap L^\infty_{\text{loc}}$, termed the generalized complex Monge Ampere, being $(dd^c u)^n$ a positive Borel measure.

Pluripotential theory is the study of M-A operator and psh functions.

Extremal psh Function

The role of the Green function is played by the solution of

$$\begin{cases} (dd^c u)^n = 0 & \text{in } \mathbb{C}^n \setminus E \\ u =_{\text{q.e.}} 0 & \text{on } E, u \in L(\mathbb{C}^n) \cap L^\infty_{\text{loc}} \end{cases}$$

Extremal psh function

The function V_E^* is locally bounded and solves the problem above

$$V_E^*(z) := \overline{\lim}_{\zeta \to z} V_E(\zeta)$$
$$V_E(\zeta) := \sup\{u(\zeta),\ u \in L(\mathbb{C}^n),\ u|_E \leq 1\}.$$

Moreover we have (as for g_E)

$$V_E(\zeta) = \sup\left\{\frac{1}{\deg p} \log|p|(\zeta),\ p \in \mathscr{P}(\mathbb{C}^n),\ \|p\|_E \leq 1\right\}.$$

$\mu_E := (dd^c V_E^*)^n$ is the pluripotential equilibrium measure.

Maximal psh functions

Maximal psh functions are defined mimicking the property that sh functions enjoy w.r.t. harmonic ones.

It turns out that they are characterized by $(\mathrm{dd^c}\, u)^n = 0$, so in particular V_E^* is maximal on $\mathbb{C}^n \setminus E$.

Recovering scv results

Despite the theory (and proofs!) is extremely different we can recover the most of the results holding in the (linear) univariate case:

1. Bernstein Walsh L^∞ and L^2 version are exactly the same.
2. Fekete measures for E converge weakly* to μ_E.
3. The same remains true for any sequence of asymptotically Fekete arrays.
4. For any Bernstein Markov measure μ we have

$$\lim_k \frac{B_k^\mu}{\dim \mathscr{P}^k(\mathbb{C}^n)}\mu = \mu_E.$$

5. For any Bernstein Markov measure μ

$$\lim_k \frac{1}{2\dim \mathscr{P}^k(\mathbb{C}^n)} \log B_k^\mu = V_E^*,$$

locally uniformly if E is L-regular.

Another strong difference

In contrast with the Log potential theory in $\mathbb{C}$ these results can be obtained only by using a weighted version of this theory, namely weighted pluripotential theory, that is the scv counterpart of log potential theory with external fields.

$\boxed{\text{A discrete approach}}$

Definition

Admissible meshes, shortly AM, are sequences $\{A_k\}$ of finite subsets of a given compact set E such that

- there exists a positive real constant C such that for any $p \in \mathscr{P}^k$ we have

$$\max_{E} |p| \leq C \max_{A_k} |p|.$$

- Card A_k increase at most polynomially.

AM vs BM measures

We associate to A_k the uniform probability measure μ_k, so we can consider $\mathscr{P}^k(\mathbb{C}^n)$ with the scalar product of $L^2_{\mu_k}$, we pick an orthonormal basis $\{q_j\}$ and we have

$$\sqrt{\|\sum_{j=1}^{N_k} |q_j|^2\|_E} = \sqrt{\|B_k^{\mu_k}\|_E} \leq C \sup_{p\in\mathscr{P}^k} \frac{\|p\|_{A_k}}{\|p\|_{L^2_{\mu_k}}} \leq C\sqrt{\text{Card}\, A_k}.$$

That is $\overline{\lim}_k \|B_k^{\mu_k}\|_E^{1/k} = 1$ hence...

$\boxed{\text{AM are discrete models of B-M measures.}}$

Indeed, any weak* limit of μ_k is a BM measure.

Let $N_k := \dim \mathscr{P}^k(\mathbb{C}^n)$.

- $\|f - \mathcal{L}_k^{\mu_k}[f]\|_E \leq (1 + C\sqrt{\mathrm{Card}\, A_k})d_k(f, E)$.
- $\lim_k \frac{B_k^{\mu_k}}{N_k}\mu_k = \mu_E$.
- $\lim_k \frac{1}{2N_k} \log B_k^{\mu_k} = V_E^*$ locally uniformly if E is L-regular.

Lastly, we can extract from an admissible mesh (a very reasonable approximation of) its Fekete points $F_k \subset A_k$, it turns out that they are asymptotically Fekete for E and thus

- $\lim_k \mu_{F_k} = \mu_E$ in the weak* sense.

Further investigations...

I am working on the Bernstein Markov property on affine algebraic subsets of $\mathbb{C}^n$, they are very thin sets but a specific version of pluripotential theory still works on them.
Aim: extending the sufficient condition for the Bernstein Markov property to this more general setting.

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There is a party upstairs!!

Thank You!