Computing optimal polynomial meshes on planar starlike domains [*]

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Abstract

We construct polynomial norming meshes with optimal cardinality growth, on planar compact starlike domains that satisfy a uniform interior ball condition. Moreover, we provide an algorithm that computes such meshes on planar C^2 convex domains by Blaschke's rolling ball theorem.

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1 Introduction

In the recent literature on multivariate polynomial approximation, the concept of *polynomial mesh* (also called *admissible mesh*) has begun to play an important role; cf., e.g., the seminal paper [11] and [7, 17]. Polynomial meshes are sequences $\{\mathcal{A}_n\}$ of *finite norming sets* (in the uniform norm) on a multidimensional polynomial determining compact $K \subset \mathbb{R}^d$ or $K \subset \mathbb{C}^d$ (i.e., a polynomial vanishing there vanishes everywhere), such that the following *polynomial inequality* holds

$$\|p\|_K \le C\,\|p\|_{\mathcal{A}_n}\ ,\ \ \forall p \in \mathbb{P}_n^d\ , \tag{1}$$

with a cardinality increasing at most like $\mathcal{O}(n^s)$, $s \ge d$ (here and below, $\|f\|_X$ denotes the sup-norm of a function f bounded on the set X). Among their properties, we recall that admissible meshes are preserved by affine

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transformations, and can be easily extended by finite union and product [11]. In the present paper, we restrict our attention to the real case, i.e., real polynomials and $K \subset \mathbb{R}^d$.

Polynomial meshes provide a "good discrete model" of a compact set for many practical purposes. For example, they are nearly optimal for uniform least square approximation [11], and contain Fekete-like interpolation subsets with the same asymptotic behavior of the continuous Fekete points of K, that can be computed by numerical linear algebra techniques (cf., e.g., [5, 6]). Such approximate Fekete points have been used within spectral element and collocation methods for PDEs (cf. [18, 25]). For a recent and deep survey on polynomial approximation and interpolation in several variables, we refer the reader to [2].

In [11, Thm.5], it has been shown that any (real) compact set which satisfies a Markov polynomial inequality with exponent r

$$\|\nabla p(x)\|_2 \leq M n^r \|p\|_K \ , \ \ \forall x \in K,\, p \in \mathbb{P}_n^d \ , \tag{2}$$

has an admissible mesh with $\mathcal{O}(n^{rd})$ cardinality (for example, $r = 2$ for compact sets which satisfy a uniform interior cone condition).

On the other hand, in the applications it is important to control the cardinality of such discrete models. Indeed, some attention has been devoted to the construction of *optimal* and *near optimal* polynomial meshes, which have cardinality $\mathcal{O}(n^d)$ and $\mathcal{O}((n \log n)^d)$, respectively, in compact sets with special geometries (observe that in (1) necessarily $\text{card}(\mathcal{A}_n) \geq \dim(\mathbb{P}_n^d) \sim n^d/d!$); cf., e.g., [9, 17, 19, 20, 22]. Moreover, the polynomial inequality (1) can be relaxed, asking that it holds with $C = C_n$, a sequence of constants increasing at most polynomially with n: in such a case, we speak of *weakly* admissible meshes. Weakly admissible meshes with $\mathcal{O}(n^d)$ cardinality and constants $C_n = \mathcal{O}((\log n)^d)$ are known in several instances, cf., e.g., [5, 14].

In the present paper we prove constructively existence of optimal polynomial meshes, i.e., with cardinality $\mathcal{O}(n^2)$, on a planar compact starlike domain (that is not restrictive to consider centered in the origin) assuming that it satisfies a classical *uniform interior ball condition* (cf., e.g., [1] and references therein). Special instances are $C^{1,1}$ planar starlike domains, thus generalizing in the planar case a recent result by A. Kroó on C^2 starlike domains [17].

Moreover, we provide an algorithm (implemented in Matlab) to compute such meshes given a regular parametrization of the boundary of a C^2 convex domain, using *Blaschke's rolling ball theorem* to determine the interior ball condition [15].

2 Constructing optimal polynomial meshes

In the sequel, the notion of *tangential Markov inequality* on a rectifiable curve Γ *with respect to a compact* K will play a key role. Given a *rectifiable* curve $\Gamma \subset K \subset \mathbb{R}^2$ (i.e., it has a continuous parametrization and finite length), this curve has a canonical Lipschitz continuous parametrization in the arclength, which is almost everywhere differentiable. Given the tangent unit vector $\boldsymbol{\tau}$ at a regular point $x \in \Gamma$ (a point where the canonical parametrization is differentiable with nonzero derivative), a tangential Markov inequality with exponent r for Γ w.r.t. K has the form

$$|\langle \nabla p(x), \boldsymbol{\tau} \rangle| \leq M n^r \|p\|_K \ , \ \ \forall p \in \mathbb{P}_n^d \ , \tag{3}$$

where $\langle \cdot, \cdot \rangle$ denotes the euclidean scalar product and M, r are independent of the (regular) point x.

The fulfillement of a tangential Markov inequality like (3) allows to construct a suitable polynomial inequality on the curve.

Lemma 1 *Let $K \subset \mathbb{R}^2$ be a compact set, and be $\Gamma \subset K$ be a rectifiable curve that satisfies (3) at its regular points. Then, for any $\alpha \in (0,1)$, there exists a mesh of equally spaced points on Γ, say $X_n = X_n(\alpha)$, such that*

$$\|p\|_\Gamma \leq \|p\|_{X_n} + \alpha \|p\|_K \ , \tag{4}$$

and card$(X_n) = \mathcal{O}(n^r)$.

Proof. Let us term $\boldsymbol{\omega}(s)$, $s \in [0, L]$ the canonical parametrization of Γ in the arclength (which is Lipschitz continuous and thus almost everywhere differentiable in $[a, b]$), where L denotes the total length of Γ. For any pair $\boldsymbol{x}, \boldsymbol{y} \in \Gamma = \boldsymbol{\omega}([0, L])$, take two values of the parameter, say $s_1, s_2 \in [0, L]$, such that $\boldsymbol{x} = \boldsymbol{\omega}(s_1)$, $\boldsymbol{y} = \boldsymbol{\omega}(s_2)$ (the curve being not necessarily simple). Then, we can write

$$|p(\boldsymbol{x}) - p(\boldsymbol{y})| = |p(\boldsymbol{\omega}(s_2)) - p(\boldsymbol{\omega}(s_1))| = \left| \int_{s_1}^{s_2} \frac{d}{ds} p(\boldsymbol{\omega}(s)) \, ds \right|$$

$$\leq \int_{s_1}^{s_2} \left| \frac{d}{ds} p(\boldsymbol{\omega}(s)) \right| ds = \int_{s_1}^{s_2} |\langle \nabla p(\boldsymbol{\omega}(s)), \boldsymbol{\omega}'(s) \rangle| \, ds \leq M n^r \, \|p\|_K \, \ell(\boldsymbol{x}, \boldsymbol{y}) \ ,$$

where $\ell(\boldsymbol{x}, \boldsymbol{y}) = s_2 - s_1$ denotes the length of the corresponding arc of Γ connecting $\boldsymbol{x}$ and $\boldsymbol{y}$.

Fix $\alpha \in (0,1)$. Taking $N + 1$ equally spaced points on Γ in the arclength

$$X_n = \{\boldsymbol{y}_k = \boldsymbol{\omega}(kL/N) \ , \ k = 0, \ldots, N\} \ , \ \ N = \left\lceil \frac{M n^r L}{2\alpha} \right\rceil \ , \tag{5}$$

and observing that for every $\boldsymbol{x} \in \Gamma$ there is a point $\boldsymbol{y}_{k(x)}$ such that $\ell(x, \boldsymbol{y}_{k(x)}) \leq \frac{1}{2}L/N$, we can write the inequality

$$|p(\boldsymbol{x})| \leq |p(\boldsymbol{y}_{k(x)})| + |p((\boldsymbol{x}) - p(\boldsymbol{y}_{k(x)})| \leq |p(\boldsymbol{y}_{k(x)})| + \alpha \|p\|_K \ ,$$

that implies (4). $\square$

Observe that if the curve is open, then the cardinality of X_n is $N+1$, whereas it is N if the curve is closed, $\boldsymbol{\omega}(0) = \boldsymbol{\omega}(L)$.

We can now state and prove the following:

Theorem 1 *Let $K \subset \mathbb{R}^2$ be a planar compact starlike domain. Assume that K satisfies a* uniform interior ball condition *(UIBC), i.e., every point of ∂K belongs to the boundary of a disk with radius $\rho > 0$, contained in K (geometrically, there is a fixed disk that can roll along the boundary remaining inside K; cf., e.g., [1]).*

Then, for every fixed $\alpha \in (0, 1/\sqrt{2})$, K possesses a sequence of finite norming sets $\{\mathcal{A}_n\}$ such that

$$\|p\|_K \leq \frac{\sqrt{2}}{1 - \alpha\sqrt{2}} \|p\|_{\mathcal{A}_n} \ , \quad \forall p \in \mathbb{P}_n^2 \ , \tag{6}$$

with

$$card(\mathcal{A}_n) \leq 2n \left\lceil \frac{n \ length(\partial K)}{2\alpha\rho} \right\rceil + 1 = \mathcal{O}(n^2) \ , \tag{7}$$

i.e., an optimal admissible mesh.

Proof. We recall that a compact set $K \subset \mathbb{R}^2$ is termed *starlike* if there exists $\boldsymbol{x}_0 \in K$ (the "star center") such that for every $\boldsymbol{x} \in K$ the segment $[\boldsymbol{x}_0, \boldsymbol{x}]$ is contained in K. By no loss of generality, up to a translation we can assume that the star center is the origin.

First, we show that K has a norming set formed by $n+1$ curves, that are suitable scalings of the boundary. Being compact and starlike with respect to the origin, K is the union of the rays $[\boldsymbol{0}, \boldsymbol{x}]$, $\boldsymbol{x} \in \partial K$. On each ray, a polynomial $p \in \mathbb{P}_n^2$ becomes a univariate polynomial of degree not greater than n, and thus by a well-known result of Ehlich and Zeller (cf. [12] and [9]), there is an admissible mesh for the ray with constant $\sqrt{2}$, given by $2n+1$ Chebyshev-Lobatto points of $[\boldsymbol{0}, \boldsymbol{x}]$, namely

$$\boldsymbol{u}_j(\boldsymbol{x}) = a_j \, \boldsymbol{x} \ , \quad a_j = \frac{1 + \xi_j}{2} \ , \tag{8}$$

where $\xi_j = \cos(j\pi/(2n))$, $j = 0, \ldots, 2n$, are the Chebyshev-Lobatto points in $[-1, 1]$. Then, the $2n+1$ curves

$$\Gamma_j = \{\boldsymbol{u}_j(\boldsymbol{x}) \, , \ \boldsymbol{x} \in \partial K\} = a_j \partial K$$

form a norming set for K, i.e.,

$$\|p\|_K \leq \sqrt{2}\,\|p\|_{\bigcup \Gamma_j} \leq \sqrt{2}\,\max_j \|p\|_{\Gamma_j}\;,\;\; \forall p \in \mathbb{P}^2_n\;. \tag{9}$$

Notice that Γ_{2n} degenerates into a singleton, $\Gamma_{2n} = \{\mathbf{0}\}$.

Second, we show that on each curve Γ_j an inequality like (4) is satisfied, with $r = 1$. Now, observe that the UIBC condition implies the (weaker) uniform interior cone condition, which in turn ensures that the boundary is a rectifiable curve (cf. [13, Thm. 4.5.11]). Moreover, the UIBC condition implies that a tangential Markov inequality w.r.t. K like (3) holds with exponent $r = 1$, at the regular points of the boundary. Indeed, for any $\boldsymbol{x} \in \partial K$ there is $\boldsymbol{x}^* \in \mathrm{int}(K)$ such $\boldsymbol{x} \in \partial D$, $D \subset K$ being the disk centered at $\boldsymbol{x}^*$ with radius ρ. If we take the parametrization of ∂D in polar coordinates centered at $\boldsymbol{x}^*$, say $\boldsymbol{z}(\phi) = \boldsymbol{x}^* + (\rho\cos(\phi), \rho\sin(\phi))$, $\phi \in [0, 2\pi]$, then every $p \in \mathbb{P}^2_n$ restricted to ∂D becomes a univariate trigonometric polynomial $t(\phi) = p(\boldsymbol{z}(\phi)) \in \mathbb{T}_n$.

Consider a regular point $\boldsymbol{x} \in \partial K$, i.e., a point where the canonical parametrization in the arclength is differentiable with nonzero derivative: then, the disk D is tangent to ∂K at $\boldsymbol{x}$. By the classical Markov inequality for trigonometric polynomials (cf., e.g., [4])

$$|t'(\phi)| \leq n\|t\|_{[0,2\pi]}\;,$$

and the fact that $|t'(\phi)| = |\langle \nabla p(\boldsymbol{z}(\phi)), \boldsymbol{z}'(\phi)\rangle|$, $|\boldsymbol{z}'(\phi))| = \rho$, and $\boldsymbol{x} = \boldsymbol{z}(\phi^*)$ for a certain ϕ^*, we get immediately

$$|\langle \nabla p(\boldsymbol{x}), \boldsymbol{\tau}\rangle| = |\langle \nabla p(\boldsymbol{z}(\phi^*)), \boldsymbol{\tau}\rangle| \leq \frac{n}{\rho}\,\|p\|_{\partial D} \leq \frac{n}{\rho}\,\|p\|_K\;, \tag{10}$$

where $\boldsymbol{\tau} = \pm \boldsymbol{z}'(\phi^*)/\rho$ are the common unit tangent vectors to ∂K and ∂D at $\boldsymbol{x}$. This shows that (3) holds with $M = 1/\rho$.

By (10) and Lemma 1 with $\Gamma = \partial K$

$$\|p\|_{\partial K} \leq \|p\|_{X_n} + \alpha\|p\|_K\;,\;\; \forall p \in \mathbb{P}^2_n\;, \tag{11}$$

from which follows setting $q(\boldsymbol{x}) = p(a_j\boldsymbol{x})$

$$\|p\|_{\Gamma_j} = \|q\|_{\partial K} \leq \|q\|_{X_n} + \alpha\|q\|_K$$

$$= \|p\|_{a_jX_n} + \alpha\|p\|_{a_jK} \leq \|p\|_{a_jX_n} + \alpha\|p\|_K\;,\;\; \forall p \in \mathbb{P}^2_n\;, \tag{12}$$

since each curve $\Gamma_j = a_j\partial K$ is an affine transformation (scaling) of the boundary.

Fix α such that $0 < \alpha < 1/\sqrt{2}$. By (12) and (9) we can write for every $p \in \mathbb{P}^2_n$

$$\|p\|_K \leq \sqrt{2}\,\max_j \left\{\|p\|_{a_jX_n}\right\} + \alpha\sqrt{2}\|p\|_K$$

$$= \sqrt{2}\,\|p\|_{\mathcal{A}_n} + \alpha\sqrt{2}\|p\|_K \ , \ \ \mathcal{A}_n = \bigcup_{j=0}^{2n} a_j X_n = \{\mathbf{0}\} \cup \bigcup_{j=0}^{2n-1} a_j X_n \ , \tag{13}$$

from which we finally get

$$\|p\|_K \leq \frac{\sqrt{2}}{1-\alpha\sqrt{2}}\,\|p\|_{\mathcal{A}_n} \ , \ \ \forall p \in \mathbb{P}_n^2 \ , \tag{14}$$

where by (5) $\mathrm{card}(\mathcal{A}_n) \leq 2n\,\mathrm{card}(X_n) + 1 = 2nN + 1 = \mathcal{O}(n^2)$. Observe, in fact, that $\mathrm{card}(\mathcal{A}_n) = 2n(N-1)+1$ if $\mathbf{0} \in X_n$, otherwise $\mathrm{card}(\mathcal{A}_n) = 2nN+1$. □

The theorem above generalizes, in the planar case, a recent result proved by A. Kroó in arbitrary dimension for C^2 starlike domains; cf. [17]. Indeed,

Corollary 1 *Let $K \subset \mathbb{R}^2$ be the closure of an open, bounded, starlike, and $C^{1,1}$ subset. Then, K has an optimal admissible mesh.*

Proof. We recall that a closed domain $K \subset \mathbb{R}^d$ (the closure of an open connected subset) is termed $C^{1,1}$ if there are a fixed radius, say $R > 0$, and a constant $L > 0$ such that for each point $\boldsymbol{\xi} \in \partial K$ there exists a $C^{1,1}$ function $f : I \to \mathbb{R}$ (I compact interval) such that after a suitable rotation, $K \cap B(\boldsymbol{\xi}, R) = \{\boldsymbol{x} = (x_1, x_2) \in K : x_2 \leq f(x_1)\}$, where $\|f'\|_I$ and the Lipschitz constant of f' are uniformly bounded by L.

Now, it is known that a closed domain is $C^{1,1}$ if and only if it satisfies a uniform two-sided (interior and exterior) ball condition; cf., e.g., [1, Cor. 3.14]. Then, all the assumptions of Theorem 1 are satisfied, and the conclusion follows. □

Remark 1 The assumptions of Theorem 1 are much weaker than those of Corollary 1. In fact, the boundary of a $C^{1,1}$ domain is a regular curve, whereas Theorem 1 allows singular points, for example inward (but not outward) corners and cusps (the domain does not even need to be Lipschitz).

Remark 2 In the special case of C^2 *convex domains*, the maximal ρ in Theorem 1 is equal to the *minimal ray of curvature*, in view of the so called *Blaschke's rolling ball theorem*, cf., e.g., [10, 15]. This fact will be used below as the basis of an algorithm for the computation of optimal polynomial meshes on C^2 convex domains.

2.1 An algorithm for C^2 convex domains

We focus on the case of a C^2 convex domain, assuming that we have at hand a Lipschitz-continuous parametrization of the boundary, say $\boldsymbol{\sigma}(t) = (\sigma_1(t), \sigma_2(t))$, $t \in [a, b]$, $\boldsymbol{\sigma}(a) = \boldsymbol{\sigma}(b)$. In this case, an optimal polynomial mesh can be computed in a simple and completely automatic way using the boundary parametrization.

In view of Remark 2, the minimal value of $M = 1/\rho$ is nothing else than the maximal curvature of the boundary. An approximate value of the latter, along with an approximate value of the total length L (both to be used in (5)), and then an optimal polynomial mesh, can be computed by the following algorithm, which relies on a polygonal approximation of the boundary.

Algorithm (computes an optimal polynomial mesh on a C^2 convex compact domain $K \subset \mathbb{R}^2$)

- *input*: $\boldsymbol{\sigma}$ (Lipschitz-continuous boundary parametrization), a, b (parameter endpoints), $\boldsymbol{c}$ (star "center"), n (polynomial degree), ε (error tolerance), α (scaling parameter), μ_0 (starting number of subdivisions)

(*i*) set $\mu ;= \mu_0$; compute the polygonal vertices $\boldsymbol{v}_i = \boldsymbol{v}_i(\mu) = \boldsymbol{\sigma}(t_i)$, $t_i = a + ih$, $i = 0, \dots, \mu - 1$ with $h = (b-a)/\mu$, by iteratively doubling μ until

$$|L_{2\mu} - L_\mu| < \varepsilon \ , \ \ L_\mu := \sum_{i=0}^{\mu-1} \|\Delta \boldsymbol{v}_i\|_2 \ ,$$

where $\Delta \boldsymbol{v}_i = \boldsymbol{v}_{i+1} - \boldsymbol{v}_i$ and ε is the given tolerance; set $L := L_{2\mu}$

(*ii*) compute the approximate curvatures

$$\tilde{\kappa}_i := \frac{\|\tilde{\boldsymbol{\tau}}_{i+1} - \tilde{\boldsymbol{\tau}}_i\|_2}{\|\Delta \boldsymbol{v}_i\|_2} \approx \kappa_i \ , \ \ \tilde{\boldsymbol{\tau}}_i = \frac{\delta \boldsymbol{v}_i}{\|\delta \boldsymbol{v}_i\|_2} \approx \boldsymbol{\tau}_i \ ,$$

where $\boldsymbol{\tau}_i$ is the unit tangent vector and κ_i the curvature at $\boldsymbol{v}_i$, and $\delta \boldsymbol{v}_i = \boldsymbol{v}_{i+1} - \boldsymbol{v}_{i-1}$; set $M := \max \tilde{\kappa}_i$ and $N := \lceil MnL/(2\alpha) \rceil$ (cf. (5) and (13)-(14))

(*iii*) compute N approximately equispaced points in the arclength on ∂K with step L/N

$$\boldsymbol{y}_k := \boldsymbol{v}_{m_k} \ , \ \ m_k = \max\{m > m_{k-1} : \sum_{i=m_{k-1}}^{m} \|\Delta \boldsymbol{v}_i\|_2 \leq L/N\} \ ,$$

$k = 0, \dots, N-1$, with $m_0 = 0$

(*iv*) set $\boldsymbol{z}_{2n} := \boldsymbol{c}$; for $j = 0, \dots, 2n-1$ compute the mesh points

$$\boldsymbol{z}_j(k) := \frac{1 + \xi_j}{2} (\boldsymbol{y}_k - \boldsymbol{c}) \ , \ \ k = 0, \dots, N-1$$

where $\xi_j = \cos(j\pi/(2n))$ are the Chebyshev-Lobatto points in $(-1, 1]$

- *output*:

$$\mathcal{A}_n = \{\boldsymbol{c}\} \cup \bigcup_{j=0}^{2n-1} \bigcup_{k=0}^{N-1} \boldsymbol{z}_j(k)$$

  in an optimal polynomial mesh on K with $\mathrm{card}(\mathcal{A}_n) = 2nN + 1$ if $\boldsymbol{c} \in \{\boldsymbol{y}_k\}$, $\mathrm{card}(\mathcal{A}_n) = 2n(N-1) + 1$ otherwise, and constant $C = \sqrt{2}/(1 - \alpha\sqrt{2})$

A Matlab code that implements the algorithm above, computing and plotting the optimal polynomial meshes, is provided in [21]. In Figure 1, we show the optimal polynomial meshes computed by the code on a planar compact, whose boundary is a convex limacon with polar representation $r = r(\theta) = a_2 + a_1\cos(\theta)$, $a_2 > 2a_1$, $\theta \in [0, 2\pi]$ (in this case, $a_2/a_1 = 2.05$; an optimal mesh is clearly preserved by any scaling of $r(\theta)$).

The input parameters are $n = 3$, $\varepsilon = 10^{-8}$, $\alpha = 1/2$, $\mu_0 = 10$, and the star center $\boldsymbol{c}$ is chosen in two different positions, internal and on the boundary (a convex domain being starlike with respect to any of its points). The resulting mesh on the boundary has cardinality $N = 23$, and thus the overall cardinality of the optimal mesh is $2nN + 1 = 139$ with the internal center, and $2n(N-1) + 1 = 133$ with the center on the boundary (since in this instance it belongs to the boundary mesh).

To the purpose of illustration, we also display $10 = \dim(\mathbb{P}_3^2)$ *approximate Fekete points*, extracted from the optimal meshes by the QR algorithm with column pivoting, applied to the corresponding transposed rectangular Vandermonde matrix in a suitable polynomial basis, as described in [23]. For the good interpolation properties of approximate Fekete points, when they are extracted from admissible polynomial meshes, we refer the reader to [5, 6]; a Matlab code that implements the extraction algorithm is provided in [24].

In Figure 2, we report the numerically evaluated Lebesgue constant of the approximate Fekete points, at a sequence of interpolation degrees, along with the infinity-norm of the discrete least square projection operator corresponding to the whole mesh. Indeed, we recall that polynomial meshes are near optimal for polynomial least square approximation [11]. Notice that the least square operator norm, as already observed in [7] for the triangle, turns out to be much smaller than the theoretical estimate provided in [11, Thm.1].

Remark 3 The algorithm above is a basic version, that could be improved along different lines. For example, it could be natural in applications to have the boundary of the C^2 convex compact in a discrete way, as a clockwise or counterclockwise ordered sampling. In such a case, the parametrization $\boldsymbol{\sigma}(t)$ can be constructed for example by *shape preserving* spline interpolation, cf., e.g., [16]. If K is a strictly convex C^4 domain, using cubic splines one can ensure that, for sufficiently small sampling steps, the approximate boundary remains a simple curve [8], and that the signed curvature remains of constant sign, since uniform convergence of first and second derivatives occurs (cf., e.g., [3]).

Moreover, notice that the mesh construction corresponds, in practice, to use the same (suitably scaled) uniform interior ball condition on the internal curves $\Gamma_j = a_j \partial K$, $j > 0$ (cf. (8)). This entails that the cardinality of the meshes on the internal curves is driven by the minimal ray of curvature along ∂K, say ρ, scaled by a_j, which could led to a large cardinality on every curve, for example when the boundary has some (smooth) narrow tip.

On the other hand, one can observe that a C^2 compact domain satisfies also a *uniform exterior ball condition*, with an arbitrary radius in the convex case. Such a property holds for the compact convex subsets bounded by the internal curves, say $K_j = a_j K$, where $\partial K_j = \Gamma_j$. This suggests that, in order to reduce the overall cardinality, one could use on K_j the exterior ball condition to obtain a tangential Markov inequality w.r.t. K, whenever $\operatorname{dist}(\partial K, \Gamma_j)$ is greater than the scaled interior ball diameter $2a_j\rho$. Indeed, for such values of j and for every $\boldsymbol{x} \in \Gamma_j$, the external tangent ball with radius $\frac{1}{2}\operatorname{dist}(\partial K, \Gamma_j)$ is contained in K. Notice that this can happen only if the center $\boldsymbol{c}$ belongs to the interior of K; a suitable choice is to take as $\boldsymbol{c}$ the barycenter of K.

These and other improvements, for example the extension of the method to general C^2 starlike compacts, finding a suitable way to estimate the maximal radius in the uniform interior ball condition, may be object of further research.

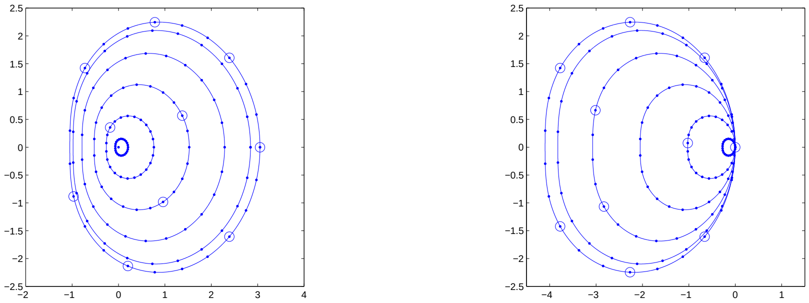

Figure 1: Optimal polynomial meshes for degree $n = 3$ and the corresponding set of norming curves on the compact whose boundary is a convex limacon, as a starlike domain centered at an interior point (left) and at a boundary point (right); in evidence (small circles) $10 = \dim(\mathbb{P}_3^2)$ approximate Fekete interpolation points extracted from the optimal meshes.

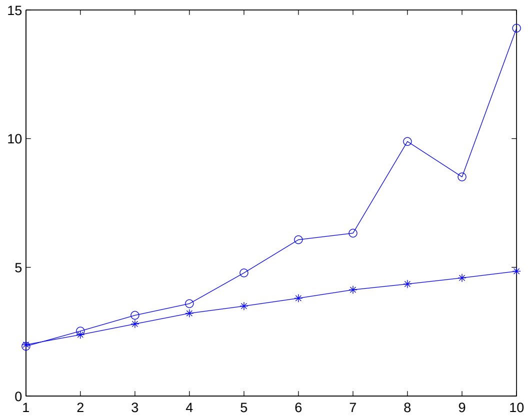

Figure 2: Lebesgue constant of the approximate Fekete points (∘) and infinity-norm of the discrete least-squares operator on the whole mesh (∗), for the optimal meshes of the convex limacon (as in Figure 1-left) at a sequence of degrees.

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