Modelling for Constraint Programming

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3. Symmetry, Viewpoints

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Symmetry in CSPs

- A symmetry transforms any solution into another
  - Sometimes symmetry is inherent in the problem (e.g. chessboard symmetry in n-queens)
  - Sometimes it's introduced in modelling
- Symmetry causes wasted search effort: after exploring choices that don't lead to a solution, symmetrically equivalent choices may be explored

Example: SONET Rings

- Split the demand graph into subgraphs (SONET rings):
  - every edge is in at least one subgraph
  - a subgraph has at most 5 nodes
  - minimize total number of nodes in the subgraphs
- Modelled using Boolean variables, x_{ij} , such that $x_{ij} = 1$ if node i is on ring j
- Introduces symmetry between the rings
  - in the problem, the rings are interchangeable

Symmetry between Values: Car sequencing

- A natural model has individual cars as the values
  - introduces symmetry between cars requiring the same option
- The model instead has *classes* of car
  - needs constraints to ensure the right number of cars in each class

cars	1	2	3	4	5	6	7	8	9	10
option 1	1	0	0	0	0	0	1	1	1	1
option 2	0	0	1	1	1	1	0	0	1	1
option 3	1	0	0	0	0	0	1	1	0	0
option 4	1	1	0	0	1	1	0	0	0	0
option 5	0	0	1	1	0	0	0	0	0	0

classes	1	2	3	4	5	6
option 1	1	0	0	0	1	1
option 2	0	0	1	1	0	1
option 3	1	0	0	0	1	0
option 4	1	1	0	1	0	0
option 5	0	0	1	0	0	0
no. of cars	1	1	2	2	2	2

Symmetry between Variables: Golfers Problem

- 32 golfers want to play in 8 groups of 4 each week, so that any two golfers play in the same group at most once. Find a schedule for *n* weeks
- One viewpoint has 0/1 variables x_{ijkl}:
  - $x_{ijkl} = 1$ if player *i* is the *j* th player in the *k* th group in week *l*, and 0 otherwise.
- The players within each group could be permuted in any solution to give an equivalent solution
  - also the groups within each week, the weeks within the schedule and the players themselves

Reformulating to avoid symmetry: Set Variables

- Eliminate the symmetry between players within a group by using set variables to represent the groups
  - G_{kl} represents the k th group in week l
  - the value of G_{kl} represents the set of players in the group.
- The constraints on these variables are that:
  - the cardinality of each set is 4
  - the sets in any week do not overlap: for all l, the sets G_{kl} , $k = 1,...,8$ have an empty intersection
  - any two sets in different weeks have at most one member in common
- Constraint solvers that support set variables allow constraints of this kind

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Symmetry Breaking

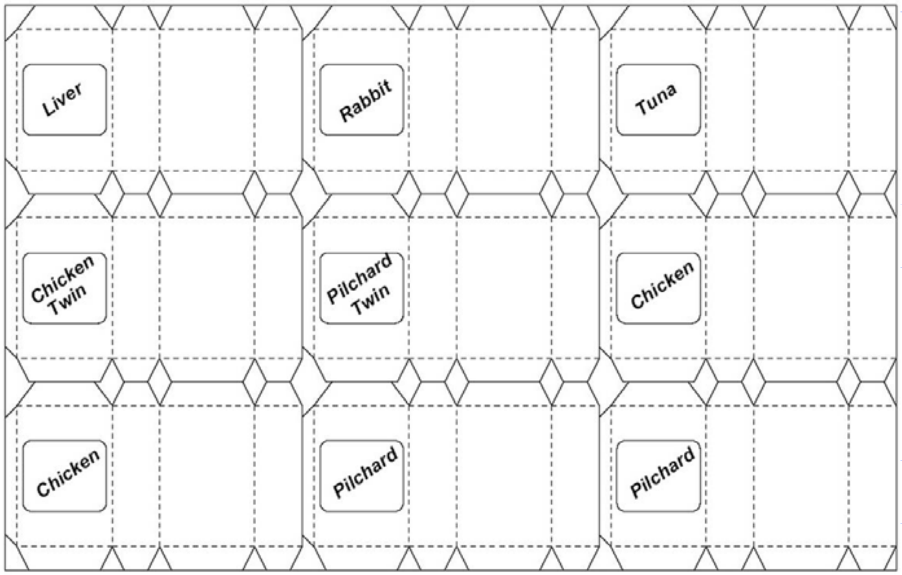
- Often, not all the symmetry can be eliminated by remodelling
- Remaining symmetry should be reduced or eliminated:
  - dynamic symmetry breaking methods (SBDS, SBDD, etc.)
  - symmetry-breaking constraints
    - unlike implied constraints, they change the set of solutions
    - can lead to further implied constraints

Example: Template Design

- Plan layout of printing templates for catfood boxes
- Each template has 9 slots
  - 9 boxes from each sheet of card
- Choose best layout for 1, 2, 3,… templates to minimize waste in meeting order
  - templates are expensive

Flavour	Order (1000s)
Liver	250
Rabbit	255
Tuna	260
Chicken Twin	500
Pilchard Twin	500
Chicken	800
Pilchard	1,100

One Template Solution



Flavour	Order (1000s)
Liver	250
Rabbit	255
Tuna	260
Chicken Twin	500
Pilchard Twin	500
Chicken	800
Pilchard	1,100

- Could meet the order using only one template
  - print it 550,000 times
  - but this wastes a lot of card

CP Model for Template Design

- For a fixed number of templates:
- x_{ij} = number of slots allocated to design j in template i
- r_i = run length for template i (number of sheets printed from this template)
- $\sum_i x_{ij} r_i \geq d_j \qquad j = 1, 2, \ldots, 7$
- minimize $p = \sum_i r_i$ (p = total sheets printed)

Symmetry Breaking & Implied Constraints

- The templates are indistinguishable
- So add $r_1 \leq r_2 \leq \ldots \leq r_t$
- If there are 2 templates:
  - at most half the sheets are printed from one template, at least half from the other
  - so $r_1 \leq p/2$; $r_2 \geq p/2$
- For 3 templates:
  - $r_1 \leq p/3$; $r_2 \leq p/2$; $r_3 \geq p/3$
- These are useful constraints
  - they allow tighter constraints on the objective to propagate to the search variables

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Changing Viewpoint

- We can *improve* a CSP model of a problem
  - express the constraints better
  - break the symmetry
  - add implied constraints
- But sometimes it's better just to use a different model
  - i.e. a different viewpoint

Different Viewpoints

- Reformulate in a standard way, e.g.
  - non-binary to binary translations
  - dual viewpoint for permutation problems
  - Boolean to integer or set viewpoints
- Find a new viewpoint by viewing the problem from a different angle
  - the constraints may express different insights into the problem

Permutation Problems

- A CSP is a permutation problem if:
  - it has the same number of values as variables
  - all variables have the same domain
  - each variable must be assigned a different value
- Any solution assigns a permutation of the values to the variables
- Other constraints determine *which* permutations are solutions
- There is a dual viewpoint in which the variables and values are swapped

Example: *n*-queens

- Standard model
  - a variable for each row, $x_1, x_2, \ldots, x_n$
  - values represent the columns, 1 to n
  - $x_i = j$ means that the queen in row i is in column j
  - n variables, n values, allDifferent($x_1, x_2, \ldots, x_n$)
- Dual viewpoint
  - a variable for each column, $d_1, d_2, \ldots, d_n$; values represent the rows
- In this problem, both viewpoints give the same CSP

Example: Magic Square

- First viewpoint:
  - variables $x_1, x_2, \ldots, x_9$
  - values represent the numbers 1 to 9
  - The assignment (x_i, j) means that the number in square i is j
- Dual viewpoint
  - a variable for each number, $d_1, d_2, \ldots, d_9$
  - values represent the squares
- Constraints are much easier to express in the first viewpoint
  - see earlier

x_1	x_2	x_3
x_4	x_5	x_6
x_7	x_8	x_9

Boolean Models

- Permutation problems: another viewpoint has a Boolean variable b_{ij} for every variable-value combination
  - e.g. in the n-queens problem, $b_{ij} = 1$ if there is a queen on the square in row i and column j, 0 otherwise
- A Boolean viewpoint can be derived from a CSP viewpoint with integer or set variables (or v.v.)
  - in an integer viewpoint, $b_{ij} = 1$ is equivalent to $x_i = j$
  - in a set-variable viewpoint, $j \in X_i$ is equivalent to $b_{ij} = 1$
- The Boolean viewpoint often gives a less efficient CSP than the integer or set model
  - the reverse translation can be useful

Different Perspectives: Example

- Constraint Modelling Challenge, IJCAI 05
- “Minimizing the maximum number of open stacks”
- A manufacturer has a number of orders from customers to satisfy
  - each order is for a number of different products, and only one product can be made at a time
  - once a customer's order is started (i.e. the first product in the order is made) a *stack* is created for that customer
  - when all the products that a customer requires have been made, the stack is closed
  - the number of stacks that are in use simultaneously i.e. the number of customer orders that are in simultaneous production, should be minimized

Minimizing Open Stacks – Example

- The product sequence shown needs 4 stacks
- But if customer 3's products are made before (or after) customer 4's, only 3 are needed
- 3 is the minimum possible because product 2 is for 3 customers

products	1	2	3	4	5	6
customer 1	0	1	0	0	0	1
customer 2	1	1	1	0	0	1
customer 3	1	0	0	1	0	1
customer 4	0	1	0	0	1	0

Open Stacks – Possible Viewpoints

- Variables are positions in production sequence, values are products
  - a permutation problem
  - so has a dual viewpoint

products	1	2	3	4	5	6
customer 1	0	1	0	0	0	1
customer 2	1	1	1	0	0	1
customer 3	1	0	0	1	0	1
customer 4	0	1	0	0	1	0

- In constructing the product sequence, at any point, products that are only for customers that already have open stacks can be inserted straightaway
  - e.g. if product 1 is first, products 3 & 4 can follow
  - the next *real* decision is whether to open a stack for customer 1 or 4 next (or both)
  - leads to a viewpoint based on customers

Open Stacks – Customer Viewpoints

- Variables are positions in *customer* sequence, values are customers
  - $r_i = j$ if the ith customer to have their order completed is j
- A variable for each customer, values are stack locations
  - customers ordering the same product cannot share a stack location
  - a graph colouring problem with additional constraints
- A Boolean variable for each *pair* of customers
  - 0 means they share a stack location, 1 means that they don't
  - NB we want to maximize the number of customers that can share a stack location

Summary

- Symmetry
  - Look out for symmetry in the CSP
    - avoid it if possible by changing the model
    - eliminate it e.g. by adding constraints
    - does this allow more implied constraints?
- Viewpoints
  - don't stick to the first viewpoint you thought of, without considering others
    - think of standard reformulations
    - think about the problem in different ways