Colloquia Patavina
On the twin prime conjecture
by
János Pintz

The twin prime conjecture is one of the oldest (if not the oldest) and most famous unsolved problem of the whole mathematics. We call a pair of primes a twin prime pair if their difference is exactly 2, that is, if they are consecutive odd numbers, which are both prime numbers.

The twin prime conjecture asserts that there are infinitely many twin primes, that is, we can find above arbitrary large bounds such pairs again and again. As an approximation of the conjecture, Hardy and Littlewood began to investigate 80 years ago, whether there are prime pairs much closer to each other than the average distance between consecutive primes. In case of a prime p the average distance is the value $\log p$. As a result of the works of several world famous mathematicians (P. Erdős, E. Bombieri, H. Davenport and others), it was shown that there are infinitely many prime pairs which are closer to each other than the $1/4$ of the average distance (H. Maier, 1988).

In 2005 we proved in a joint work with D.A. Goldston and C.Y. Yıldırım, that the above ratio $1/4$ can be substituted by an arbitrarily small positive constant, that is the distance of the closest prime pairs can be arbitrarily small compared to the average distance. In particular, denoting p_n as the nth prime, we proved that

Theorem (Goldston-Pintz-Yıldırım). $p_{n+1} - p_n < \sqrt{\log n}\,(\log \log n)^2$ *for infinitely many values* n.

Moreover, denoting as **H** the hypothesis that the primes have an admissible distribution level greater than $1/2$, which means that the validity of the Bombieri-Vinogradov theorem can be extended beyond the exponent $1/2$ to a fixed exponent $\theta > 1/2$, we were also able to prove

Theorem (Goldston-Pintz-Yıldırım). *If* **H** *is true then there exists a bound* C *such that* $p_{n+1} - p_n < C$ *for infinitely many values* n. *In other words, there are infinitely many bounded gaps between consecutive primes.*

Stronger results were also obtained for the distribution of the distances between integers having exactly two prime factors (semiprimes); for example in this case we, jointly with S.W. Graham, unconditionally proved that the difference between two consecutive semiprimes is infinitely often at most 6.

In a series of recent papers we were also able to improve such results by getting

Theorem (Goldston-Pintz-Yıldırım). *For any* $c > 0$ *we have* $p_{n+1} - p_n < c \log n$ *for a positive proportion of all gaps* $p_{n+1} - p_n$, *that is, the set of such indices* n *has positive lower density.*

Theorem (J. Pintz). *There exists a constant* C *such that* $p_{n+1} - p_n < (\log n)^{3/7}(\log \log n)^C$ *for infinitely many values* n.

Finally we mention the following celebrated result:

Green-Tao Theorem. *For every* k *the primes contain infinitely many* k-*term arithmetic progressions.*

In connection with the above theorem we were able to prove

Theorem (J. Pintz). *If* **H** *is true then there exists a positive even number* h *such that for every* k *the primes contain infinitely many* k-*term arithmetic progressions with the property that* $p' = p + h$ *for all elements of the arithmetic progression.*

In this Colloquium, after some hystorical remark, we will describe the underlying ideas of the method we developed to study the distribution of prime gaps and that lead us to prove the above mentioned results.