**Parameter selection strategies for the Arnoldi-Tikhonov method**  
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**Introduction**

Consider a linear system

\[ Ax = b, \quad A \in \mathbb{R}^{m \times n}. \]

We assume that

- the singular values \( s_i \geq s_{i+1} \geq \cdots \geq s_{n} > 0 \) of \( A \) quickly decay and cluster at zero;
- the available right-hand side is affected by additive Gaussian white noise, i.e., \( b = b^0 + e \); and
- the Discrete Picard Condition (DPC) holds: \( \| R(A) \| \approx s_n \).

Problems like this arise in a variety of applications, especially solving inverse problems, i.e., when one wants to reconstruct the cause of an observed effect. Here, we focus on the case of \( A \) and the errors in \( b \); some sort of regularization should be employed in order to find a meaningful approximation of the exact solution \( x^0 \).

**Arnoldi-Tikhonov regularization method**

To regularize the available system we employ the well-established Tikhonov method that, in its general form, consists in solving the problem

\[ \min_{\tilde{x} \in \mathbb{R}^n} \{ \| A \tilde{x} - b \|_2^2 + \lambda \| W \tilde{x} \|_2^2 \}, \]

where \( L \in \mathbb{R}^{m \times n} (P \leq N) \) is the regularization parameter and \( x_0 \in \mathbb{R}^n \) is an initial guess for the solution, in the special case \( L = I \) and \( \lambda = 0 \) the problem is said to be in standard form. The Arnoldi-Tikhonov (AT) method solves the regularized system by projecting it into a Krylov subspace

\[ K_m(x_0, r_0) = \text{span} \{ r_0, Ar_0, \ldots, A^{m-1}r_0 \}, \quad \text{with } r_0 = b - Ax_0, \quad m \in \mathbb{N}, \]

of increasing dimension. An orthonormal basis for \( K_m(x_0, r_0) \) is computed by the Arnoldi algorithm that, in matrix form, leads to the decomposition

\[ AV_m = W_mR_m, \]

where \( W_m = [w_0, w_1, \ldots, w_{m-1}] \in \mathbb{R}^{m \times m} \) has orthonormal columns that span \( K_m(x_0, r_0) \). The matrix \( R_m \in \mathbb{R}^{m \times m} \) is upper Hessenberg. Imposing the regularized solution \( x_m \) to belong to the space \( \text{span} \{ r_0, \ldots, A^{m+1}r_0 \} \), i.e., substituting \( x_m = x + W_m \tilde{x} \) into the regularized problem, we obtain

\[ x_m = \arg\min_{\tilde{x} \in \mathbb{R}^n} \left\{ \| R_m \tilde{x} \|_2 + \lambda \| W_m \tilde{x} \|_2 \right\}. \]

**Numerical experiments**

**Theoretical estimates**

**Convergence of the Arnoldi algorithm**

**Theorem 1.** Let us assume that \( A \) is severely ill-conditioned (i.e., \( \sigma_1(A) \approx 0 \)) and that \( \lambda^* \) satisfies the DPC. Then, if \( m^* \) is the starting vector of the Arnoldi process, we have

\[ h_{m^*} = O\left(m^* \lambda^* \right). \]

**Approximation of the SVD**

Let us denote the SVD of \( \bar{A} = U \Sigma V^T \) by

\[ A = U \sqrt{\lambda_0} \Sigma V^T, \]

where \( \lambda_0 \) is the norm of the GMRES residual and \( \beta_m = \lambda_0 \| W_m \|_2 \). Then

\[ \lambda_i = \lambda_0 \left( \lambda_i - \lambda_{i-1} \right), \quad i = 1, \ldots, m. \]

**Proposition 2.** Let \( \left( \lambda_1, \sigma_1 \right), \ldots, \left( \lambda_m, \sigma_m \right) \) denote the residual of the GMRES applied to \( A = \sqrt{\lambda^*} \). one can express

\[ \| R_m \|_2 \approx \sigma_m \| W_m \|_2 \| x_m \|_2. \]

**Behavior of \( |\lambda_m| \)**

Let us define the \( N \times m \) matrices \( U \Sigma_m = \text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_m}) \) and \( \bar{V}^T = V_m^T \). \( \| \bar{V}^T \|_2 \) denotes the residual of the GMRES applied to \( A = \sqrt{\lambda^*} \). one can express

\[ \| \bar{V}^T \|_2 \approx \| x_m \|_2. \]

**Proposition 3.** The norm of the GMRES residual satisfies

\[ \| x_m \|_2^2 \leq \min_{\tilde{x} \in \mathbb{R}^n} \left\{ 1 + \left( \| x_0 \|_2 + \| x_1 \|_2 + \cdots + \| x_m \|_2 \right)^2 \right\}. \]

We choose the following stopping criterion

\[ \| \bar{V}^T \|_2 - \| x_m \|_2 < 10^{-5}. \]

**References**


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**Image deburring & denoising**

In this test problem, the blur is Gaussian symmetric and the corresponding matrix \( A \) is block Toeplitz with Toeplitz blocks. The size of the original image is 256 pixels, therefore \( A \in \mathbb{R}^{256 \times 256} \). As is generated by the function \( \text{blur} \), whose parameters are \( \alpha = 0.1, \sigma = 1 \). The regularization matrix employed is a discretization of the first derivative operator. The noise level is \( 10^{-5} \).