Automatic parameter setting for Arnoldi-Tikhonov methods

Silvia Gazzola
Joint work with **Paolo Novati**

Department of Mathematics
University of Padova - Italy

3rd Dolomites Workshop on
Constructive Approximation and Application

Outline

Outline

Outline

1. The Arnoldi-Tikhonov method
   - The problem
   - The (standard) Arnoldi-Tikhonov (AT) method
   - The Generalized AT (GAT) method
2. The parameter selection strategy
   - Geometric interpretation
3. Examples
   - Common test problems
   - Image restoration
4. Noise level detection
   - The algorithm
   - Example
5. Final remarks

Outline

Outline

The problem

We consider linear systems of equations

$$Ax = b, \quad A \in \mathbb{R}^{N\times N}, \quad b \in \mathbb{R}^{N},$$

in which the matrix A is assumed to have singular values that rapidly decay and cluster near zero[1].
We assume that

- the available right-hand side vector b is affected by noise, that is

$$b = \overline{b} + e,$$

  where $\overline{b}$ represents the unknown noise-free right-hand side;
- the quantity $\varepsilon \approx \|e\|$ is known;

[1] Hansen(1998), *Rank-deficient and Discrete Ill-Posed Problems*

The problem

We consider linear systems of equations

$$Ax = b, \quad A \in \mathbb{R}^{N\times N}, \quad b \in \mathbb{R}^{N},$$

in which the matrix A is assumed to have singular values that rapidly decay and cluster near zero[1].
We assume that

- the available right-hand side vector b is affected by noise, that is
$$b = \overline{b} + e,$$
where $\overline{b}$ represents the unknown noise-free right-hand side;
- the quantity $\varepsilon \approx \|e\|$ is known;

[1] HANSEN(1998), *Rank-deficient and Discrete Ill-Posed Problems*

The problem

We consider linear systems of equations

$$Ax = b, \quad A \in \mathbb{R}^{N\times N}, \quad b \in \mathbb{R}^N,$$

in which the matrix A is assumed to have singular values that rapidly decay and cluster near zero[1].
We assume that

- the available right-hand side vector b is affected by noise, that is

$$b = \overline{b} + e,$$

  where $\overline{b}$ represents the unknown noise-free right-hand side;
- the quantity $\varepsilon \approx \|e\|$ is known;

[1] HANSEN(1998), *Rank-deficient and Discrete Ill-Posed Problems*

The Arnoldi-Tikhonov (AT) method

Given λ, L, x_0 consider the Tikhonov regularization

$$\min_{x\in\mathbb{R}^N}\left\{\|Ax-b\|^2+\lambda\|L(x-x_0)\|^2\right\}.$$

For the special case of $L = I_N$ and $x_0 = 0$, the Arnoldi-Tikhonov method[2] is based on the reduction to a problem of much smaller dimension, projecting the matrix A onto the Krylov subspaces generated by A and the vector b,

$$\mathcal{K}_m(A,b)=\mathrm{span}\{b,Ab,\dots,A^{m-1}b\},\quad m\ll N.$$

For the construction of the Krylov subspaces the AT method uses the Arnoldi algorithm.

[2] CALVETTI-MORIGI-REICHEL-SGALLARI(2000), *Tikhonov regularization and the L-curve for large discrete ill-posed problems*

The Arnoldi-Tikhonov (AT) method

Given λ, L, x_0 consider the Tikhonov regularization

$$\min_{x\in\mathbb{R}^N} \left\{ \|Ax-b\|^2 + \lambda\|L(x-x_0)\|^2 \right\}.$$

For the special case of $L = I_N$ and $x_0 = 0$, the Arnoldi-Tikhonov method[2] is based on the reduction to a problem of much smaller dimension, projecting the matrix A onto the Krylov subspaces generated by A and the vector b,

$$\mathcal{K}_m(A,b) = \mathrm{span}\{b, Ab, \dots, A^{m-1}b\}, \quad m \ll N.$$

For the construction of the Krylov subspaces the AT method uses the Arnoldi algorithm.

[2] Calvetti-Morigi-Reichel-Sgallari(2000), *Tikhonov regularization and the L-curve for large discrete ill-posed problems*

The Arnoldi-Tikhonov (AT) method

Given λ, L, x_0 consider the Tikhonov regularization

$$\min_{x \in \mathbb{R}^N} \left\{ \|Ax - b\|^2 + \lambda \|L(x - x_0)\|^2 \right\}.$$

For the special case of $L = I_N$ and $x_0 = 0$, the Arnoldi-Tikhonov method[2] is based on the reduction to a problem of much smaller dimension, projecting the matrix A onto the Krylov subspaces generated by A and the vector b,

$$\mathcal{K}_m(A, b) = \mathrm{span}\{b, Ab, \dots, A^{m-1}b\}, \quad m \ll N.$$

For the construction of the Krylov subspaces the AT method uses the Arnoldi algorithm.

[2] Calvetti-Morigi-Reichel-Sgallari(2000), *Tikhonov regularization and the L-curve for large discrete ill-posed problems*

The Arnoldi algorithm yields the decomposition

$$AV_m = V_{m+1}\overline{H}_m,$$

where

- $V_{m+1} = [v_1, ..., v_{m+1}] \in \mathbb{R}^{N\times(m+1)}$ has orthonormal columns which span the Krylov subspace $\mathcal{K}_{m+1}(A, b)$;
  v_1 is defined as $b/\|b\|$.
- $\overline{H}_m \in \mathbb{R}^{(m+1)\times m}$ is an upper Hessenberg matrix.

The AT method searches for approximations belonging to $\mathcal{K}_m(A, b)$.
Taking $x = V_m y_m$ ($y_m \in \mathbb{R}^m$) we obtain the reduced minimization problem

$$\min_{y_m\in\mathbb{R}^m}\left\{\left\|\overline{H}_m y_m - V_{m+1}^T b\right\|^2 + \lambda\|y_m\|^2\right\}.$$

Parameter choice strategy: L_m-curve criterium.

The Arnoldi algorithm yields the decomposition

$$AV_m = V_{m+1}\overline{H}_m,$$

where

- $V_{m+1} = [v_1, ..., v_{m+1}] \in \mathbb{R}^{N\times(m+1)}$ has orthonormal columns which span the Krylov subspace $\mathcal{K}_{m+1}(A, b)$;
  v_1 is defined as $b/\left\|b\right\|$.
- $\overline{H}_m \in \mathbb{R}^{(m+1)\times m}$ is an upper Hessenberg matrix.

The AT method searches for approximations belonging to $\mathcal{K}_m(A, b)$.
Taking $x = V_m y_m$ ($y_m \in \mathbb{R}^m$) we obtain the reduced minimization problem

$$\min_{y_m\in\mathbb{R}^m}\left\{\left\|\overline{H}_m y_m - V_{m+1}^T b\right\|^2 + \lambda\left\|y_m\right\|^2\right\}.$$

Parameter choice strategy: L_m-curve criterium.

The Arnoldi algorithm yields the decomposition

$$AV_m = V_{m+1}\overline{H}_m,$$

where

- $V_{m+1} = [v_1, ..., v_{m+1}] \in \mathbb{R}^{N\times(m+1)}$ has orthonormal columns which span the Krylov subspace $\mathcal{K}_{m+1}(A, b)$;
  v_1 is defined as $b/\|b\|$.
- $\overline{H}_m \in \mathbb{R}^{(m+1)\times m}$ is an upper Hessenberg matrix.

The AT method searches for approximations belonging to $\mathcal{K}_m(A, b)$.
Taking $x = V_m y_m$ ($y_m \in \mathbb{R}^m$) we obtain the reduced minimization problem

$$\min_{y_m\in\mathbb{R}^m} \left\{ \left\|\overline{H}_m y_m - V_{m+1}^T b\right\|^2 + \lambda \|y_m\|^2 \right\}.$$

Parameter choice strategy: L_m-curve criterium.

The Arnoldi algorithm yields the decomposition

$$AV_m = V_{m+1}\overline{H}_m,$$

where

- $V_{m+1} = [v_1, \dots, v_{m+1}] \in \mathbb{R}^{N\times(m+1)}$ has orthonormal columns which span the Krylov subspace $\mathcal{K}_{m+1}(A, b)$;
  v_1 is defined as $b/\|b\|$.
- $\overline{H}_m \in \mathbb{R}^{(m+1)\times m}$ is an upper Hessenberg matrix.

The AT method searches for approximations belonging to $\mathcal{K}_m(A, b)$.
Taking $x = V_m y_m$ ($y_m \in \mathbb{R}^m$) we obtain the reduced minimization problem

$$\min_{y_m\in\mathbb{R}^m} \left\{ \left\|\overline{H}_m y_m - V_{m+1}^T b\right\|^2 + \lambda \|y_m\|^2 \right\}.$$

Parameter choice strategy: L_m-curve criterium.

The Arnoldi algorithm yields the decomposition

$$AV_m = V_{m+1}\overline{H}_m,$$

where

- $V_{m+1} = [v_1, ..., v_{m+1}] \in \mathbb{R}^{N\times(m+1)}$ has orthonormal columns which span the Krylov subspace $\mathcal{K}_{m+1}(A, b)$;
  v_1 is defined as $b/\|b\|$.
- $\overline{H}_m \in \mathbb{R}^{(m+1)\times m}$ is an upper Hessenberg matrix.

The AT method searches for approximations belonging to $\mathcal{K}_m(A, b)$.
Taking $x = V_m y_m$ ($y_m \in \mathbb{R}^m$) we obtain the reduced minimization problem

$$\min_{y_m\in\mathbb{R}^m} \left\{ \left\| \overline{H}_m y_m - V_{m+1}^T b \right\|^2 + \lambda \|y_m\|^2 \right\}.$$

Parameter choice strategy: L_m-curve criterium.

The Generalized Arnoldi-Tikhonov (GAT) method

Extension of the AT method in order to work with a general regularization operator $L \neq I_N$ and with an arbitrary starting vector x_0.

Starting from

$$\min_{x \in \mathbb{R}^N} \left\{ \|Ax - b\|^2 + \lambda \|L(x - x_0)\|^2 \right\},$$

we search for approximations of the type

$$x_m = x_0 + V_m y_m,$$

where the columns of $V_m \in \mathbb{R}^{N \times m}$ span the Krylov subspace $\mathcal{K}_m(A, r_0)$, where $r_0 = b - Ax_0$.

We obtain the reduced minimization problem

$$\begin{aligned} & \min_{y_m \in \mathbb{R}^m} \left\{ \|AV_m y_m - r_0\|^2 + \lambda \|LV_m y_m\|^2 \right\} \\ = & \min_{y_m \in \mathbb{R}^m} \left\{ \left\|\overline{H}_m y_m - \|r_0\| e_1\right\|^2 + \lambda \|LV_m y_m\|^2 \right\} \\ = & \min_{y_m \in \mathbb{R}^m} \left\| \begin{pmatrix} \overline{H}_m \\ \sqrt{\lambda} LV_m \end{pmatrix} y_m - \begin{pmatrix} \|r_0\| e_1 \\ 0 \end{pmatrix} \right\|^2. \end{aligned}$$

The Generalized Arnoldi-Tikhonov (GAT) method

Extension of the AT method in order to work with a general regularization operator $L \neq I_N$ and with an arbitrary starting vector x_0.
Starting from

$$\min_{x \in \mathbb{R}^N} \left\{ \|Ax - b\|^2 + \lambda \|L(x - x_0)\|^2 \right\},$$

we search for approximations of the type

$$x_m = x_0 + V_m y_m,$$

where the columns of $V_m \in \mathbb{R}^{N \times m}$ span the Krylov subspace $\mathcal{K}_m(A, r_0)$, where $r_0 = b - Ax_0$.
We obtain the reduced minimization problem

$$\begin{aligned} & \min_{y_m \in \mathbb{R}^m} \left\{ \|AV_m y_m - r_0\|^2 + \lambda \|LV_m y_m\|^2 \right\} \\ = & \min_{y_m \in \mathbb{R}^m} \left\{ \left\|\overline{H}_m y_m - \|r_0\| e_1\right\|^2 + \lambda \|LV_m y_m\|^2 \right\} \\ = & \min_{y_m \in \mathbb{R}^m} \left\| \begin{pmatrix} \overline{H}_m \\ \sqrt{\lambda} LV_m \end{pmatrix} y_m - \begin{pmatrix} \|r_0\| e_1 \\ 0 \end{pmatrix} \right\|^2. \end{aligned}$$

The Generalized Arnoldi-Tikhonov (GAT) method

Extension of the AT method in order to work with a general regularization operator $L \neq I_N$ and with an arbitrary starting vector x_0.
Starting from

$$\min_{x \in \mathbb{R}^N} \left\{ \|Ax - b\|^2 + \lambda \|L(x - x_0)\|^2 \right\},$$

we search for approximations of the type

$$x_m = x_0 + V_m y_m,$$

where the columns of $V_m \in \mathbb{R}^{N \times m}$ span the Krylov subspace $\mathcal{K}_m(A, r_0)$, where $r_0 = b - Ax_0$.
We obtain the reduced minimization problem

$$\begin{aligned} & \min_{y_m \in \mathbb{R}^m} \left\{ \|AV_m y_m - r_0\|^2 + \lambda \|LV_m y_m\|^2 \right\} \\ = & \min_{y_m \in \mathbb{R}^m} \left\{ \left\|\overline{H}_m y_m - \|r_0\| e_1\right\|^2 + \lambda \|LV_m y_m\|^2 \right\} \\ = & \min_{y_m \in \mathbb{R}^m} \left\| \begin{pmatrix} \overline{H}_m \\ \sqrt{\lambda} LV_m \end{pmatrix} y_m - \begin{pmatrix} \|r_0\| e_1 \\ 0 \end{pmatrix} \right\|^2 . \end{aligned}$$

The parameter selection strategy

The discrepancy principle is satisfied as soon as

$$\phi_m(\lambda) := \|b - Ax_{m,\lambda}\| \leq \eta\varepsilon, \quad \eta \gtrapprox 1.$$

For the GAT method the approximations are $x_{m,\lambda} = x_0 + V_m y_{m,\lambda}$ and the discrepancy can be rewritten as

$$\|b - Ax_{m,\lambda}\| = \|r_0 - AV_m y_{m,\lambda}\| = \|c - \overline{H}_m y_{m,\lambda}\|,$$

where $c = V_{m+1}^T r_0 = \|r_0\| e_1 \in \mathbb{R}^{m+1}$.
Since $y_{m,\lambda}$ solves the normal equation

$$(\overline{H}_m^T \overline{H}_m + \lambda V_m^T L^T L V_m) y_{m,\lambda} = \overline{H}_m^T c,$$

we obtain

$$\phi_m(\lambda) = \left\| \overline{H}_m (\overline{H}_m^T \overline{H}_m + \lambda V_m^T L^T L V_m)^{-1} \overline{H}_m^T c - c \right\|.$$

The parameter selection strategy

The discrepancy principle is satisfied as soon as

$$\phi_m(\lambda) := \|b - Ax_{m,\lambda}\| \leq \eta\varepsilon, \quad \eta \gtrapprox 1.$$

For the GAT method the approximations are $x_{m,\lambda} = x_0 + V_m y_{m,\lambda}$ and the discrepancy can be rewritten as

$$\|b - Ax_{m,\lambda}\| = \|r_0 - AV_m y_{m,\lambda}\| = \|c - \overline{H}_m y_{m,\lambda}\|,$$

where $c = V_{m+1}^T r_0 = \|r_0\| e_1 \in \mathbb{R}^{m+1}$.

Since $y_{m,\lambda}$ solves the normal equation

$$(\overline{H}_m^T \overline{H}_m + \lambda V_m^T L^T L V_m) y_{m,\lambda} = \overline{H}_m^T c,$$

we obtain

$$\phi_m(\lambda) = \left\| \overline{H}_m (\overline{H}_m^T \overline{H}_m + \lambda V_m^T L^T L V_m)^{-1} \overline{H}_m^T c - c \right\|.$$

The parameter selection strategy

The discrepancy principle is satisfied as soon as

$$\phi_m(\lambda) := \|b - Ax_{m,\lambda}\| \leq \eta\varepsilon, \quad \eta \gtrapprox 1.$$

For the GAT method the approximations are $x_{m,\lambda} = x_0 + V_m y_{m,\lambda}$ and the discrepancy can be rewritten as

$$\|b - Ax_{m,\lambda}\| = \|r_0 - AV_m y_{m,\lambda}\| = \|c - \overline{H}_m y_{m,\lambda}\|,$$

where $c = V_{m+1}^T r_0 = \|r_0\| e_1 \in \mathbb{R}^{m+1}$.
Since $y_{m,\lambda}$ solves the normal equation

$$(\overline{H}_m^T \overline{H}_m + \lambda V_m^T L^T L V_m) y_{m,\lambda} = \overline{H}_m^T c,$$

we obtain

$$\phi_m(\lambda) = \left\| \overline{H}_m (\overline{H}_m^T \overline{H}_m + \lambda V_m^T L^T L V_m)^{-1} \overline{H}_m^T c - c \right\|.$$

- Standard approach[3] [4]:
  solve with respect to λ the nonlinear equation

$$\phi_m(\lambda) = \eta\varepsilon.$$

- New approach:
  Basic hypothesis: the discrepancy can be well approximated by

$$\phi_m(\lambda) \approx \alpha_m + \lambda\beta_m$$

  in which $\alpha_m, \beta_m \in \mathbb{R}$ can be easily computed.

Definition of α_m. The Taylor expansion of $\phi_m(\lambda)$ suggests to chose

$$\alpha_m = \phi_m(0) = \left\| \overline{H}_m (\overline{H}_m^T \overline{H}_m)^{-1} \overline{H}_m^T c - c \right\|,$$

which is just the residual of the GMRES (computed working in reduced dimension).

[3] Reichel-Shyshkov(2008), *A new zero-finder for Tikhonov regularization.*
[4] Lewis-Reichel(2009), *Arnoldi-Tikhonov regularization methods.*

- Standard approach[3] [4]:
  solve with respect to λ the nonlinear equation

$$\phi_m(\lambda) = \eta\varepsilon.$$

- New approach:
  Basic hypothesis: the discrepancy can be well approximated by

$$\phi_m(\lambda) \approx \alpha_m + \lambda\beta_m$$

  in which $\alpha_m, \beta_m \in \mathbb{R}$ can be easily computed.

Definition of α_m. The Taylor expansion of $\phi_m(\lambda)$ suggests to chose

$$\alpha_m = \phi_m(0) = \left\| \overline{H}_m(\overline{H}_m^T\overline{H}_m)^{-1}\overline{H}_m^T c - c \right\|,$$

which is just the residual of the GMRES (computed working in reduced dimension).

[3] Reichel-Shyshkov(2008), *A new zero-finder for Tikhonov regularization.*
[4] Lewis-Reichel(2009), *Arnoldi-Tikhonov regularization methods.*

- Standard approach[3 4]:
  solve with respect to λ the nonlinear equation

$$\phi_m(\lambda) = \eta\varepsilon.$$

- New approach:
  Basic hypothesis: the discrepancy can be well approximated by

$$\phi_m(\lambda) \approx \alpha_m + \lambda\beta_m$$

  in which $\alpha_m, \beta_m \in \mathbb{R}$ can be easily computed.

Definition of α_m. The Taylor expansion of $\phi_m(\lambda)$ suggests to chose

$$\alpha_m = \phi_m(0) = \left\| \overline{H}_m(\overline{H}_m^T\overline{H}_m)^{-1}\overline{H}_m^T c - c \right\|,$$

which is just the residual of the GMRES (computed working in reduced dimension).

[3] Reichel-Shyshkov(2008), *A new zero-finder for Tikhonov regularization.*
[4] Lewis-Reichel(2009), *Arnoldi-Tikhonov regularization methods.*

- Standard approach[3 4]:
  solve with respect to λ the nonlinear equation

$$\phi_m(\lambda) = \eta\varepsilon.$$

- New approach:
  Basic hypothesis: the discrepancy can be well approximated by

$$\phi_m(\lambda) \approx \alpha_m + \lambda\beta_m$$

  in which $\alpha_m, \beta_m \in \mathbb{R}$ can be easily computed.

Definition of α_m. The Taylor expansion of $\phi_m(\lambda)$ suggests to chose

$$\alpha_m = \phi_m(0) = \left\| \overline{H}_m(\overline{H}_m^T\overline{H}_m)^{-1}\overline{H}_m^T c - c \right\|,$$

which is just the residual of the GMRES (computed working in reduced dimension).

[3] Reichel-Shyshkov(2008), *A new zero-finder for Tikhonov regularization.*
[4] Lewis-Reichel(2009), *Arnoldi-Tikhonov regularization methods.*

- Standard approach[3 4]:
  solve with respect to λ the nonlinear equation

$$\phi_m(\lambda) = \eta\varepsilon.$$

- New approach:
  Basic hypothesis: the discrepancy can be well approximated by

$$\phi_m(\lambda) \approx \alpha_m + \lambda\beta_m$$

in which $\alpha_m, \beta_m \in \mathbb{R}$ can be easily computed.

Definition of α_m. The Taylor expansion of $\phi_m(\lambda)$ suggests to chose

$$\alpha_m = \phi_m(0) = \left\| \overline{H}_m(\overline{H}_m^T\overline{H}_m)^{-1}\overline{H}_m^T c - c \right\|,$$

which is just the residual of the GMRES (computed working in reduced dimension).

[3] Reichel-Shyshkov(2008), *A new zero-finder for Tikhonov regularization.*
[4] Lewis-Reichel(2009), *Arnoldi-Tikhonov regularization methods.*

- Standard approach[3] [4]:
  solve with respect to λ the nonlinear equation

$$\phi_m(\lambda) = \eta\varepsilon.$$

- New approach:
  Basic hypothesis: the discrepancy can be well approximated by

$$\phi_m(\lambda) \approx \alpha_m + \lambda\beta_m$$

  in which $\alpha_m, \beta_m \in \mathbb{R}$ can be easily computed.

Definition of

α_m. The Taylor expansion of $\phi_m(\lambda)$ suggests to chose

$$\alpha_m = \phi_m(0) = \left\| \overline{H}_m(\overline{H}_m^T\overline{H}_m)^{-1}\overline{H}_m^T c - c \right\|,$$

which is just the residual of the GMRES (computed working in reduced dimension).

[3] Reichel-Shyshkov(2008), *A new zero-finder for Tikhonov regularization.*

[4] Lewis-Reichel(2009), *Arnoldi-Tikhonov regularization methods.*

β_m. Suppose that, at step m, we have used the parameter λ_{m-1} (computed at the previous step or, if $m = 1$, given by the user) to compute $y_{m,\lambda_{m-1}}$, by solving the reduced minimization, and the corresponding discrepancy

$$\phi_m(\lambda_{m-1}) = \left\| c - \overline{H}_m y_{m,\lambda_{m-1}} \right\| .$$

Using the linear approximation, we obtain

$$\beta_m \approx \frac{\phi_m(\lambda_{m-1}) - \alpha_m}{\lambda_{m-1}}.$$

To select λ_m for the next step of the Arnoldi algorithm we impose $\phi_m(\lambda_m) = \eta\varepsilon$, and force the approximation

$$\phi_m(\lambda_m) \approx \alpha_m + \lambda_m \beta_m.$$

Hence we define

$$\lambda_m = \frac{\eta\varepsilon - \alpha_m}{\phi_m(\lambda_{m-1}) - \alpha_m} \lambda_{m-1}.$$

β_m. Suppose that, at step m, we have used the parameter λ_{m-1} (computed at the previous step or, if $m = 1$, given by the user) to compute $y_{m,\lambda_{m-1}}$, by solving the reduced minimization, and the corresponding discrepancy

$$\phi_m(\lambda_{m-1}) = \left\| c - \overline{H}_m y_{m,\lambda_{m-1}} \right\| .$$

Using the linear approximation, we obtain

$$\beta_m \approx \frac{\phi_m(\lambda_{m-1}) - \alpha_m}{\lambda_{m-1}}.$$

To select λ_m for the next step of the Arnoldi algorithm we impose $\phi_m(\lambda_m) = \eta\varepsilon$, and force the approximation

$$\phi_m(\lambda_m) \approx \alpha_m + \lambda_m \beta_m.$$

Hence we define

$$\lambda_m = \frac{\eta\varepsilon - \alpha_m}{\phi_m(\lambda_{m-1}) - \alpha_m}\lambda_{m-1}.$$

β_m. Suppose that, at step m, we have used the parameter λ_{m-1} (computed at the previous step or, if $m = 1$, given by the user) to compute $y_{m,\lambda_{m-1}}$, by solving the reduced minimization, and the corresponding discrepancy

$$\phi_m(\lambda_{m-1}) = \left\| c - \overline{H}_m y_{m,\lambda_{m-1}} \right\|.$$

Using the linear approximation, we obtain

$$\beta_m \approx \frac{\phi_m(\lambda_{m-1}) - \alpha_m}{\lambda_{m-1}}.$$

To select λ_m for the next step of the Arnoldi algorithm we impose $\phi_m(\lambda_m) = \eta\varepsilon$, and force the approximation

$$\phi_m(\lambda_m) \approx \alpha_m + \lambda_m \beta_m.$$

Hence we define

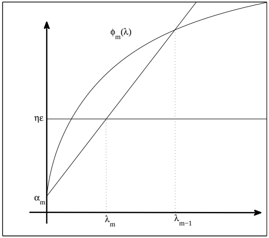
$$\lambda_m = \frac{\eta\varepsilon - \alpha_m}{\phi_m(\lambda_{m-1}) - \alpha_m} \lambda_{m-1}.$$

Geometric interpretation

We know that $\phi_m(\lambda)$ is a monotonically increasing function such that $\phi_m(0) = \alpha_m$.
Hence, the linear function

$$f(\lambda) = \alpha_m + \lambda \left(\frac{\phi_m(\lambda_{m-1}) - \alpha_m}{\lambda_{m-1}} \right),$$

interpolates $\phi_m(\lambda)$ at 0 and λ_{m-1}, and the new parameter λ_m is obtained by solving $f(\lambda) = \eta\varepsilon$.



secant update method.

- The method is just a secant method in which the leftmost point is $(0, \alpha_m)$.
- In the very first steps we may have $\alpha_m > \eta\varepsilon$. In this situation the result of the method may be negative and therefore we use

$$\lambda_m = \left| \frac{\eta\varepsilon - \alpha_m}{\phi_m(\lambda_{m-1}) - \alpha_m} \right| \lambda_{m-1}.$$

Examples

Test problem shaw[5]

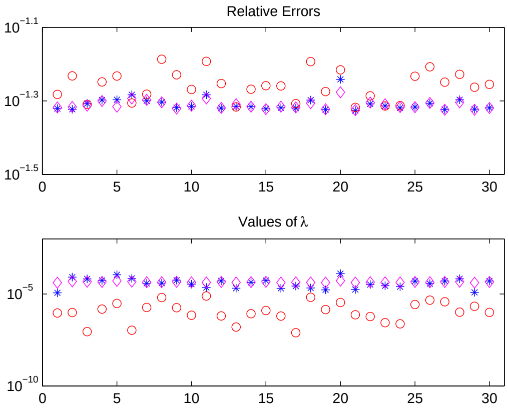
- $A \in \mathbb{R}^{200\times 200}$ symmetric; $\mathrm{cond}(A) \simeq 10^{20}$;
- noise level $\widetilde{\varepsilon} = \|e\|/\|\overline{b}\| = 10^{-3}$;
- $L = I_{200}$;
- $\eta = 1.001$, $\lambda_0 = 1$ $x_0 = 0$ (as always).

......

$*$ new method

○ L_m-curve method

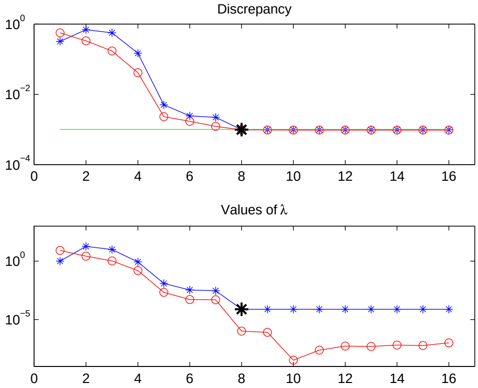
◇ $\phi_8(\lambda) = \eta\varepsilon$ by Newton's method



$x_{m,\lambda} \in \mathcal{K}_8(A, b)$.

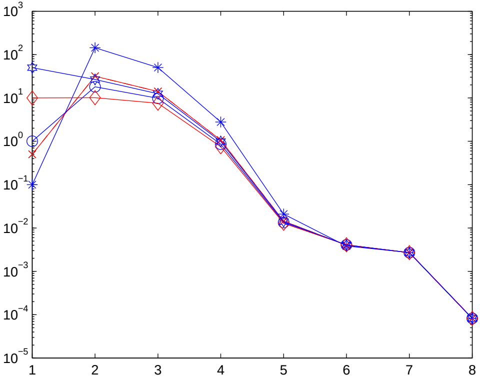
[5] HANSEN(1994), *Regularization Tools*

Comparison of the L_m-curve method and of the secant update approach at each iteration.
Discrepancy principle satisfied after 8 iterations; we compute 8 extra iterations. The secant approach exhibits a very stable behavior.



o– is the L_m-curve method, ∗– is the new method.

Stability with respect to the choice of the initial value λ_0.
We choose $\lambda_0 = 0.1,\ 0.5,\ 1,\ 10,\ 50$.



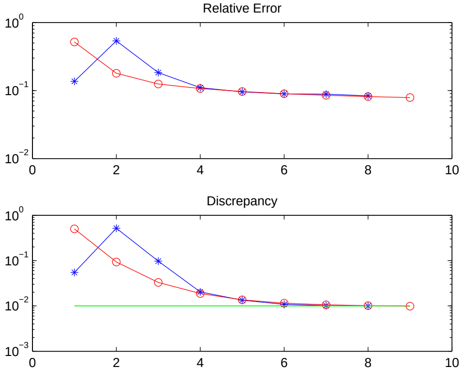
Remark: at the beginning we just force λ_m to be positive; after a few iterations (when $\alpha_m < \eta\varepsilon$) the new approach is a zero-finder.

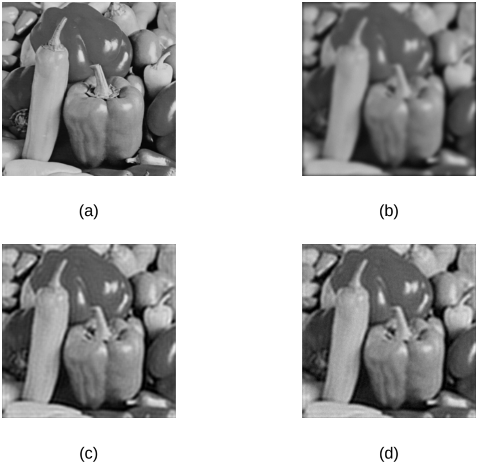
Image restoration

- Test image: `peppers.png`, size 256×256 pixels; $N = 65536$.
- Noise level: $\widetilde{\varepsilon} = 10^{-2}$.
- Gaussian blur: $\sigma = 2.5$, $q = 6$.
- Regularization matrix: $D_2 \in \mathbb{R}^{(N-2)\times N}$.

o– L_m method

∗– secant update





(a) original image (b) blurred and noisy image

(c) restored with L_m-curve approach (d) restored with secant approach

Noise level detection

Assumption: ε is overestimated by a quantity $\overline{\varepsilon}$.

Therefore, applying the GAT method we can fully satisfy the discrepancy principle (even with $\eta = 1$),

$$\phi_m(\lambda_{m-1}) < \overline{\varepsilon}.$$

Applying the secant update method the discrepancy would then stabilize around $\overline{\varepsilon}$.

We define $\overline{\varepsilon} = \phi_m(\lambda_{m-1})$ as the new approximation of the noise.

We restart the GAT method immediately with the Krylov subspace $\mathcal{K}_\ell(A, b - Ax_{m,\lambda_{m-1}})$, where $x_{m,\lambda_{m-1}}$ is the last approximation obtained.

We proceed until the discrepancy is almost constant.

Noise level detection

Assumption: ε is overestimated by a quantity $\overline{\varepsilon}$.
Therefore, applying the GAT method we can fully satisfy the discrepancy principle (even with $\eta = 1$),

$$\phi_m(\lambda_{m-1}) < \overline{\varepsilon}.$$

Applying the secant update method the discrepancy would then stabilize around $\overline{\varepsilon}$.
We define $\overline{\varepsilon} = \phi_m(\lambda_{m-1})$ as the new approximation of the noise.
We restart the GAT method immediately with the Krylov subspace $\mathcal{K}_\ell(A, b - Ax_{m,\lambda_{m-1}})$, where $x_{m,\lambda_{m-1}}$ is the last approximation obtained.
We proceed until the discrepancy is almost constant.

Noise level detection

Assumption: ε is overestimated by a quantity $\overline{\varepsilon}$.
Therefore, applying the GAT method we can fully satisfy the discrepancy principle (even with $\eta = 1$),

$$\phi_m(\lambda_{m-1}) < \overline{\varepsilon}.$$

Applying the secant update method the discrepancy would then stabilize around $\overline{\varepsilon}$.
We define $\overline{\varepsilon} = \phi_m(\lambda_{m-1})$ as the new approximation of the noise.
We restart the GAT method immediately with the Krylov subspace $\mathcal{K}_\ell(A, b - Ax_{m,\lambda_{m-1}})$, where $x_{m,\lambda_{m-1}}$ is the last approximation obtained.
We proceed until the discrepancy is almost constant.

Noise level detection

Assumption: ε is overestimated by a quantity $\overline{\varepsilon}$.
Therefore, applying the GAT method we can fully satisfy the discrepancy principle (even with $\eta = 1$),

$$\phi_m(\lambda_{m-1}) < \overline{\varepsilon}.$$

Applying the secant update method the discrepancy would then stabilize around $\overline{\varepsilon}$.
We define $\overline{\varepsilon} = \phi_m(\lambda_{m-1})$ as the new approximation of the noise.
We restart the GAT method immediately with the Krylov subspace $\mathcal{K}_\ell(A, b - Ax_{m,\lambda_{m-1}})$, where $x_{m,\lambda_{m-1}}$ is the last approximation obtained.
We proceed until the discrepancy is almost constant.

Noise level detection

Assumption: ε is overestimated by a quantity $\overline{\varepsilon}$.
Therefore, applying the GAT method we can fully satisfy the discrepancy principle (even with $\eta = 1$),

$$\phi_m(\lambda_{m-1}) < \overline{\varepsilon}.$$

Applying the secant update method the discrepancy would then stabilize around $\overline{\varepsilon}$.
We define $\overline{\varepsilon} = \phi_m(\lambda_{m-1})$ as the new approximation of the noise.
We restart the GAT method immediately with the Krylov subspace $\mathcal{K}_\ell(A, b - Ax_{m,\lambda_{m-1}})$, where $x_{m,\lambda_{m-1}}$ is the last approximation obtained.
We proceed until the discrepancy is almost constant.

The algorithm

Input: A, b, L, $\lambda^{(0)}$, η, δ (threshold parameter), and $\varepsilon_0 = \overline{\varepsilon} > \varepsilon$.
Define $x^{(0)} = 0$.
For $k = 1, 2, \ldots$ until

$$\frac{\|\varepsilon_k - \varepsilon_{k-1}\|}{\|\varepsilon_{k-1}\|} \leq \delta$$

1. Apply GAT method with $x_0 = x^{(k-1)}$, $\varepsilon = \varepsilon_{k-1}$, $\lambda_0 = \lambda^{(k-1)}$. Let $x^{(k)}$ be the last approximation achieved, $\phi^{(k)}$ the corresponding discrepancy norm, and $\lambda^{(k)}$ the last parameter value;
2. Define $\varepsilon_k = \phi^{(k)}$;
3. Define $\lambda^{(k)} = \frac{\phi^{(k)}}{\phi^{(k-1)}}\lambda^{(k)}$.

The algorithm

Input: A, b, L, $\lambda^{(0)}$, η, δ (threshold parameter), and $\varepsilon_0 = \overline{\varepsilon} > \varepsilon$.
Define $x^{(0)} = 0$.
For $k = 1, 2, \ldots$ until

$$\frac{\|\varepsilon_k - \varepsilon_{k-1}\|}{\|\varepsilon_{k-1}\|} \leq \delta$$

1. Apply GAT method with $x_0 = x^{(k-1)}$, $\varepsilon = \varepsilon_{k-1}$, $\lambda_0 = \lambda^{(k-1)}$. Let $x^{(k)}$ be the last approximation achieved, $\phi^{(k)}$ the corresponding discrepancy norm, and $\lambda^{(k)}$ the last parameter value;
2. Define $\varepsilon_k = \phi^{(k)}$;
3. Define $\lambda^{(k)} = \frac{\phi^{(k)}}{\phi^{(k-1)}} \lambda^{(k)}$.

The algorithm

Input: A, b, L, $\lambda^{(0)}$, η, δ (threshold parameter), and $\varepsilon_0 = \overline{\varepsilon} > \varepsilon$.
Define $x^{(0)} = 0$.
For $k = 1, 2, \dots$ until

$$\frac{\|\varepsilon_k - \varepsilon_{k-1}\|}{\|\varepsilon_{k-1}\|} \leq \delta$$

1. Apply GAT method with $x_0 = x^{(k-1)}$, $\varepsilon = \varepsilon_{k-1}$, $\lambda_0 = \lambda^{(k-1)}$. Let $x^{(k)}$ be the last approximation achieved, $\phi^{(k)}$ the corresponding discrepancy norm, and $\lambda^{(k)}$ the last parameter value;
2. Define $\varepsilon_k = \phi^{(k)}$;
3. Define $\lambda^{(k)} = \frac{\phi^{(k)}}{\phi^{(k-1)}}\lambda^{(k)}$.

The algorithm

Input: A, b, L, $\lambda^{(0)}$, η, δ (threshold parameter), and $\varepsilon_0 = \overline{\varepsilon} > \varepsilon$.
Define $x^{(0)} = 0$.
For $k = 1, 2, \dots$ until

$$\frac{\|\varepsilon_k - \varepsilon_{k-1}\|}{\|\varepsilon_{k-1}\|} \leq \delta$$

1. Apply GAT method with $x_0 = x^{(k-1)}$, $\varepsilon = \varepsilon_{k-1}$, $\lambda_0 = \lambda^{(k-1)}$. Let $x^{(k)}$ be the last approximation achieved, $\phi^{(k)}$ the corresponding discrepancy norm, and $\lambda^{(k)}$ the last parameter value;
2. Define $\varepsilon_k = \phi^{(k)}$;
3. Define $\lambda^{(k)} = \frac{\phi^{(k)}}{\phi^{(k-1)}}\lambda^{(k)}$.

The algorithm

Input: A, b, L, $\lambda^{(0)}$, η, δ (threshold parameter), and $\varepsilon_0 = \overline{\varepsilon} > \varepsilon$.
Define $x^{(0)} = 0$.
For $k = 1, 2, \ldots$ until

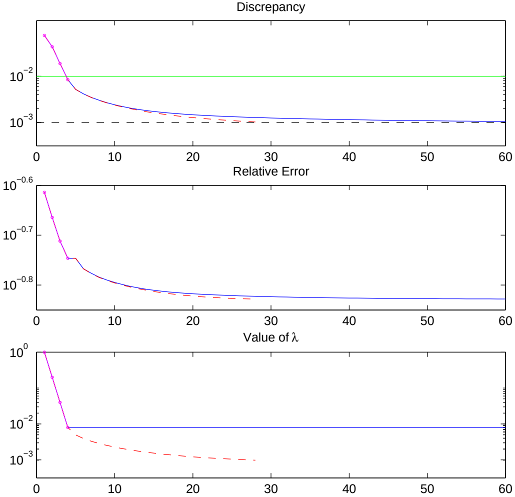
$$\frac{\|\varepsilon_k - \varepsilon_{k-1}\|}{\|\varepsilon_{k-1}\|} \leq \delta$$

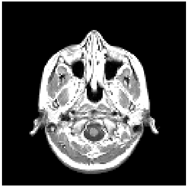
1. Apply GAT method with $x_0 = x^{(k-1)}$, $\varepsilon = \varepsilon_{k-1}$, $\lambda_0 = \lambda^{(k-1)}$. Let $x^{(k)}$ be the last approximation achieved, $\phi^{(k)}$ the corresponding discrepancy norm, and $\lambda^{(k)}$ the last parameter value;
2. Define $\varepsilon_k = \phi^{(k)}$;
3. Define $\lambda^{(k)} = \frac{\phi^{(k)}}{\phi^{(k-1)}}\lambda^{(k)}$.

Example

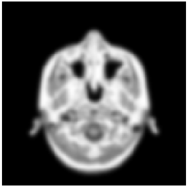
- test image `mri.tif`, 128 pixels.
- Gaussian blurring, $\sigma = 1.5$ and $q = 6$.
- $\varepsilon / \|b\| = 10^{-3}$
  $\overline{\varepsilon} / \|b\| = 10^{-2}$.
- $\delta = 0.01$
- Regularization matrix: D_1.

- without step 3; 56 restarts, $\varepsilon_{24} / \|b\| = 1.03 \cdot 10^{-3}$.
- - with step 3; 24 restarts, $\varepsilon_{56} / \|b\| = 1.05 \cdot 10^{-3}$.

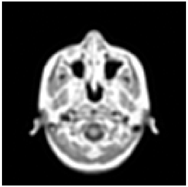




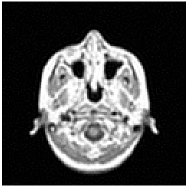
(i)



(ii)



(iii)



(iv)

(i) original; (ii) noisy and blurred; (iii) after 4 steps; (iv) after 24 restarts.

Final remarks

- simple and efficient (all the extra computations are performed in reduced dimension);
- simultaneously determine the regularization parameter and the number of iterations;
- can be generalized to the multi-parameter case (S.G., P.Novati, *Multi-parameter Arnoldi-Tikhonov methods*, submitted) and to the Range-Restricted methods.

Final remarks

- simple and efficient (all the extra computations are performed in reduced dimension);
- simultaneously determine the regularization parameter and the number of iterations;
- can be generalized to the multi-parameter case (S.G., P.Novati, *Multi-parameter Arnoldi-Tikhonov methods*, submitted) and to the Range-Restricted methods.

Final remarks

- simple and efficient (all the extra computations are performed in reduced dimension);
- simultaneously determine the regularization parameter and the number of iterations;
- can be generalized to the multi-parameter case (S.G., P.Novati, *Multi-parameter Arnoldi-Tikhonov methods*, submitted) and to the Range-Restricted methods.

Final remarks

- simple and efficient (all the extra computations are performed in reduced dimension);
- simultaneously determine the regularization parameter and the number of iterations;
- can be generalized to the multi-parameter case (S.G., P.Novati, *Multi-parameter Arnoldi-Tikhonov methods*, submitted) and to the Range-Restricted methods.