Regularizing inverse problems by Krylov subspace methods

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Outline.

The problem.

Solution of

$$Ax = b, \quad A \in \mathbb{R}^{N\times N}, \quad b \in \mathbb{R}^{N},$$

coming from suitable discretization of

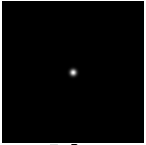
$$\int_{\Omega} k(s,t)f(t)dt = g(s).$$

Modeling inverse problems:

- the process k, the output g ($g = g^{ex} + \varepsilon$) are known;
- the input f is unknown.

Ill-posed problems; appear in a variety of applications.

Our main concern: image deblurring and denoising.

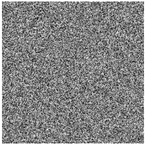


PSF

$*$

exact

$+$



noise

$=$

available

Features of the problem & the need of regularization.

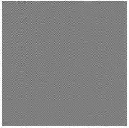
$$Ax = b$$

where

- $b = b^{ex} + e = Ax^{ex} + e$, e Gaussian white noise.
- Considering the SVD: $A = U\Sigma V^T$, $\Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_N)$.
  Behavior of the σ_i's and v_i's
- Discrete Picard Condition
  DPC.

"Naive" solution

$$\begin{aligned} x &= V\Sigma^{-1}U^T b = \sum_{i=1}^{N} \frac{u_i^T b}{\sigma_i} v_i \\ &= \underbrace{\sum_{i=1}^{N} \frac{u_i^T b^{ex}}{\sigma_i} v_i}_{x^{ex}} + \underbrace{\sum_{i=1}^{N} \frac{u_i^T e}{\sigma_i} v_i}_{\text{error}}. \end{aligned}$$



Bibliography.

J. Chung, S. Knepper, and J. Nagy.
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Ch. 2, Handbook of Mathematical Methods in Imaging, Springer, 2011.

M. Hanke and P. C. Hansen.
Regularization methods for large-scale problems.
Surv. Math. Ind., 1993.

P. C. Hansen, J. G. Nagy, and D. P. O'Leary.
Deblurring Images: Matrices, Spectra and Filtering.
SIAM, 2006.

P. C. Hansen.
Discrete Inverse Problems: Insight and Algorithms.
SIAM, 2010.

P. C. Hansen. *Regularization Tools Version 4.1.*
`http://www2.compute.dtu.dk/ pcha/Regutools/`

J. G. Nagy. *RestoreTools.*
`http://www.mathcs.emory.edu/ nagy/RestoreTools/index.html`

Tikhonov regularization method.

Tikhonov regularization method (direct regularization):

$$x_\lambda = \arg\min_{x\in\mathbb{R}^N} \left\{ \|b - Ax\|^2 + \lambda \|L(x - x^*)\|^2 \right\},$$

- $\lambda \in \mathbb{R}^+$ reg. parameter;
- $L \in \mathbb{R}^{q\times N}$ reg. matrix;
- $x^* \in \mathbb{R}^N$ initial guess.

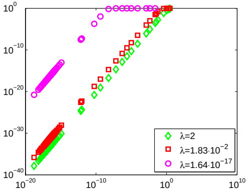
Equivalent formulations ($x^* = 0$):

- $\min_{x\in\mathbb{R}^n} \left\| \begin{pmatrix} A \\ \sqrt{\lambda} L \end{pmatrix} x - \begin{pmatrix} b \\ 0 \end{pmatrix} \right\|^2$
- $(A^T A + \lambda L^T L)x = A^T b$

Regularized solution ($L = I$):

$$x_\lambda = \sum_{i=1}^{N} \underbrace{\frac{\sigma_i^2}{\sigma_i^2 + \lambda}}_{\phi_i^\lambda} \frac{u_i^T b}{\sigma_i} v_i$$

ϕ_i^λ



Iterative Regularization: Krylov subspace methods.

Early termination of the iterations.
Projection onto Krylov subspaces:

$$\mathcal{K}_m(C,d) = \mathrm{span}\{d, Cd, \ldots, C^{m-1}d\},$$

$$x_m \in \mathcal{K}_m(C,d), \quad r_m = b - Ax_m \perp \mathcal{K}_m(C',d')$$

Typically:

- $C,\ C' = A,\ A^T,\ A^TA,\ AA^T$;
- $d,\ d' = b,\ A^Tb,\ Ab,\ A^\ell b$.

In general, at step m:

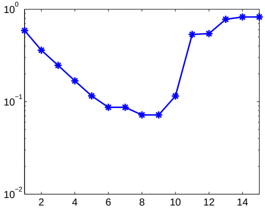
- update $AW_m = Z_{m+1}\bar{D}_m$ ($\bar{D}_m \in \mathbb{R}^{(m+1)\times m}$, $\mathcal{R}(W_m) = \mathcal{K}_m(C,d)$)
- compute $y_m = \arg\min_{y\in\mathbb{R}^m} \left\| d - \bar{D}_m y \right\|$
- take $x_m = W_m y_m$.

Examples: methods based on Arnoldi, Golub-Kahn bidiagonalization, (nonsymmetric) Lanczos algorithms.

Y. Saad. *Iterative Methods for Sparse Linear Systems*. SIAM, 2003.

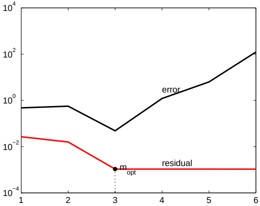
Iterative Regularization.

Error History



Semiconvergence!

- Reliable stopping criterion!



- Hybrid approach: $x_m = W_m y_m$

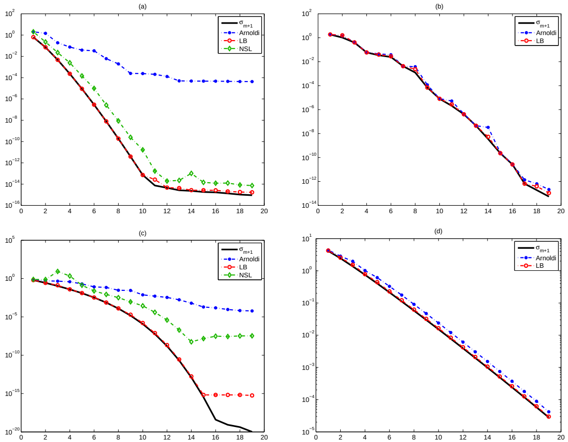
$$y_m = \arg\min_{y\in\mathbb{R}^m} \left\{ \left\| d - \bar{D}_m y \right\| + \lambda_m \left\| y \right\| \right\}.$$

J. Chung, J. G. Nagy, and D. P. O'Leary. A w-GCV Method for Lanczos Hybrid Regularization. ETNA, 2008.

S. Gazzola, P. Novati, and M. R. Russo. On Krylov projection methods and Tikhonov regularization. Submitted.

D. P. O'Leary and J. A. Simmons. A bidiag.-reg. procedure for large scale ill-posed pbs. SISC, 1981.

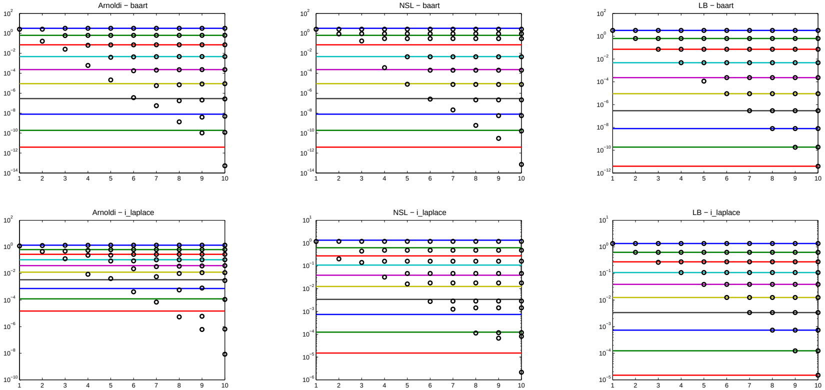
Approximation by Krylov subspace methods.



Test problems: (a) `baart`, (b) `shaw`, (c) `i_laplace`, (d) `gravity`.

S. Gazzola, P. Novati, and M. R. Russo. On Krylov projection methods and Tikhonov regularization. Submitted.

Approximation by Krylov subspace methods.



Approximation of the dominating singular values.

S. Gazzola, P. Novati, and M. R. Russo. On Krylov projection methods and Tikhonov regularization. Submitted.

Krylov-Tikhonov methods.

Basic idea: projection of the Tikhonov-regularized problem

$$\min_{x\in\mathbb{R}^N}\left\{\|b-Ax\|^2+\lambda\|x\|^2\right\}$$

onto Krylov subspaces.

The Arnoldi-Tikhonov (AT) method:

At step m, projection onto $\mathcal{K}_m(A,b)$, whose basis is computed by the Arnoldi alg.:

$$AW_m = W_{m+1}\bar{H}_m.$$

Derivation: $x_{m,\lambda} = W_m y_{m,\lambda}$

$$\begin{aligned} y_{m,\lambda} &= \arg\min_{y\in\mathbb{R}^m}\left\{\|b-AW_my\|^2+\lambda\|W_my\|^2\right\} \\ &= \arg\min_{y\in\mathbb{R}^m}\left\{\big\|\|b\|W_{m+1}e_1-W_{m+1}\bar{H}_my\big\|^2+\lambda\|W_my\|^2\right\} \\ &= \arg\min_{y\in\mathbb{R}^m}\left\|\begin{pmatrix}\bar{H}_m\\ \sqrt{\lambda}I_m\end{pmatrix}y-\begin{pmatrix}\|b\|e_1\\ 0\end{pmatrix}\right\|^2. \end{aligned}$$

Calvetti, Morigi, Reichel, and Sgallari. Tikhonov reg. for large ill-posed pbs. *JCAM*, 2000.

Generalizations of the Arnoldi-Tikhonov method.

- Incorporating an initial guess x^*.
  Projection of
  $$\min_{x\in\mathbb{R}^N}\left\{\|b-Ax\|^2+\lambda\|x-x^*\|^2\right\}$$
  onto $\mathcal{K}_m(A,r^*)$, $r^*=b-Ax^*$.
  $x_{m,\lambda}=x^*+W_m y_{m,\lambda}$, where
  $$y_{m,\lambda}=\arg\min_{y\in\mathbb{R}^m}\left\{\left\|\|r^*\|e_1-\bar{H}_m y\right\|^2+\lambda\|y\|^2\right\}.$$
- Incorporating a generic regularization matrix, $L\in\mathbb{R}^{q\times N}$
  - $L_m=LW_m$.
    Possibly: QR factorization $L_m=Q_mR_m$ (eventually: $L_m=R_m$).
  - $L\in\mathbb{R}^{N\times N}$: $L_m=W_m^TLW_m$.

$$x_{m,\lambda}=W_m y_{m,\lambda},\quad y_{m,\lambda}=\arg\min_{y\in\mathbb{R}^m}\left\{\left\|\|b\|e_1-\bar{H}_m y\right\|^2+\lambda\|L_m y\|^2\right\}.$$

Gazzola, Novati. Automatic par. setting for AT methods. *JCAM*, 2014.

Parameter choice methods: discrepancy-based strategies.

At step m, definition of the discrepancy function:

$$\phi_m(\lambda) = \|b - Ax_{m,\lambda}\| = \left\| \|b\|e_1 - \bar{H}_m y_{m,\lambda} \right\|.$$

Explicit expression with respect to λ:

$$\phi_m(\lambda) = \left\| \|b\|e_1 - \bar{H}_m(\bar{H}_m^T \bar{H}_m + \lambda L_m^T L_m)^{-1} \bar{H}_m^T \|b\|e_1 \right\|.$$

Knowing $\varepsilon = \|e\|$, discrepancy principle:

$$\phi_m(\lambda) < \eta\varepsilon, \quad \eta > 1 \ (\eta \simeq 1).$$

Usual approaches: solving the nonlinear (w.r.t. λ) equation $\phi_m(\lambda) = \eta\varepsilon$.

Gazzola, Novati. Automatic par. setting for AT methods. *JCAM*, 2014.

Reichel, Shyshkov. A new zero-finder for Tikhonov regularization. *BIT*, 2008.

The secant update method.

At step m, linear (w.r.t.λ) approximation:

$$\phi_m(\lambda) \simeq \phi_m(0) + \lambda d_m,$$

- $\phi_m(0)$ is the norm of the GMRES residual: $x_{m,0} = W_m y_{m,0}$, $y_{m,0} = \arg\min_{y\in\mathbb{R}^m} \left\| \|b\| e_1 - \bar{H}_m y \right\|$.
- $d_m = {}^{\phi_m(\lambda_m) - \phi_m(0)}/_{\lambda_m}$,
  where λ_m employed at the previous step.
  Note: d_m obtained by imposing the linear approximation.

Choosing the (next) λ_{m+1}:

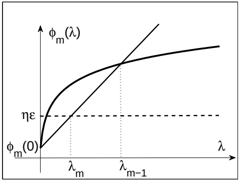
$$\phi_m(\lambda_{m+1}) = \phi_m(0) + \lambda_{m+1} d_m = \eta\varepsilon,$$

and therefore

$$\lambda_{m+1} = \left| \frac{\eta\varepsilon - \phi_m(0)}{\phi_m(\lambda_m) - \phi_m(0)} \right| \lambda_m.$$

The secant update method: remarks.

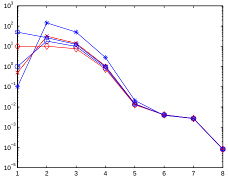
- GEOMETRICAL INTERPRETATION
  - $\phi_m(\lambda)$ monotonically increasing
  - $f(\lambda) = \phi_m(0) + \left(\phi_m(\lambda_m)-\phi_m(0)/\lambda_m\right)\lambda$ interpolates $\phi_m(\lambda)$ at 0 and λ_m
  - λ_{m+1} obtained by solving $f(\lambda) = \eta\varepsilon$.



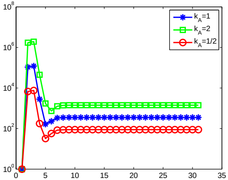
"Secant" method.

- CHOICE OF λ_1 (default: $\lambda_1 = 1$).

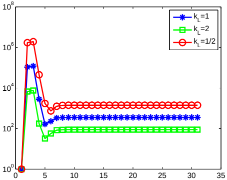
varying λ_1



varying A



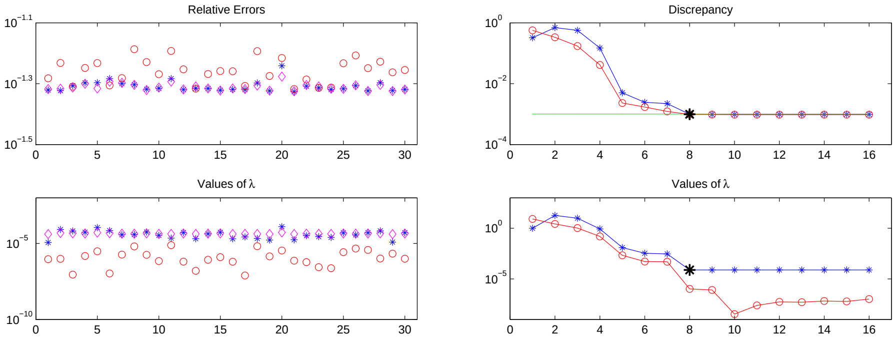
varying L



- PARAMETER CHOICE, STOPPING CRITERION ($\mathcal{K}_{m+1} \approx \mathcal{K}_m$).

Numerical experiments.

Test problem: `shaw` ($N = 200$, $\|e\|/\|b^{ex}\| = 10^{-3}$, $L = I$).



Numerical experiments.

Test problem: `blur(256,6,2.5)` ($N = 256^2$, $\|e\|/\|b^{ex}\| = 10^{-2}$, $L = D_2$).

Exact.

Blurred & noisy.

Restored: L_m-curve.

Restored: secant.

$\|e\|$-free secant update: overestimation of $\|e\|$.

Assumption: $\bar{\varepsilon} > \varepsilon$.

Inner-outer strategy: at the kth restart (as soon as $\phi_m^{(k-1)}(\lambda_m^{(k-1)}) < \bar{\varepsilon}$)

$$\bar{\varepsilon} := \varepsilon^{(k)} = \phi_m^{(k-1)}(\lambda_m^{(k-1)}), \quad \lambda_1 := \lambda_m^{(k-1)}, \quad x^* := x_{m,\lambda}^{(k-1)}.$$

Algorithm:

Inputs: A, b, L, $\lambda_1 = \lambda^{(0)}=1$, $x^* = x^{(0)} = 0$, $\varepsilon^{(0)} = \bar{\varepsilon}$, η, δ (threshold parameter).

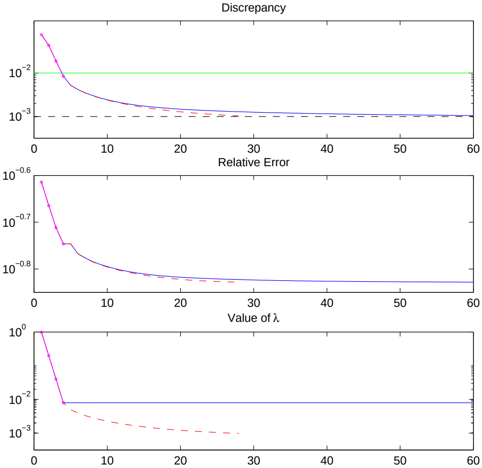
For $k = 1, 2, \ldots$ until

$$\frac{\|\varepsilon^{(k)} - \varepsilon^{(k-1)}\|}{\|\varepsilon^{(k-1)}\|} \leq \delta$$

1. Apply the secant-AT method with $x^* = x_{m,\lambda}^{(k-1)}$, $\bar{\varepsilon} = \varepsilon^{(k-1)}$, $\lambda_1 = \lambda_m^{(k-1)}$. Let $\phi^{(k)}$ the last discrepancy, and $\lambda^{(k)}$ the last parameter.
2. Define $\bar{\varepsilon} = \varepsilon^{(k)} = \phi^{(k)}$.
3. HEURISTIC IMPROVEMENT: define $\lambda^{(k)} = {}^{\phi^{(k)}}/_{\phi^{(k-1)}}\lambda^{(k)}$.

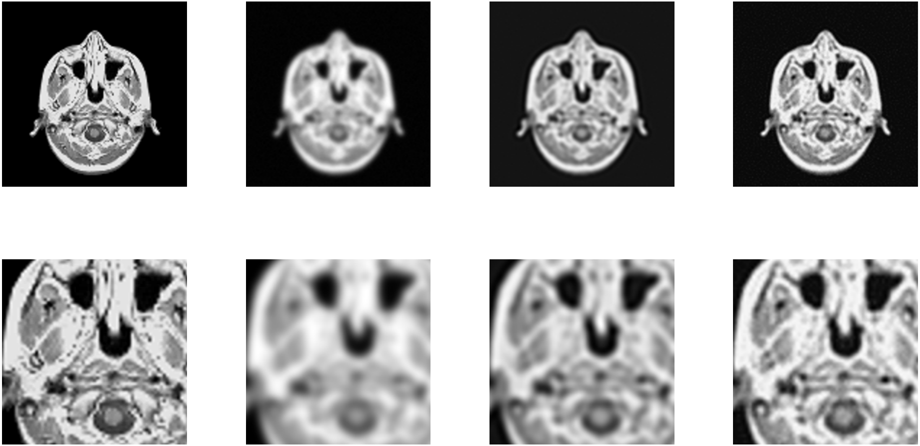
Numerical experiments.

Test problem: `blur(128,6,1.5)`
$(N = 128^2,\ \|e\|/\|b^{ex}\| = 10^{-3},\ \bar{\varepsilon}/\|b^{ex}\| = 10^{-2},\ L = D_1)$.



Numerical experiments.

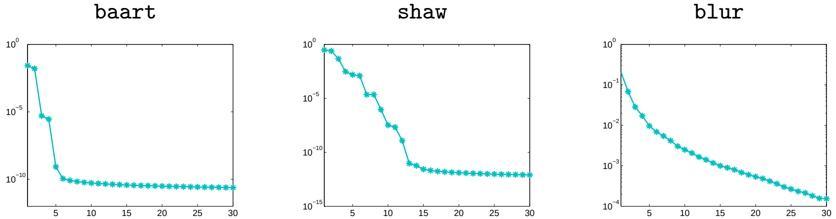
Exact. | Blurred & noisy. | Intermediate. | Final.



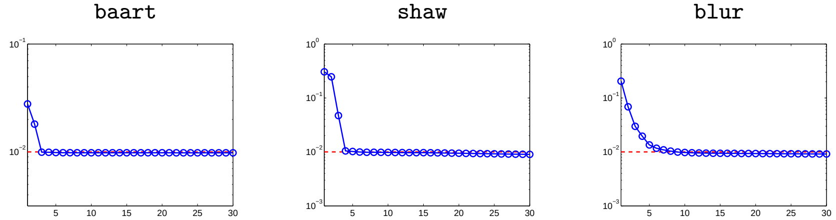
||e||-free secant update: theoretical "break".

Behavior of the GMRES residual:

- Unperturbed problem



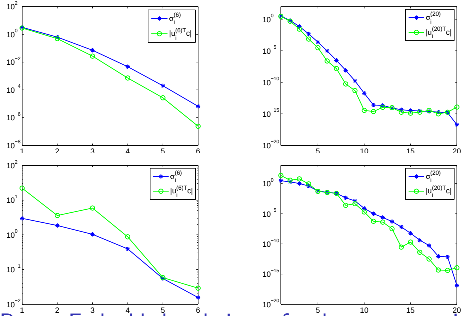
- Perturbed problem



$\|e\|$-free secant update: theoretical "break".

Behavior of the GMRES (unperturbed problem, $b = b^{ex}$).
Framework: the DPC is satisfied, $\sigma_j = O(e^{-\alpha j})$, $\alpha > 0$.
Theorem. Assuming that the DPC is "inherited",

$$\|r_m\| = O\left(m\,\sigma_m\right).$$



Gazzola, Novati, Russo. Embedded techniques for the parameter in Tikhonov reg. *Num. LA Appl.*, to appear.

Gazzola, Novati,. On the inheritance of the DPC. Almost ready!

$\|e\|$-free secant update: theoretical "break".

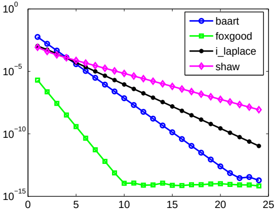
Behavior of the GMRES (perturbed problem, $b = b^{ex} + e$).
Framework: the DPC is satisfied, $\sigma_j = O(e^{-\alpha j})$, $\alpha > 0$.
Theorem.

Let $\widetilde{V}_m^{ex} = \left[A^k b^{ex} / \|A^k b^{ex}\|\right]_{k=0,\dots,m-1}$, $\widetilde{V}_m = \left[A^k b / \|A^k b\|\right]_{k=0,\dots,m-1}$.

Then $\|r_m\| \leq \eta(m)\|e\|$, where

$$\eta(m) = 1 + \frac{\|r_m^{ex}\| + \left\|\left(\widetilde{V}_{m+1} - \widetilde{V}_{m+1}^{ex}\right) s^{ex}\right\|}{\|e\|},$$

$$\|r_m^{ex}\| = \left\|b^{ex} - \widetilde{V}_{m+1}^{ex} s^{ex}\right\| \quad ([s^{ex}]_1 = 0).$$



Gazzola, Novati, Russo. Embedded techniques for the parameter in Tikhonov reg. *Num. LA Appl.*, to appear.

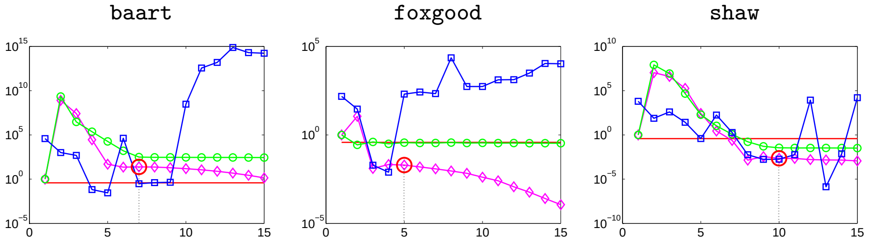
$\|e\|$-free secant update: embedded approach.

The following strategy is derived: if $m \geq 2$

$$\lambda_{m+1} = \frac{\hat{\eta} r_{m-1} - r_m}{\phi_m(\lambda_m) - r_m}\lambda_m, \quad \hat{\eta} > 1$$

As stopping criteria: given two tolerances $\tau_{\mathrm{res}}, \tau_{\mathrm{discr}} > 0$

$$\frac{r_m - r_{m-1}}{r_{m-1}} < \tau_{\mathrm{res}} \quad \text{and} \quad \frac{\phi_m(\lambda_m) - \phi_{m-1}(\lambda_{m-1})}{\phi_{m-1}(\lambda_{m-1})} < \tau_{\mathrm{discr}}.$$

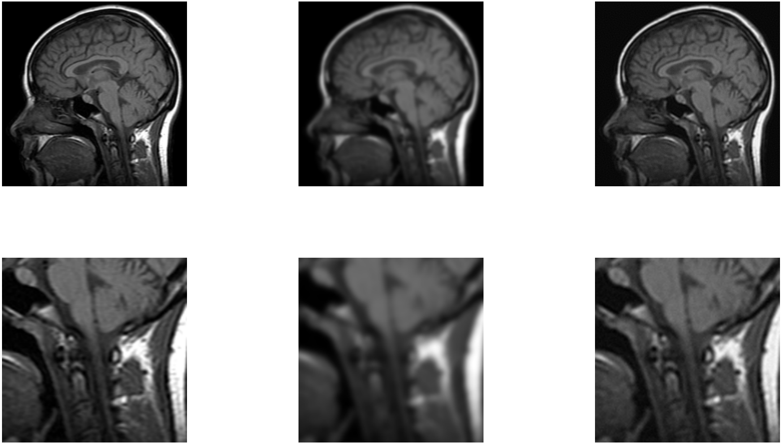


Gazzola, Novati, Russo. Embedded techniques for the parameter in Tikhonov reg. *Num. LA Appl.*, to appear.

Numerical experiments.

Test problem: `blur(256,9,2.5)` ($N = 256^2$, $\|e\|/\|b^{ex}\| = 10^{-1}$, $L = D_1$).

Exact. Blurred & noisy. Restored.



Sparse Reconstruction.

Better result obtained solving

$$\min_{x \in \mathbb{R}^N} \left\{ \|b - Ax\|_p^p + \lambda \ell(x) \right\}$$

- $p = 2$;
- $\ell(x) = \|x\|_q^q$ ($q = 1$ to force sparsity in the solution);
- $\ell(x) = TV(x) = \|\sqrt{(D_h x)^2 + (D_v x)^2}\|_1$.

ISSUE: nonlinearity!

- Oliveira, Bioucas-Dias, Figueiredo. Adaptive total vatiation image deblurring: a major.-minim. approach. *Sig. Proc.*, 2009.
- Rodríguez, Wohlberg. An Efficient Alg. for Sp. Repr. with ℓ^p Data Fidelity Term. *Proceedings of IEEE ANDESCON*, 2008.
- Wohlberg, Rodriguez. An Iteratively Reweighted Norm Alg. for Minim. of TV Funct. *IEEE Sig. Proc. Letters*, 2007.
- Wright, Nowak, Figueiredo. SpaRSA. *IEEE Trans. Sig. Proc.*, 2009.

Sparse Reconstruction.

Better result obtained solving

$$\min_{x\in\mathbb{R}^N}\left\{\|b-Ax\|_p^p+\lambda\ell(x)\right\}$$

- $p=2$;
- $\ell(x)=\|x\|_q^q$ ($q=1$ to force sparsity in the solution);
- $\ell(x)=TV(x)=\|\sqrt{(D_hx)^2+(D_vx)^2}\|_1$.

ISSUE: nonlinearity!

Adopting an iteratively reweighted LS strategy: iterative solution of

$$\min_x\|b-Ax\|_s^s=\min_x\|r(x)\|_s^s=\min_x\|\underbrace{\mathrm{diag}\left(|r(x)|^{\frac{s-2}{2}}\right)}_{\mathcal{W}}r(x)\|_2^2.$$

An iteratively reweighted norm approach

Exploiting the iterative setting of the AT method:

$$\ell(x) \approx \|\mathcal{W}x\|_2^2 = \|\mathcal{W}_m x\|_2^2,$$

where

- $\ell(x) = \|x\|_1$
$$\mathcal{W}_m = L_m = \mathrm{diag}\left(\frac{1}{\sqrt{|x_{m-1}|}}\right)$$
- $\ell(x) = TV(x) = \|\sqrt{(D_h x)^2 + (D_v x)^2}\|_1,\ x \in \mathbb{R}^N = \mathbb{R}^{n^2}$

$$\mathcal{W}_m = L_m = S_m D_{hv}$$

$$D = \begin{pmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{pmatrix} \in \mathbb{R}^{(n-1)\times n}, \quad D_{hv} = \begin{pmatrix} D_h \\ D_v \end{pmatrix} = \begin{pmatrix} D \otimes I_n \\ I_n \otimes D \end{pmatrix}$$

$$\widetilde{x}_{m-1} = D_{hv} x_{m-1}, \quad \widetilde{S}_m = \mathrm{diag}\left(\frac{1}{\sqrt[4]{\sum_{i=1}^{2(N-n)} (\widetilde{x}_{m-1})_i}}\right), \quad S_m = \begin{pmatrix} \widetilde{S}_m & 0 \\ 0 & \widetilde{S}_m \end{pmatrix}$$

Gazzola, Nagy. Generalized AT method for sparse reconstruction. *SISC*, to appear.

The $\|\cdot\|_1$ case

Standard form transformation

$$\min_x \left\{\|b - Ax\|_2^2 + \lambda\|L_m(x - x^*)\|_2^2\right\}$$

$$\min_{\widetilde{x}}\{\|b - \widetilde{A}_m\widetilde{x}\|_2^2 + \widetilde{\lambda}\|\widetilde{x} - \widetilde{x}^*\|_2^2\}$$

- $\widetilde{A}_m = AL_m^{-1}$
- $\widetilde{x}^* = L_m x^*$
- $\widetilde{x} = L_m x$

Features:

- Matrix associated to the projected LS problem completely reduced.
- Preconditioned Krylov subspaces, with variable preconditioning.

  Saad. A flexible prec. GMRES. *SISC*, 1993.

  Flexible Arnoldi algoritm:

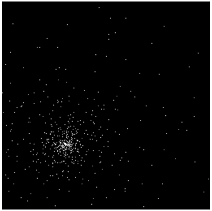
$$AZ_m = \check{V}_{m+1}\check{H}_m$$

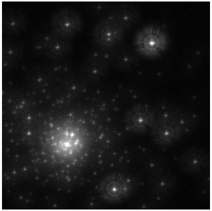
$$x_{m,\lambda} = Z_m y_{m,\lambda}, \quad Z_m = span\{L_1^{-1}\check{v}_1, L_2^{-1}\check{v}_2, \dots, L_m^{-1}\check{v}_m\}.$$

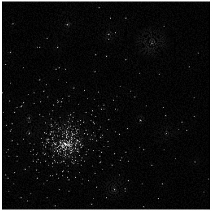
$$y_{m,\lambda} = \arg\min_{y\in\mathbb{R}^m}\left\{\left\|\|b\|e_1 - \check{H}_m y\right\|_2^2 + \lambda\|y\|_2^2\right\}$$

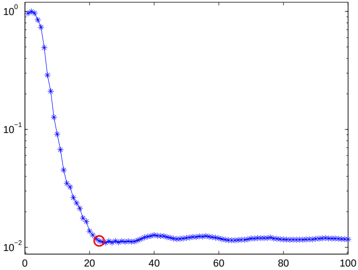
- The secant update method can still be adopted.

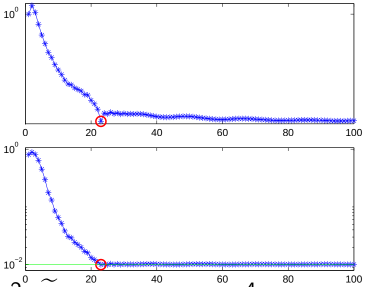
Numerical Experiments.











Iteration #23; Relative Error $1.1349 \cdot 10^{-2}$; $\widetilde{\lambda} = 1.1976 \cdot 10^{-4}$.

The *TV* case.

Problem: $L_m = S_m D_{hv}$ not easily "invertible".

$$\min_{x\in\mathbb{R}^N} \left\{ \|b - Ax\|^2 + \lambda \|L(x - x^*)\|^2 \right\}, \quad x_m \in x^* + \mathcal{K}_m(A, r^*)$$

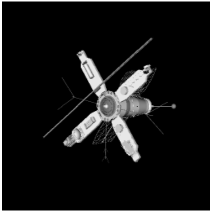
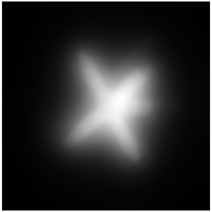
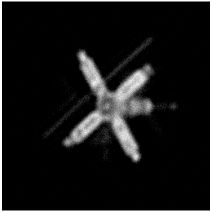
Given a meaningful initial guess $x^{*(k)}$, restarting (inner-outer iterations)

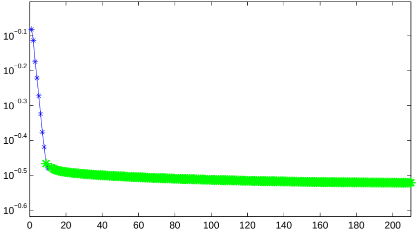
$$\min_{y\in\mathbb{R}^m} \left\| \begin{pmatrix} \bar{H}_m \\ \sqrt{\lambda} S^{(k)} D_{hv} W_m \end{pmatrix} y - \begin{pmatrix} \|r^{*(k)}\|_2 e_1 \\ 0 \end{pmatrix} \right\|^2, \quad x_m = x^{*(k)} + W_m y$$

- "Wait" for the discrepancy principle to be satisfied
- Regularization Matrix updated at each outer cycle
- Initial guess, at the kth outer cycle
  - $x^{*(k+1)} = 0$;
  - $x^{*(k+1)} = x_m^{(k)}$;
  - $x^{*(k+1)} = P(x_m^{(k)})$, projection onto $\{x \in \mathbb{R}^N : (x)_i \geq 0,\ i = 1, \dots, N\}$;

Main difference: unpreconditioned Krylov subspaces

Numerical Experiments.

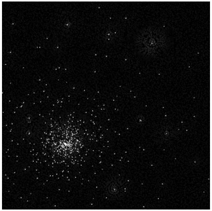




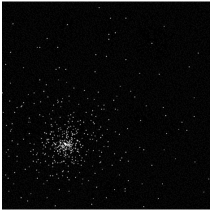


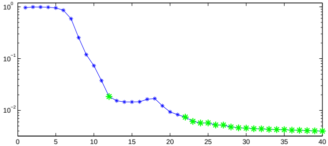
Outer Iterations: 200; Total Iterations: 210; Relative Error: $3.0115 \cdot 10^{-1}$.

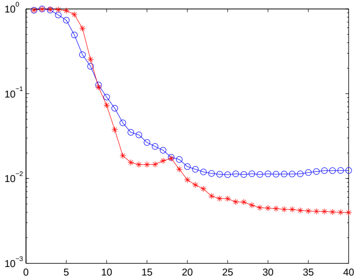
The $\|\cdot\|_1$ case revisited.

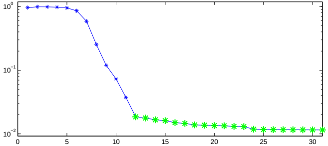
Std form transformation; update and nonnegativity at each restart.

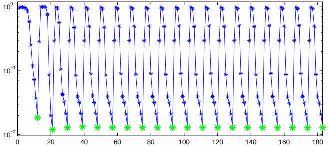












Comparisons.

$\ell(x) = \|x\|_1$ (star cluster)

Method	Relative Error	Iterations	Total Time	Average Time
SpaRSA	$1.1081 \cdot 10^{-2}$	343	45.16	0.13
TwIST	$1.1105 \cdot 10^{-2}$	102	16.39	0.16
l1_l2	$1.1146 \cdot 10^{-2}$	307	378.61	1.23
IRN-BPDN	$1.1146 \cdot 10^{-2}$	791	112.33	0.14
AT	$1.8609 \cdot 10^{-2}$	12	0.65	0.05
Flexi-AT	$1.1610 \cdot 10^{-2}$	100	5.03	0.05
NN-ReSt-GAT	$4.0606 \cdot 10^{-3}$	40	2.56	0.06

Bioucas-Dias, Figueiredo. A new TwIST: two-step it.e shrinkage/thresholding alg. *IEEE Trans. Im Proc.*, 2007.

Gazzola, Nagy. Generalized AT method for sparse reconstruction. *SISC*, to appear.

Ki, Koh, Lustig, Boyd, Gorinvesky. An interior-point method for large-scale ℓ_1-regularized least squares. *IEEE Top. Im. Proc.*, 2007.

Rodríguez, Wohlberg. An Efficient Alg. for Sp. Repr. with ℓ^p Data Fidelity Term. *Proceedings of IEEE ANDESCON*, 2008.

Wright, Nowak, Figueiredo. SpaRSA. *IEEE Trans. Sig. Proc.*, 2009.

Comparisons.

$\ell(x) = TV(x)$ (satellite)

Method	Relative Error	Iterations	Total Time	Average Time
aMM-TV	$2.8834 \cdot 10^{-1}$	631	4754.54	7.53
IRN-TV(0)	$3.2136 \cdot 10^{-1}$	418	15.65	0.04
IRN-TV(wr)	$3.2141 \cdot 10^{-1}$	190	7.66	0.04
NN-ReSt-TV	$3.0110 \cdot 10^{-1}$	210	11.36	0.05
AT	$3.4176 \cdot 10^{-1}$	9	0.17	0.02
GAT	$3.4809 \cdot 10^{-1}$	9	0.36	0.04

Oliveira, Bioucas-Dias, Figueiredo. Adaptive total vatiation image deblurring: a major.-minim. approach. *Sig. Proc.*, 2009.

Gazzola, Nagy. Generalized AT method for sparse reconstruction. *SISC*, to appear.

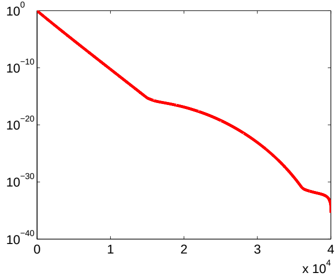
Rodríguez, Wohlberg. An Efficient Alg. for Sp. Repr. with ℓ^p Data Fidelity Term. *Proceedings of IEEE ANDESCON*, 2008.

Future work.

- incorporating other general regularization terms into the AT framework;
- incorporating physical constraints on the solution into the AT framework;
- exploring the performance of recent Krylov subspaces methods.

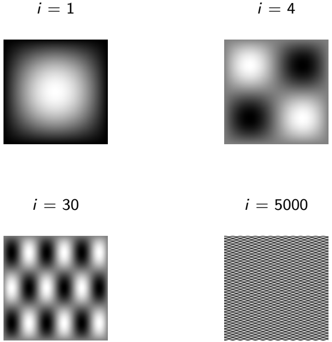
Thanks for your attention!!!

σ_i



The singular values quickly cluster at zero.

v_i



More oscillations as the singular values decrease.

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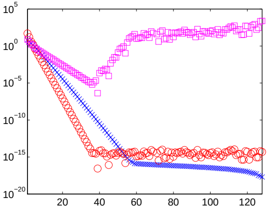
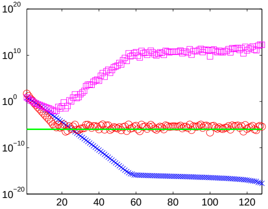
Without noise



×: σ_i; o: $u_i^T b^{ex}$; □: $u_i^T b^{ex} / \sigma_i$.

With noise



×: σ_i; o: $u_i^T b$; □: $u_i^T b / \sigma_i$.

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