The inner quantum logic of Bell's states

Giulia Battilotti
Paola Zizzi
Dipartimento di Matematica Pura ed Applicata,
Università di Padova
via G. Belzoni n.7, I–35131 Padova, Italy

Abstract

We introduce logical judgements for the internal logic of a quantum computer with two qubits, in the two limit cases of non-entanglement (separable states) and maximal entanglement (Bell's states). To this aim, we consider an internal (reversible) measurement which preserves the probabilities by mirroring the states. We then obtain logical rules, obeying the reflection principle of basic logic, which illustrate the different computational behaviour of separable and Bell's states.

1 Introduction

The main aim of our work is to look for the inner logic of quantum computation [15], illustrating the point of view of a hypotetical "internal observer" who lives inside the quantum black box. Such an observer, introduced in [22], can perform internal reversible measurements in the quantum system. Recently, the geometrical approach of [22] has been algebraically developed in [24]. Furthermore, in a logical framework, the idea is that internal measurements give rise to logical assertions ([1], [2]), which are then treated following the reflection principle as in basic logic [19]. Indeed, by the reflection principle, logical connectives are the result of importing some pre-existing metalinguistic links between assertions into the formal language. Our final purpose is to obtain adequate connectives, corresponding to the *physical* links in the quantum black box, and the associated inference rules. The latter should witness the process of quantum computation.

In [2] whe have considered a toy-model quantum computer with one qubit and have obtained an interpretation of the *superposition* of the two basis states in terms of the additive conjunction "&" (and, dually, with the additive disjunction "$\oplus$"). The resulting logic is paraconsistent (see also [8], [7]), and symmetric, like basic logic.

In the present paper, we introduce a model of two qubits. This makes it possible to deal with two different physical links occurring between the two qubits of the register: *maximal entanglement* (the two qubits are a Bell state) and *non entanglement* (the two qubits state is separable).

One of the main advantages of quantum computation versus its classical counterpart is quantum computational speed up, due to massive quantum parallelism. Entanglement has been proved to be the sufficient and necessary condition for quantum computational speed up, at least for pure states [13]. Up to now, however, entanglement has been considered just as the physical support to the quantum computational process rather than an intrinsical formal feature of the process itself. The reason for this can be easily be attributed to history. In fact, as it is well known, entanglement was first investigated in the thirties, while quantum computation was introduced in the eighties by R. Feynman [10]. So, one is lead to adopt the traditional algebraic characterizations. However, disregarding the historical background of quantum mechanics, and focusing on quantum computation, one might identify entanglement with quantum computational speed up. Then, if one is interested in the internal logic of quantum computation, she/he will be automatically lead to look for the internal logic of entanglement. So one could describe computational speed up in logical terms rather than the state in algebraic terms (C-NOT and control). To do this, it is necessary to develop a logical calculus for entanglement which seems to require unshared contexts ([23]). This approach is under study, but it is also the underlying idea in the present work.

In this framework see also [9] concerning the quantum computational flow. Moreover see [16] and [6] (the latter in category theory) concerning the teleportation protocol. Up to now, people have been generally more interested in finding quantum algorithms "ad hoc" exploiting entanglement rather than giving an axiomatic formalization of quantum computational speed up itself. However, we think that only an axiomatic approach could lead to a deep understanding of this process, and probably, provide a formal scheme to generate many more quantum algorithms.

Finally, we think that the lack of formalization of entanglement is also due to the fact that, up to now, the most common interpretations of a quantum phenomenon are settled in a subjective perspective. The subjective interpretation, in our opinion, leads to disregard entanglement as a computational process, since in that view it cannot be treated as a "fact". Indeed, only very few authors have been involved in an objective interpretation ([14], [12]). Our attitude, in particular, is strongly objectivist, as we rely on the notion of internal measurement [22], [24], and then we are able to "measure" an entangled state from "inside".

2 Measurements and Mirrors

To obtain the judgements for the two qubits model, we extend the definition of the internal measurement to the case of two qubits. We remind that, in a Hilbert space C^2, the internal measurement of one qubit is given by a unitary 2×2 complex matrix [22]. In such a model, the judgements are obtained by means of a particular internal measurement, called "mirror measurement" [2], given by the matrices:

$$M = e^{i\phi} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix} \tag{1}$$

where $\alpha\alpha^* = |\alpha|^2 = 1$. We have:

$$M(a|0\rangle + b|1\rangle) = e^{i\phi}(\alpha a|0\rangle + \alpha^* b|1\rangle) \tag{2}$$

So, our mirrors are "quasi-identities"; actually, they modify the longitude of the qubit in the Bloch sphere, that is, the probability amplitudes:

$$a \to a' = e^{i\phi}\alpha a$$

$$b \to b' = e^{i\phi}\alpha^* b$$

and preserve the "internal truth" given by the probabilities, since $|a'|^2 = |a|^2$ and $|b'|^2 = |b|^2$. For this reason, we have chosen mirror-matrices to witness the internal truth and the consequent logical judgements, as we shall see in the next section.

We now extend the mirror-matrices to C^4. If $M_1 = e^{i\phi_1} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix}$ and $M_2 = e^{i\phi_2} \begin{pmatrix} \beta & 0 \\ 0 & \beta^* \end{pmatrix}$, the tensor product $M = M_1 \otimes M_2$ given by:

$$M = e^{i(\phi_1+\phi_2)} \begin{pmatrix} \alpha\beta & 0 & 0 & 0 \\ 0 & \alpha\beta^* & 0 & 0 \\ 0 & 0 & \alpha^*\beta & 0 \\ 0 & 0 & 0 & \alpha^*\beta^* \end{pmatrix} = e^{i\phi} \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & \delta^* & 0 \\ 0 & 0 & 0 & \gamma^* \end{pmatrix} \tag{3}$$

is also a mirror matrix. In fact, the most general state of C^4 in the computational basis is:

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \tag{4}$$

and one has:

$$M|\psi\rangle = e^{i\phi}(\gamma a|00\rangle + \delta b|01\rangle + \delta^* c|10\rangle + \gamma^* d|11\rangle) \tag{5}$$

and again we have: $a \to a' = e^{i\phi}\gamma a$... and so on, then probabilities are preserved: $|a'|^2 = |a|^2$... and so on.

Note that, if $|\psi\rangle$ is one of the Bell states $|\psi_\pm\rangle = 1/\sqrt{2}e^{i\phi}|00\rangle \pm 1/\sqrt{2}e^{i\phi}|11\rangle$, its mirroring is: $M|\psi_\pm\rangle = \gamma 1/\sqrt{2}e^{i\phi}|00\rangle \pm \gamma^* 1/\sqrt{2}e^{i\phi}|11\rangle$. Similarly, for the Bell's state $|\phi_\pm\rangle = 1/\sqrt{2}e^{i\phi}|01\rangle \pm 1/\sqrt{2}e^{i\phi}|10\rangle$, we have $M|\phi_\pm\rangle = \delta 1/\sqrt{2}e^{i\phi}|01\rangle \pm \delta^* 1/\sqrt{2}e^{i\phi}|10\rangle$. Then Bell states behave as a single particle in the mirroring, since the result of applying a mirror to a Bell state has the same form as (2). This fact will be shown in Sect.4 in logical terms.

3 From Mirrors to Judgements in the Black Box

We recall the line of thought followed for the case of the one qubit model (see [2]). Inside the Black Box, a hypothetical internal observer **P** is equipped with mirror-matrices and then can perform reversible measurements. Outside the Black Box, instead, an external observer **G** can perform standard quantum measurements, represented by projectors.

It is well known that performing a standard quantum measurement in the given basis, e.g. $|0\rangle, |1\rangle$, on a qubit $|q\rangle = a|0\rangle + b|1\rangle$, means to apply one of the two projectors P_0 or P_1, breaking the superposition and obtaining one of the two basis states. So the observer **G** can "read" the value of the qubit as $|0\rangle$, asserting: "$|0\rangle$ is true" or $|1\rangle$, asserting: "$|1\rangle$ is true". So, let us suppose that a standard quantum measurement is applied and a result $A \in \{|0, |1\rangle\rangle\}$ is obtained. Then **G** asserts "A is true", written as:

$$\vdash A$$

Conversely, denoting by $A^\perp$ the opposite result, **G** asserts "$A^\perp$ is true", written as:

$$\vdash A^\perp$$

By the no cloning theorem [21], after the measurement, **G** can assert *only one* of the two. The same does not happen to the internal observer **P**, who applies a mirror to $|q\rangle$. In fact, any mirror is the sum of the two projectors:

$$M = e^{i\phi} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^* \end{pmatrix} = e^{i\phi}\alpha P_0 + e^{i\phi}\alpha^* P_1 \tag{6}$$

so that $M|q\rangle = e^{i\phi}\alpha P_0|q\rangle + e^{i\phi}\alpha^* P_1|q\rangle$. Hence **P** obtains a *superposition* of the two results obtainable by **G**. We write then both the above judgements together:

$$\vdash A \qquad\qquad \vdash A^\perp$$

What is a couple of possibilities for **G** is instead a unique fact for **P**! By the reflection principle, a connective corresponds to a link between judgements. As in [19], we make the connective "&" correspond to the above couple, putting

$$\vdash A \& A^\perp \equiv \qquad \vdash A \qquad \vdash A^\perp \tag{7}$$

Then

$$\vdash A \& A^\perp \tag{8}$$

"$A \& A^\perp$ is true" is the judgement put by **P** inside the Black Box, concerning the value of the qubit $|q\rangle$.

Now, let us consider a two-qubit model, that is a Black Box equipped with a register $|\psi\rangle$ of two qubits $|q_1\rangle, |q_2\rangle$. Fixed a basis of C^4, e.g. the computational basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, the external observer, who performs a standard quantum measurement in that basis, applies one of the four projectors $P_{00}, P_{01}, P_{10}, P_{11}$. Let us suppose that she finds an answer $A \in \{|0\rangle, |1\rangle\}$ for $|q_1\rangle$

and an answer $B \in \{|0\rangle, |1\rangle\}$ for $|q_2\rangle$. Then she has a register of two classical bits, and her assertion is:

$$\vdash A, B \tag{9}$$

where the comma stands for the register link between the two classical bits. We interpret the link by the multiplicative connective on the right of the sequent, which is called "par" in linear logic, here written as "$\odot$". "Par" has the same physical meaning of the tensor product "times", which, however, is used in linear logic and basic logic to interpret the comma on the left of the sequent.

Then, as in [19] we put the equation:

$$\vdash A \oslash B \quad \equiv \quad \vdash A, B \tag{10}$$

What are all the possible judgements? If the measurements of the two qubits are independent, four combinations are possible:

$$\vdash A, B \quad \vdash A, B^{\perp} \quad \vdash A^{\perp}, B \quad \vdash A^{\perp}, B^{\perp}$$

and so four judgements are obtainable:

$$\vdash A \oslash B \qquad \vdash A \oslash B^{\perp} \qquad \vdash A^{\perp} \oslash B \qquad \vdash A^{\perp} \oslash B^{\perp} \tag{11}$$

This is the case of a pair of unentangled qubits. On the contrary, let us consider a Bell pair. In such a case the two measurements are related, thus not all combinations are possible: if $\vdash A, B$ is a result, then $\vdash A^{\perp}, B^{\perp}$ is the only other. Note that **G** is in general unaware of the link existing between $|q_1\rangle$ and $|q_2\rangle$ inside the Black Box, so that the same register link is used outside. Then, in the case of entanglement, two judgements are possible outside:

$$\vdash A \oslash B \qquad \vdash A^{\perp} \oslash B^{\perp} \tag{12}$$

As in the case of one qubit, the external observer can put only one of the possible judgements. Again, the judgement of the internal observer is given by a mirror measurement, and mirrors of C^4 are obtainable as a linear combination of the four projectors:

$$M = e^{i\phi}(\gamma P_{00} + \delta P_{01} + \delta^* P_{10} + \gamma^* P_{11}) \tag{13}$$

So **P**, who applies M to the register $|\psi\rangle$, obtains:

$$M|\psi\rangle = e^{i\phi}(\gamma P_{00}|\psi\rangle + \delta P_{01}|\psi\rangle + \delta^* P_{10}|\psi\rangle + \gamma^* P_{11}|\psi\rangle) \tag{14}$$

that is the superposition of the possible values obtainable outside. If $|\psi\rangle$ is not a maximally entangled state, every projector gives a result and the judgement of the internal observer is obtained as a superposition of the four judgements in (11), that is:

$$\vdash (A \oslash B) \& (A \oslash B^{\perp}) \& (A^{\perp} \oslash B) \& (A^{\perp} \oslash B^{\perp}) \tag{15}$$

If $|\psi\rangle$ is a Bell state, for example $|\psi_\pm\rangle$, one has $M|\psi_\pm\rangle = e^{i\phi}(\gamma P_{00}|\psi_\pm\rangle + \gamma^* P_{11}|\psi_\pm\rangle)$. The same holds for the other Bell states. The result of the measurement of a maximally entangled state is the superposition of the two judgements (12), that is:

$$\vdash (A \oslash B) \& (A^\perp \oslash B^\perp) \tag{16}$$

Now, let us consider the internal measurement without any reference to the external one. We know that **P** achieves a superposition $A\&A^\perp$ measuring $|q_1\rangle$ and another $B\&B^\perp$ measuring $|q_2\rangle$. The two results are linked by two different register links present in the black box, that is non entanglement and maximal entanglement. For non entanglement we write $\asymp$, while, for maximal entanglement $\bowtie$. The mirror measurement gives one of the two judgements:

$$\vdash (A\&A^\perp) \asymp (B\&B^\perp) \tag{17}$$

and

$$\vdash (A\&A^\perp) \bowtie (B\&B^\perp) \tag{18}$$

respectively.

We put the two reflection principles, writing $\oslash_0$ (no correlation) and $\oslash_1$ (maximum correlation) for the two corresponding binary connectives:

$$\vdash (A\&A^\perp) \oslash_0 (B\&B^\perp) \equiv \;\; \vdash (A\&A^\perp) \asymp (B\&B^\perp) \tag{19}$$

and

$$\vdash (A\&A^\perp) \oslash_1 (B\&B^\perp) \equiv \;\; \vdash (A\&A^\perp) \bowtie (B\&B^\perp) \tag{20}$$

So we have two kinds of internal judgements concerning our register of two qubits:

$$\vdash (A\&A^\perp) \oslash_0 (B\&B^\perp) \tag{21}$$

and

$$\vdash (A\&A^\perp) \oslash_1 (B\&B^\perp) \tag{22}$$

As for the states that are neither separable nor maximally entangled, we argue that connectives $\oslash_x$, $x \in (0,1)$ (where x is a suitable degree of correlation), should be introduced, perhaps leading to a kind of fuzzy logic.

4 Towards a calculus of judgements

Of course, the internal judgements (21) and (22) must be equivalent to the superposition of the external ones (15) and (16), that is we have to prove the following equivalence:

$$\vdash (A\&A^\perp) \odot_0 (B\&B^\perp) \Longleftrightarrow \vdash (A \odot B)\&(A \odot B^\perp)\&(A^\perp \odot B)\&(A^\perp \odot B^\perp) \tag{23}$$

in the non entangled case, and the equivalence:

$$\vdash (A\&A^\perp) \odot_1 (B\&B^\perp) \Longleftrightarrow \vdash (A \odot B)\&(A^\perp \odot B^\perp) \tag{24}$$

in the maximally entangled case.

In order to prove the equivalences, we first remind that, in basic logic, which is a quantum linear logic, the additive connective & is obtained putting the following definitional equation:

$$\Gamma \vdash A \& B \quad \equiv \quad \Gamma \vdash A \quad \Gamma \vdash B \tag{25}$$

which does not admit any context besides the active formulae A and B (visibility of basic logic). On the contrary, the same equation in any non-quantum logic can be written with a context C:

$$\Gamma \vdash A \& B, C \quad \equiv \quad \Gamma \vdash A, C \quad \Gamma \vdash B, C \tag{26}$$

As it is well known, in this case, one case derive distributivity of the multiplicative disjunction (here $\odot$) with respect to the additive conjunction &, that is $(A \odot C) \& (B \odot C) = (A \& B) \odot C$. Here, we see a derivation of distributivity which exploits only definitional equations. As we have seen in basic logic, the definitional equations give rise to "formation" and "implicit reflection" rules, each one corresponding to a direction of the equivalence. Such rules can be found in the following derivation, which can be read top-down and bottom-up.

$$\cfrac{\cfrac{\cfrac{\Gamma \vdash (A \odot C) \& (B \odot C)}{\cfrac{\Gamma \vdash A \odot C}{\Gamma \vdash A, C}\,\odot \qquad \cfrac{\Gamma \vdash B \odot C}{\Gamma \vdash B, C}\,\odot}\,\&}{\Gamma \vdash (A \& B), C}\,\&cont}{\Gamma \vdash (A \& B) \odot C}\,\odot \tag{27}$$

Then, putting $\Gamma = (A \odot C) \& (B \odot C)$ and reading the above top-down, one derives the sequent $(A \odot C) \& (B \odot C) \vdash (A \& B) \odot C$, while putting $\Gamma = (A \& B) \odot C$ and reading the above bottom-up, one derives the sequent $(A \& B) \odot C \vdash (A \odot C) \& (B \odot C)$. So one proves distributivity.

Inside the black box, we have (at least!) two kinds of distinct links for registers and hence we can deal with two different versions of the equation for &:

$$\Gamma \vdash (A \& A^{\perp}) \bowtie (B \& B^{\perp}) \quad \equiv \quad \Gamma \vdash A, B \quad \Gamma \vdash A^{\perp}, B^{\perp} \tag{28}$$

for maximal entanglement, and:

$$\Gamma \vdash (A \& A^{\perp}) \asymp (B \& B^{\perp}) \quad \equiv \quad \Gamma \vdash A, B \quad \Gamma \vdash A^{\perp}, B \quad \Gamma \vdash A, B^{\perp} \quad \Gamma \vdash A^{\perp}, B^{\perp} \tag{29}$$

for non entanglement. Note that this last equation could be derived from the classical one, considering the separation link $\asymp$ instead of the comma, putting $B = A^{\perp}$, $C = B \& B^{\perp}$ and expanding twice (one needs also to assume commutativity or to put the definitional equations for contexts on the left too). That is, the unentangled case corresponds to a classical use of the context. On the contrary, in the equation (28), we have an odd use of the context, corresponding to entanglement. Notice moreover that, due to visibility, the basic logic equation (25) represents a lower bound for both the equations valid in the black box, that is (28) and (29). This means again that the external observer, unaware of the actual kind of correlations present in the black box, but aware of her unawareness, must, as Scepticism suggests, suspend judgement. Hence, for her judgements, no context at all.

Now let us prove the equivalences. This can be achieved again disassembling the judgements and then assembling them the other way around, as we have already done above for distributivity. We have the following pair of derivations:

$$\dfrac{\dfrac{\dfrac{\vdash (A\odot B)\&(A^{\perp}\odot B^{\perp})}{\dfrac{\vdash A\odot B}{\vdash A,B}\,\odot \qquad \dfrac{\vdash A^{\perp}\odot B^{\perp}}{\vdash A^{\perp},B^{\perp}}\,\odot}\;\&}{\vdash (A\&A^{\perp})\bowtie(B\&B^{\perp})}\;\&congr}{\vdash (A\&A^{\perp})\odot_1(B\&B^{\perp})}\;\odot_1 \tag{30}$$

for the maximally entangled case, and

$$\dfrac{\dfrac{\dfrac{\vdash (A\odot B)\&(A\odot B^{\perp})\&(A^{\perp}\odot B)\&(A^{\perp}\odot B^{\perp})}{\dfrac{\vdash A\odot B}{\vdash A,B}\,\odot \qquad \dfrac{\vdash A\odot B^{\perp}}{\vdash A,B^{\perp}}\,\odot \qquad \dfrac{\vdash A^{\perp}\odot B}{\vdash A^{\perp},B}\,\odot \qquad \dfrac{\vdash A^{\perp}\odot B^{\perp}}{\vdash A^{\perp},B^{\perp}}\,\odot}\;\&}{\vdash (A\&A^{\perp})\asymp(B\&B^{\perp})}\;\&cont}{\vdash (A\&A^{\perp})\odot_0(B\&B^{\perp})}\;\odot_0 \tag{31}$$

for the non entangled case.

The above couple of derivations can be read top-down and bottom-up, providing the required equivalences between the judgements. In particular, in (30) we have the new rule "&*congr*", which follows from the definitional equation (28), where "*congr*" is for "congruence", as equation (28) resembles a congruence rule in algebraic terms. It shows in logical terms that entanglement is a particular form of superposition. In (31) the rule "&*cont*" (for "context"), coming from the definitional equation (29), that is equivalent to the classical equation (26), is a form of a classical &-rule of sequent calculus.

The premise Γ, used in the definitional equations, is not present in our judgements yet, because its full justification inside the Black Box is still under study. Anyway, once a premise can be adopted, putting convenient premises in 30 and in 31, as seen in the classical case, one can prove the following pairs of sequents, respectively:

$$(A\&A^{\perp})\odot_1(B\&B^{\perp})\vdash(A\odot B)\&(A^{\perp}\odot B^{\perp})$$

$$(A\odot B)\&(A^{\perp}\odot B^{\perp})\vdash(A\&A^{\perp})\odot_1(B\&B^{\perp})$$

and

$$(A\&A^{\perp})\odot_0(B\&B^{\perp})\vdash(A\odot B)\&(A\odot B^{\perp})\&(A^{\perp}\odot B)\&(A^{\perp}\odot B^{\perp})$$

$$(A\odot B)\&(A\odot B^{\perp})\&(A^{\perp}\odot B)\&(A^{\perp}\odot B^{\perp})\vdash(A\&A^{\perp})\odot_0(B\&B^{\perp}).$$

The first pair is "distributivity" of $\odot_0$ w.r.t. &, that is the equality $(A\&A^\perp)\odot_0 (B\&B^\perp) = (A \odot B)\&(A \odot B^\perp)\&(A^\perp \odot B)\&(A^\perp \odot B^\perp)$, the second is "odd distributivity" of $\odot_1$ w.r.t. &, that is the equality $A\&A^\perp)\odot_1 (B\&B^\perp) = A \odot B)\&(A^\perp \odot B^\perp)$ (we remind that the subscripts 1 and 0 disappear outside).

Furtherly, trivializing the premises in the equivalences 28 and 29, as in basic logic, one obtains the so called "reflection axioms", namely the pair

$$(A\&A^\perp)\odot_1 (B\&B^\perp) \vdash A, B \qquad (A\&A^\perp)\odot_1 (B\&B^\perp) \vdash A^\perp, B^\perp$$

and the quartet

$$(A\&A^\perp)\odot_0 (B\&B^\perp) \vdash A, B \quad A\&A^\perp)\odot_0 (B\&B^\perp) \vdash A^\perp, B$$

$$A\&A^\perp)\odot_0 (B\&B^\perp) \vdash A, B^\perp \quad A\&A^\perp)\odot_0 (B\&B^\perp) \vdash A^\perp, B^\perp$$

respectively. Pair and quartet are to be considered as a whole inside, while, when they are splitted, they give rise to the collapse of the wave function, represented in logical terms as follows (we write down only one of the possible cases):

$$\frac{\vdash (A\&A^\perp)\odot_1 (B\&B^\perp) \qquad (A\&A^\perp)\odot_1 (B\&B^\perp) \vdash A, B}{\vdash A, B}\,cut$$

where the conclusion $\vdash A, B$ is the representation of the external measurement of our pair of qubits, as we have seen in the previous section. So the cut rule represents the projective measurement, that is the irreversible moment of the computation. For the one qubit case, see [2].

In our opinion, the derivations (30) and (31), despite their simplicity, are already quite informative for a logical calculus which aims to grasp the efficiency of quantum computation. In fact, they show how the "quantum parallelism", in the entangled case, can be obtained by only one half of the derivation branches with respect to the non entangled case! This is achieved thanks to the rule "&*congr*". As for the non entangled case, the rule "&*cont*" realizes a classical parallelism (superposition without entanglement). One could object that, even in (27), that is the case of classical distributivity, we have half of the branches, but in such case half of the information is hidden and so parallelism is not achieved. Then the so called "massive quantum parallelism" requires entanglement.

Note that derivation (30) has the same form of the following one, that shows how the judgements $\vdash A$ and $\vdash A^\perp$ can be assembled and disassembled in the one qubit case (cf. [2]):

$$\frac{\dfrac{\vdash A\&A^\perp}{\vdash A \quad \vdash A^\perp}}{\vdash A\&A^\perp} \tag{32}$$

In this sense we think that a Bell's state seen from inside the quantum computer can be assimilated to a single particle. Notice that here "inside" means that the quantum computer is embedded in a non-commutative geometry background [22]. In other words a 2-qubits register is a fuzzy sphere with four elementary cells each one encoding a two-qubits string $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ (see fig. (1)). In the maximally entangled case, the fuzzy sphere has two cells, each one with doubled surface area, and encoding the two-bits string $|00\rangle$ and $|11\rangle$ (see fig. (2)). The latter situation resembles the case of one qubit, where the fuzzy sphere has two elementary cells, each one encoding the one bit string $|0\rangle$ and $|1\rangle$ (see fig. (3)).

The fact that a Bell's state (as seen from inside a quantum computer) "pretends" to be a single particle, while we think it is not, is at the origin of all paradoxes related to entanglement. For example, "non-locality" is just a problem of the external observer, who lives in a local space-time. Instead, the Bell's state, as seen from inside a quantum computer, lives in a non-local space, which is the fuzzy sphere. Moreover, as far as causality is concerned, let us consider the cut rule:

$$\frac{A \vdash B \quad B \vdash C}{A \vdash C}\, cut$$

which is a causal relation. But the cut rule as such is not admissible inside the Black Box (as we have seen above and in [2]), since it corresponds to a projective measurement performed in the external world. Thus we argue that the usual meaning of causality is absent in the case of a quantum computer on a fuzzy sphere (internal logic). This is the reason why an external observer, who lives in a causal world, sees as a paradox the non-causal behaviour of a Bell's state.

Conclusions

Notice that the model of a quantum computer in a non-commutative geometry can be identified with a model of Computational Loop Quantum Gravity (CLQG) [23]. For a review on Loop Quantum Gravity (LQG) see for example [18]. This is equivalent to consider a quantum computer at the Planck scale. Causality at the Planck scale is a very controversial issue, and some authors, like Sorkin and collaborators [4] are inclined to believe in a sort of micro-causality that they discuss in terms of causal sets. However, the fact is that the light cone at the Plank scale might be "smeared" by the very strong quantum fluctuations of the metric field (the "quantum foam" [20]) and this would indicate that (micro) causality is lost at that scale at least in its usual setting. At the light of our logical result, i.e., that the cut rule as such is not admissible inside the Black Box, we are now lead to argue that causality is absent at the fundamental scale.

One can realize that it is just the intrinsic non-locality of non-commutative geometry which leads to a modification of micro-causality (see for example [5] in support of this idea). Then, it is non-commutative geometry itself which "trivializes" at once both the "paradoxes" of non-locality and non-causality of entanglement. Recall that non-commutative geometry was exploited in [22] to describe the inner quantum world, and here some logical consequences are analyzed, producing an internal logic. We think that our internal approach can lead to a logical explanation of the different computational behaviour (speed up) of quantum computers with respect to classical ones.

Finally, we are confident that the present results are susceptible of significant developments. We foresee, for example, that it will be possible to complete the rules of the sequent calculus for our inner quantum logic. This will require the complete solution of the definitional equations, like in basic logic.

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Fig. 1

Two unentangled qubits.

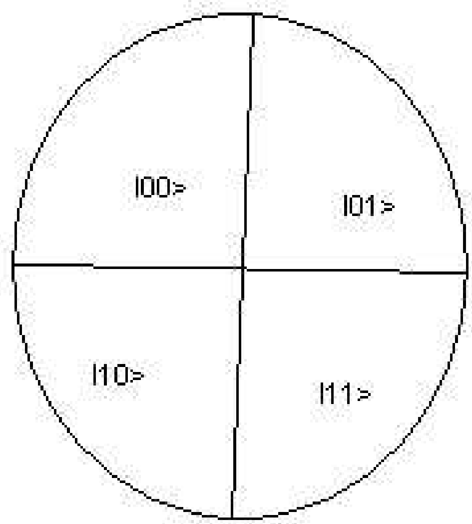

Fig. 2

A Bell's state

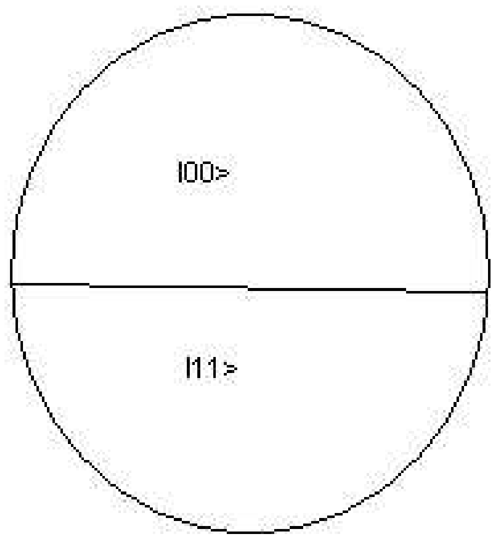

Fig. 3

One qubit