A modality for spin states:
overcoming the constraints of first order language

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Abstract

On the basis of a characterization of quantum states by predicative
formulae, we derive the idea of “universal spin state” of a particle, that is
obtained introducing a modality. Then we investigate about the logic of
such universal spin states.

Introduction

Classical physics was developed assuming, a priori, that sharp identification of
objects is possible. On the contrary, in quantum mechanics, the possibility of
separating and identifying objects cannot be always assumed. However, the
mathematical formalism, that had already been developed to describe physics,
considered the identification as an ”a priori”. In turn, the logical foundations
were based on the characterization of first order language, which included the
notions of closed term and of variable, and did not bring the problem into
question otherwise. Then, on one side, incompleteness of first order language in
presence of very elementary mathematical axioms was proved, and, on the other
side, quantum mechanics was accused to be “incomplete”. For a very recent
careful analysis of the epistemological problems arised from the unawareness of
the strong metaphysical presuppositions at the origin of the theory of quantum
physics, we refer to the work of C. De Ronde, e.g. [DeR16].

Investigating a model of quantum states by first order variables [Ba12], we
have seen how closed terms and variables are related to the identification of
states. In the present paper, we shortly review the model. Then, we consider
the spin observable only, that permits us to avoid the no-go contained in the
Kochen-Specker theorem. Extending our previous definition, we introduce a
modal operator, which eliminates variables and can be interpreted as something
describing “the universal spin state” of a particle. We prove that it coincides
with the modal operator of $S_4$, that was Gödel’s original proposal to overcome
incompleteness [Go’30]. However, differently from Gödel’s system, our elemen-
tary propositions cannot be considered literals of classical propositional logic,
on which one can define an intuitionistic system given by the modality. Indeed,
as shown in [Ba12], the representation of spin states shows in particular that negation is not well defined on spin states.

We have found [Ba13, Ba14] that the model with variables corresponds to a “symmetric” logic, namely a different logical mode, termed symmetric mode, completely unstructured, from a standard logical point of view, but also very rich, proper of the unconscious thinking [MB75].

In the last section, we begin to see how the modal operator can recover a well defined negation and logical consequence from the symmetric mode. This represents a challenge also for the study of the unconscious thinking: it would be very important to find a model controlling the way in which the symmetric mode collapses into the mode of conscious thinking. Matte Blanco terms the latter bivalent mode, since it can distinguish two truth values. Here, once again, the insight from quantum mechanics could reveal very useful, since the Kochen Specker theorem characterizes a bivalent mode as the only one in which complete determination is conceivable.

In the future, in order to develop our proposal, we need to see better how to extend our view to entangled states and identical particles. We think that the problem is related to the exponentials operators of linear logic [Gi87], that is a strong alternative logical proposal to go beyond the constraints of first order language, keeping the basis of a classical logical approach.

1 States by predicative formulae

We have shown [Ba12] that the state of a particle with respect to an observable can be represented by a universal proposition. The measurement of a particle \( \mathcal{A} \), considering a certain observable, gives a set of outcomes

\[
D_Z = \{(s_i, p\{Z = s_i\}) : i \in I\}
\]

where \( Z \) is the random variable associated to the measurement, whose outcomes are the states \( s_i \) with their frequencies. Then, if \( \Gamma \) is a set of assumptions for the measurement of particle \( \mathcal{A} \), we can say that:

\[
\Gamma \text{ yields } \mathcal{A} \text{ is found in state } s_i \text{ with probability } p\{Z = s_i\}.
\]

Let us write yields by the sequent sign for consequence: \( \vdash \). Let us consider a first order variable \( z \) whose range is \( D_Z \). Then we write

\[
\forall z \in D_Z, \Gamma(-z) \vdash A(z)
\]

The notation \( \Gamma(-z) \) indicates that the set of assumptions \( \Gamma \) cannot depend on the measurement outcome \( z \) (that is true, if the measurement is correct), namely \( \Gamma \) is closed with respect to variable \( z \). The premise \( z \in D_Z \) is at a metalinguistic level. We import it into the language of sequents and write:

\[
\Gamma(-z), z \in D_Z \vdash A(z)
\]

Then we put the following equation, defining \( \forall [MS05] \):

\[
\Gamma(-z) \vdash (\forall x \in D_Z)A(x) \text{ if and only if } \Gamma(-z), z \in D_Z \vdash A(z) \quad (1)
\]
Notice that introducing the quantifier one closes the formula $A$ with respect to
the variable $z$.
By the above definition, we attribute a state to the particle by the proposition

$$(\forall x \in D_Z)A(x)$$

1.1 Incompleteness

Above, we have first considered the assertion "forall $z \in D_Z$, $\Gamma(-z) \vdash A(z)$" describing formally the result of a measurement, where the variable is used at
a metalevel, as a parameter. Putting:

$\Gamma(-z) \vdash (\forall \omega x \in D_Z)A(x)$ if and only if forall $z \in D_Z : \Gamma(-z), \vdash A(z)$

one defines a connective $\forall\omega$ (this is traditionally termed “$\omega$-rule”). It coincides
with the conjunction $\&$ of $n$ conjuncts $A(t_i)$ when $D_Z = \{t_1, \ldots, t_n\}$ is finite.
It describes the overall result of a measurement, namely the mixed state after
measurement. The failure of the sequent

$$(\forall\omega x \in D_Z)A(x) \vdash (\forall x \in D_Z)A(x)$$

is a way to see, on one side, the incompleteness of Quantum Mechanics, on the
other, the incompleteness of Arithmetic.

1.2 Infiniteness

Let us assume $D_Z = \{t_1, \ldots, t_n\}$ is a finite domain. One can prove that the
above sequent $(\forall\omega x \in D_Z)A(x) \vdash (\forall x \in D_Z)A(x)$ is equivalent to

$$z \in D_Z \vdash z = t_1 \lor \ldots \lor z = t_n$$

Since one can count $n$ elements only if one can distinguish them, one can con-
clude that the pure state “prior to measurement” (even if it is “sharp”!) is
infinite [Ba12].

2 Spin states

We now consider the spin observable. The measurement outcome of the spin of
the particle w.r.t. a certain direction $d$ is “spin down” with probability $\alpha^2$ and
“spin up” with probability $\beta^2$, $\alpha^2 + \beta^2 = 1$.

The set $D_Z$ of outcomes is $\{(\downarrow, \alpha^2), (\uparrow, \beta^2)\}$ and then the state is attributed
to $A$ by means of the predicative formula $(\forall x \in D_Z)A(x)$ that is obtained
putting the equation 1.

This can be said for every direction $d$. However, different spin observables
are incompatible, so our formula should be able to attribute an objective state
to the particle only for one direction at a time.
What could we say about the overall spin of the particle? Since the spin is
two-valued, we have no constraint from the Kochen-Specker theorem. We think
that, in different worlds, they measure the spin of $A$ in different directions. The
overall equation, which generalizes equation 1, should have the form

$\square\Gamma \vdash \square A$ if and only if $\square \Gamma \vdash A$ \hspace{1cm} (2)

The domain and the variable have disappeared since no reference to a specific set
of measurement outcomes is involved. The notation $\square$ indicates that formulae
are closed with respect to any variable for any direction $d$.

The above equation characterizes a unary operator $\square$ on formulae. It enables
us to find rules for $\square$ in sequent calculus, following the pattern first proposed
in [SBF00]. In the following result we show the features of the operator $\square$ so
characterized.

**Lemma 2.1** The above equivalence is valid in the modal system $S_4$. Conversely,
it entails the rules of $\square$ in $S_4$.

Proof: Let us assume the rules of $\square$ in $S_4$. From $\square \Gamma \vdash A$ one derives $\square \square \Gamma \vdash \square A$
by $K$ of $S_4$. Then, one derives $\square \Gamma \vdash \square A$ since $\square A \vdash \square \square A$ for every $A$.
Conversely, from $\square \Gamma \vdash \square A$, since $\square A \vdash \square A$, one has $\square \Gamma \vdash A$.

Conversely, let us assume that equivalence 2 holds. Then the necessitation
rule of $S_4$ is true assuming $\Gamma = \emptyset$. The clause $\square A \vdash A$ follows from the axiom
$\square A \vdash \square A$ by the only if direction of the equivalence and the clause $\square A \vdash \square \square A$
follows from the same axiom by the if direction. Finally, in order to derive $K$,
we first see that $\Gamma \vdash A$ entails $\square \Gamma \vdash \square A$: assuming $\Gamma \vdash A$ one derives $\square \Gamma \vdash A$
since for every $A \square A \vdash A$ as just seen, then $\square \Gamma \vdash \square A$ by our equivalence. So,
since $A \rightarrow B, A \vdash B$ is a provable sequent, one derives $\square (A \rightarrow B), \square A \vdash \square B$,
from which $K$.

3 The logic of universal spin states

By the above result, our $\square$ has the desired feature of “universal projector”, in
this sense: $\square A \vdash \square \square A$ and $\square A \vdash A$ together give idempotency:

$\square \square A = \square A$

Then, it renders the value of the spin of particle $A$ stable, whatever direction
is chosen. Then $\square A$ is a formula which associates to $A$ its “universal state” for
the spin.

What could we say about the logic associated to our formulae describing
universal states? As discussed in [Ba12], the logic associated to different spin
directions shows opposite aspects: the unitary operator which switches the two
opposite sharp states $\uparrow$ and $\downarrow$, with respect to, say, the direction of the $z$ axis,
that represents negation in such case, is not effective with respect to the sharp
states $\uparrow$ and $\downarrow$ with respect to the $x$ direction, since they are its eigenvectors!
Then, if one considered simply a set of assertions describing the spin state of quantum systems, with no reference to the direction, the underlying logic should be completely unstructured.

In the following we discuss a possible way to extend the definition of □ and some ideas to give a structure to the logic underlying propositions on spin states. We consider propositions $A_i$ which refer to the spin state of a given particle $A_i$, or more complex quantum systems. They will be considered as literals to define a logic. Then, we exploit our $S_4$ modality, following the pattern first proposed by Kurt Gödel just after his incompleteness theorem. In that case, intuitionistic logic is found from a set of classical literals equipped with classical logic, that would be our case if we had bits rather than qubits. We have seen [Ba13] that qubits better fit with the symmetric mode of Matte Blanco [MB75] whereas bits correspond to the bivalent mode. The first is an infinite mode, the second is finite. In Gödel, the modality was conceived to represent the predicate “provable by any possible means” to be distinguished from his provability predicate in the incompleteness theorem, that was a provability by finitary methods. Then, the modality was conceived “to capture the infinite”. In the same period, Gödel proved also that intuitionistic logic, which is found by means of the modality, has infinite many truth values [Go’30]. In our model, the original objects are infinite, and the modality wants to keep “as much infiniteness as possible” in order to define a logic.

We need to introduce propositions for states of well known quantum systems more complex that one-particle systems. We first consider the mixed state $\frac{1}{2} |\uparrow\rangle + \frac{1}{2} |\downarrow\rangle$. It is the same whatever direction is considered. Then we interpret any system in such a state by a constant, \bot, for which

$$\square \bot \equiv \bot$$

consistently with our interpretation of □ and with Gödel’s interpretation. The singlet state does not depend on the chosen direction too. So, we can introduce a constant, 1, to denote any system in such a state, for which:

$$\square 1 \equiv 1$$

The availability of a falsum gives the opportunity to define a negation by means of its intuitionistic definition:

$$\neg A \equiv \square A \supset \bot$$

Namely, the negation of any $A$ is the intuitionistic negation of $\neg \square A$. Notice that, in Gödel, intuitionistic negation $\neg A$ on a literal $A$ was defined adopting classical negation $\sim$ and putting: $\neg A \equiv \sim \square A$. Here we have no definite classical negation, however, classically, the two definitions are equivalent. Indeed, Gödel’s interpretation of the formula $\square A \supset \bot$ was obtained by the following translation of intuitionistic implication adopting classical implication and then classical negation and disjunction: $A \supset B \equiv \square A \rightarrow \square B \equiv \neg \square A \lor B$. In our case, $\square A \supset \bot$ is $\sim \square \square A \lor \bot$, that is $\sim \square A \lor \bot$, that is classically $\sim \square A$. 

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One could object that even intuitionistic implication is not available in our unstructured set of literals. However, one can derive a meaning from quantum mechanics, considering systems of two particles $A_1$ and $A_2$ in the singlet state: 
$\frac{1}{\sqrt{2}}|\downarrow\uparrow\rangle + \frac{1}{\sqrt{2}}|\uparrow\downarrow\rangle$. Now, let us try to discover “the objective state” of one of the two particles $A_1$ and $A_2$. Quantum Mechanics says that, once one of the two is traced out, the other is necessarily found in the mixed state $\frac{1}{2}P|\uparrow\rangle + \frac{1}{2}P|\downarrow\rangle$. In our terms, we write this

$$\Box A_i \equiv \Box A_j \supset \bot$$

for $i, j = 1, 2$ and $i \neq j$.

The universal spin state for each of the two particles is interpreted as the negation of the other: the single particle in the singlet state has no meaning, whereas the two particles together do have a meaning! Since

$$\Box A_j \equiv \Box A_i \supset \bot \equiv (\Box A_j \supset \bot) \supset \bot$$

negation acts classically in such a case.

The definition can be extended to the case of propositions for isolated particles. In such a case, since the particle has no companion, negation amounts to an irreversible loss of state and acts intuitionistically.

The above considerations lead to the attempt to find an interpretation to the truth content of a singlet state, or other entangled states, that is more detailed than the constant 1, adopted for any singlet state. Considering a specific couple in the singlet state: $A_1 + A_2$, the system as a whole could be interpreted as the classical disjunction: $\neg \Box A \lor \Box A$. It is equivalent to 1, provided 1 is identified with the $T$ constant of classical logic. Moreover, $\neg \Box A \lor \Box A$ is equivalent to $\Box A \rightarrow \Box A$, that in turn can be read as $\Box A \supset \Box A$ or even $A \supset A$ by Gödel’s translation. Such an interpretation is very intriguing, since the entanglement link would be converted into the implication link when the unstructured set of propositions on quantum states has to find its logical truth content. The loss would be symmetry, but the added value would be asymmetry, here read in terms of the “directionality” of logical consequence, that renders logic significant.

In [Ba12] we have described how to write assertions on the state of entangled particles, and how to define a generalized quantifier to obtain propositions on entangled states, by a suitable generalization of 1. The analysis of how such assertions could be treated by a suitable extension of the modality $\Box$, namely an extension of equivalence 2, and of what could be obtained from such an extension, deserves further study and will be object of future research.
References


