Basic logic:
reflection, symmetry, visibility.

Giovanni Sambin - Giulia Battilotti - Claudia Faggian*

Abstract

We introduce a sequent calculus **B** *for a new logic, named basic logic. The aim of basic logic is to find a structure in the space of logics. Classical, intuitionistic, quantum and non-modal linear logics, are all obtained as extensions in a uniform way and in a single framework. We isolate three properties, which characterize* **B** *positively: reflection, symmetry and visibility.*

A logical constant obeys to the principle of reflection if it is characterized semantically by an equation binding it with a metalinguistic link between assertions, and if its syntactic inference rules are obtained by solving that equation. All connectives of basic logic satisfy reflection.

To the control of weakening and contraction of linear logic, basic logic adds a strict control of contexts, by requiring that all active formulae in all rules are isolated, that is visible. From visibility, cut-elimination follows. The full, geometric symmetry of basic logic induces known symmetries of its extensions, and adds a symmetry among them, producing the structure of a cube.

Contents

1 Introduction

Up to the beginning of the century, there was only one logic, Aristotle's classical logic, which was conceived as a metaphysical absolute. Starting with Brouwer's revolution, which introduced intuitionistic logic, a number of different new logics have been developed. Each of them aimed to capture some of the distinctions which can be observed in a specific field of interpretation, but which are ignored by classical logic. Excluding intensional logics (in which modalities are considered), all such logics can be grouped under three main headings: intuitionistic logic (absence of the principle of double negation), quantum logic (absence of distributivity between conjunction and disjunction), and relevance and linear[1] logic (finer control of structural rules).

*The first author has been the inspirer for the philosophical foundations, in particular for the principle of reflection. The discovery and development of basic logic as presented in this paper is the outcome of real collaboration among all the three authors.

[1] To simplify terminology in this paper by linear logic we mean the system sometimes called MALL, that is Girard's linear logic [23] deprived of the modalities ! and ? he calls exponentials.

Although all of these logics are derived from classical logic, they have been considered as mutually incompatible. Basic logic was developed in order to provide a common foundation and to show that they share a common structure. This was achieved in the first version of basic logic [6], which is a common denominator, but only in terms of provable formulae. The present version of basic logic is completely new and can be characterized more positively as the logic which obeys three general principles: reflection, symmetry and visibility. These principles are introduced below and will be demonstrated in pratice in later sections.[2]

The common explanation of the truth of a compound proposition like $A\&B$ is that $A\&B$ is true if and only if A is true *and* B is true. In our terms, a connective $\circ$ between propositions, like & above, reflects at the level of object language a *link* between assertions in the meta-language, like *and* above. The semantical equivalence

$$A \circ B \ true \quad \text{if and only if} \quad A \ true \ link \ B \ true$$

which we call *definitional equation* for $\circ$, gives all we need to know about it. $A \circ B$ is semantically *defined* as that proposition which, when asserted true, behaves exactly as the compound assertion $A \ true \ link \ B \ true$. The inference rules for $\circ$ are easily obtained by *solving* the definitional equation, and they provide an explicit definition. We then say that $\circ$ is introduced according to the *principle of reflection*.

All logical constants of basic logic are introduced according to the principle of reflection. We show that all inference rules are justified by solving suitable definitional equations, in which $A \circ B$ is allowed to appear also as an assumption. Moreover, we show that only two types of link among assertions are sufficient.

In this way we bring the interplay between language and metalanguage into explicit consideration. The construction of basic logic is thus seen as a product of the dynamics between meta-language and its formalization at the level of objects[3]. No external notion of truth is invoked, not even in the form of an a priori choice of connectives.

A symmetry among logical constants of classical logic was pointed out by Gentzen [22] in his calculus of sequents **LK**. Later, J-Y. Girard has reached with classical linear logic a deeper symmetry, which allows the definition of negation for all formulae starting from negation on atoms. On the other hand, intuitionistic logic is commonly considered as intrinsically asymmetric. Basic logic is again fully symmetric in a strong sense, but still it admits both intuitionistic and linear logic as natural extensions.

To transform symmetry into a conceptual tool, one has to abandon the traditional scheme which says that the rule introducing a connective is always the rule operating on the right and that the rule on the left is always the elimination rule. On the contrary, all logical constants are divided into "left" and "right" constants. A "left" connective has a formation rule which operates on the left, on assumptions, and a second rule, called reflection, which operates on the right, on conclusions. Any left connective is accompanied by its symmetric right connective, governed by the rules obtained by interchanging antecedent with succedent.

The symmetry of basic logic is not internal, but rather a simple geometric symmetry which is evident at the meta-level: whatever action is taken on the right side can be symmetrically performed on the left side of sequents. So basic logic is simultaneously a logic of derivations as well as a logic of refutations, and one is not definable from the other. Any proof has a symmetric proof with identical structure, apart from the swap left-right. This fact is essential to prove cut-elimination also when negation is added, in the style of Girard, on top of structural rules to obtain quantum-like logics (see [20] and [17]).

One of the main principles of proof theory, put forward by Gentzen and clarified mainly by D. Prawitz, is that the meaning of a connective is determined by rules dealing exclusively with it. This discovery is manifested technically in the theorems on normalization of derivations. One of the principles of contemporary proof theory, promoted by Girard, is that

[2] We plan to write a more conceptual discussion [31], in particular of the principle of reflection.

[3] A forceful though brief and general description of the dynamics between formal language and metalanguage in the development of mathematics is given by N. G. de Bruijn [13].

a careful control of structural rules of weakening and contraction permits a finer analysis of the structure of derivations. Basic logic pushes both such principles to their ultimate consequences.

One of the main discoveries of basic logic is that the meaning of a connective is determined also by contexts in its rules, which can bring in latent information on the behaviour of the connective, possibly in combination with other connectives.

We thus say that a rule satisfies visibility if it operates on a formula (or two formulae) only if it is (they are) the only formula(e), either in the antecedent or in the succedent of a sequent. Formally, visibility is the property that all active formulae (secondary or principal formulae, in Gentzen's terminology) are isolated, or visible, all passive contexts (not on the same side of any active formula) are free.

The main technical novelty of basic logic is that all its rules satisfy visibility. In other terms, basic logic keeps under control not only the rules of weakening and contraction, but also the presence of contexts on the same side of active formulae. This is done in a very drastic way, namely by suppressing them systematically.

When the principle of reflection is met, visibility is characterized more intrinsically by the presence of only one parameter for contexts in definitional equations, rather than two as it happens for other usual logics.

As with control of weakening and contraction a new class of connectives – multiplicatives in Girard's terminology – comes to the surface, so with strict control of contexts the only reasonable way to allow movement from one side of a sequent to the other is to treat implication (and its symmetric) as primitive. This is what gives to basic logic its intuitionistic flavour, even in the presence of symmetry.

Summing up in superficial terms, the sequent calculus for basic logic looks like a (two-sided) calculus for linear logic, except for the absence of all contexts at the side of active formulae and for the presence of two symmetric connectives of movement.

Once basic logic is introduced, it is straightforward to realize that linear logic is regained by relaxing visibility on both sides, that is by adding contexts at the side of all active formulae. Relaxing visibility only on the left gives an intuitionistic linear logic with "par" (treated in detail in [3]); adding also weakening and contraction gives the usual intuitionistic logic, but with an extra primitive connective called exclusion, and symmetric of implication. Symmetrically, liberalizing contexts only on the right gives symmetric copies of intuitionistic linear logic and of intuitionistic logic with exclusion (similar to the "dual-intuitionistic" logic of [26]). The simultaneous control of contexts on both sides, namely visibility, allows us to block the derivation of all distributive laws and to obtain logics in which the deduction theorem fails. This allows us to bring the field of quantum and, more generally, non-distributive logics under the realm of proof theory (cf. [19], [17], [2], and, for a survey, [5]).

The structure of extensions of basic logic becomes quite easy to grasp if it is pictured as a cube in which each vertex corresponds to the sequent calculus obtained by performing a combination of the three actions called **L**, for liberalize contexts on the Left, **R**, for liberalize contexts on the Right, and **S**, for add the Structural rules of weakening and contraction (see the figure in section 3).

It is an ambition of basic logic to offer a new perspective and new tools to the search for unity in logic. Differently from [25], our plan is to look for the basic principles and structures common to many different logics.

So one aim is to obtain each specific logic by the addition of rules concerning exclusively the structure (i.e. structural rules dealing only with assertions), while keeping the logic of propositions (i.e. operational rules dealing with logical constants) absolutely fixed. Note that the extensions described above are not pure in this sense; but this aim has been obtained for some "symmetric" logics, including linear and classical logic (cf. [18],[20]), while it seems within reach for "asymmetric" logics, including intuitionistic logic (cf. [4] and [8]). A second aim is to embed each logic in basic logic, once it is provided with some additional kinds of assertion, or modalities; some cases have been obtained (cf. [1]) and other cases are expected.

More generally, it is our belief that only some of the potentialities of basic logic and its principles are discussed in the present paper, and certainly not in an exhaustive way. We do not even touch here themes like possible applications, inside and outside logic itself, or the new perspective and new problems offered to philosophical investigations. We postpone also a more detailed discussion of the connections with the literature, mainly with linear logic of Jean-Yves Girard [23], display logic of Nuel Belnap [11] and with the work of Kosta Došen (see for instance his recent [14]).

The picture is not complete, yet we hope that what we present here makes it possible to see that the name we have chosen, basic logic, is justified.

Several people have played an important role in the authors' discovery and development of basic logic (in chronological order): John L. Bell, for raising our interest in the non-distributive side[4], Silvio Valentini, for helpful conversations and suggestions at the early stage, Jean-Yves Girard, for the creation of linear logic and all the ideas coming with it, Marisa Dalla Chiara, for inviting us to Florence three times to speak about basic logic, Per Martin-Löf, for his semantical justification of the logical laws and for suggesting to define negation by means of implication, Grigori Mints, for calling to our attention that two different forms of cut could be helpful, Rajeev Goré, for conversations on display logic. We are grateful to all of them. We are also very grateful to Morgan E. Kernow for her effort to correct and enrich our use of English and to Silvia Gebellato for her help with the drawing.

2 Basic logic and the principle of reflection

The best way to justify the very choice of logical constants and the determination of their rules of inference in basic logic is to see them as the result of a very general principle, which we have called the *principle of reflection*. Suppose we want to create a new connective $\circ$ by laying down the truth equation it must satisfy. This equation is called the *definitional equation* for $\circ$. For example, assume we wish the equation

$$A \circ B \vdash \Delta \;\; \textit{if and only if} \;\; A, B \vdash \Delta$$

to hold for any Δ. A direction of this equation is the rule

$$\frac{A, B \vdash \Delta}{A \circ B \vdash \Delta}$$

It says which conditions are sufficient to assert $A \circ B$. The other direction, however, describes only what we should be allowed to deduce from $A \circ B \vdash \Delta$, when we have already $A \circ B$. So it contains information on $\circ$, but only in implicit form, or backwards. To define $\circ$ with no vicious circle, we must find an equivalent rule in which $A \circ B$ appears in the conclusion, rather than in the premiss. To do that, we first trivialize the assumption $A \circ B \vdash \Delta$ by considering Δ to be $A \circ B$ itself. Then we obtain the axiom $A, B \vdash A \circ B$, with the benefit that $A \circ B$ is now on the right side. So, assuming that $\Gamma \vdash A$ and $\Gamma' \vdash B$ hold, from the axiom we obtain $\Gamma, \Gamma' \vdash A \circ B$ by cut. Hence the rule

$$\frac{\Gamma \vdash A \quad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \circ B}$$

is derivable from the direction of the equation: $A \circ B \vdash \Delta \;\; \textit{only if} \;\; A, B \vdash \Delta$. To see that it is actually equivalent to it, we trivialize the premises by considering Γ and Γ' to be A and B respectively. Then the axiom $A, B \vdash A \circ B$ is derived, which in turn gives the claim by cut. The two inference rules say under which premises, involving A and B but not $A \circ B$, the compound proposition $A \circ B$ can be asserted. Taken together, they are equivalent to the definitional equation. We then say that $\circ$ is introduced according to the principle of reflection.

[4] By observing in [10] and in later conversations that the representation of complete Boolean algebras of [30], which we obtained as a special case of representation of quantales through pretopologies [9], [7], is a special case also of that of ortholattices through Birkhoff's polarities; trying to find a general representation theory including both was the catalyst for basic logic.

In basic logic all logical constants are introduced via the principle of reflection. Moreover, the solution of definitional equations is always found by following a single simple pattern. To solve equations in a non-circular way, the means to be used must be fixed beforehand. So the first task is to declare explicitly which meta-level properties are assumed to be known. Thus here it is the meta-level which comes first, and based only on it the formal system is built up. To make the pattern of solution clear, and thus save many details, we also need to introduce some new terminology.

Assume from now on that $A, B, C \ldots$ denote propositions. A proposition A must be distinct from the assertion on it. At this basic level, rather than A *is true* (as in [27] and [28]), we prefer to adopt for assertions a more neutral notation like A *is* (to recall the common form of A *is true*, A *is available*, A *is measured*, etc.), which shares with A *is true* only the fact it is an assertion, rather than a proposition. This is to recall that no specific meaning is at the moment attached to the assertion of A.

We need to consider also more complex metalinguistic statements, which are built up from atomic assertions of the form A *is* by means of some metalinguistic *links* and which we call *compound assertions*. The first discovery here is that the compound assertions used in any sequent calculus (lists Γ, sequents $\Gamma \vdash \Delta$, rules, derivations, etc.) can be seen as obtained from atomic assertions by means of *only two* metalinguistic links, namely *and* and *yields*[5].

A conjunction of atomic assertions C_1 *is and* $\ldots$ *and* C_n *is* is abbreviated by $C_1, \ldots, C_n$, where comma takes the place both of *and* and of *is*. Like Gentzen, we write Γ for any conjunction of atomic assertions, either empty or $C_1, \ldots, C_n$. Similarly, for Δ and $D_1, \ldots, D_m$ and for other capital Greek letters.

The meaning of a sequent $\Gamma \vdash \Delta$ is that Δ is a consequence of Γ, that is Γ *yields* Δ or (C_1 *is and* $\ldots$ *and* C_n *is*) *yields* (D_1 *is and* $\ldots$ *and* D_m *is*). Thus the usual sign $\vdash$ is a shorthand for *yields* in such a compound assertion. The meaning of a rule of inference

$$\frac{\Gamma \vdash \Delta}{\Gamma' \vdash \Delta'}$$

is also clear: it says that we can move from the assertion $\Gamma \vdash \Delta$ to the assertion $\Gamma' \vdash \Delta'$, that is, that (Γ *yields* Δ) *yields* (Γ' *yields* Δ'). So Gentzen's horizontal bar is also a shorthand of *yields*, but in a different form of assertion.

Inference rules sometimes have more than one premiss, which are all put above the horizontal bar and separated by a blank space. Thus the blank space is a notation for *and* in this case, and so for instance

$$\frac{\Gamma \vdash \Delta \quad \Gamma' \vdash \Delta'}{\Gamma'' \vdash \Delta''}$$

is just a shorthand for ((Γ *yields* Δ) *and* (Γ' *yields* Δ')) *yields* (Γ'' *yields* Δ''). Finally, also the usual notation for derivations is clearly just a convenient shorthand for even more complex combination of *and* and *yields*.

The metalinguistic links *and* and *yields* are therefore sufficient to produce all compound assertions used in a sequent calculus for a propositional logic. The absence of a formal definition of compound assertion is too important to explain it simply by silence. In fact, a formal definition would turn assertions into objects, and thus *ipso facto* the aim of making the meta-language explicit would vanish[6].

The way the links *and* and *yields* are meant to behave is made explicit by laying down a few rules. When *and* is the outermost link, its behaviour is what one would expect, that is, for instance, the assertion $\Gamma \vdash \Delta$ *and* $\Gamma' \vdash \Delta'$ is just the same thing as the assertion $\Gamma \vdash \Delta$ together with the assertion $\Gamma' \vdash \Delta'$. In formal terms, in this case *and* behaves at the meta-level like the usual conjunction of propositions at object level.

This immediately loses any meaning when *and* is inside the scope of *yields*, that is, comma under turnstile; for instance, by no way is (A *is and* B *is*) $\vdash \Delta$ equivalent to A *is* $\vdash \Delta$

[5] After reading a first draft of the present paper, Kosta Došen has suggested that actually *and* and *yields* could be more deeply characterized as absence of a link or presence of a link, respectively.

[6] This attitude, developed in [29] and divergent from the common one, in the very end originates with Brouwer (see [12]).

and B *is* $\vdash \Delta$. In basic logic nothing is assumed about *and* in this case, except that it behaves well with respect to composition of derivations, as explained below. The same applies for *and* inside the scope of turnstile, at the right, and this is why no link interpreted as disjunction is needed.

When *yields* is the principal link, its behaviour is also nothing more than what one expects. So for instance to know that $\Gamma \vdash \Delta$ *yields* $\Gamma' \vdash \Delta'$ is the same thing as to know that from $\Gamma \vdash \Delta$ we can get to know that $\Gamma' \vdash \Delta'$. By the same reason, $A \vdash \Delta$ means that from A *is* we can get to know that Δ. Written in symbols, this looks like a cut, or composition:

$$\frac{\vdash A \quad A \vdash \Delta}{\vdash \Delta}$$

We actually assume that it holds in a generalized form, both in the sense that A *is* can be a consequence of some Γ in the first premiss and that in the second premiss it can appear in conjunction with other assertions, say in $\Gamma' \vdash \Delta$. Note that this is the only assumption about *and* when occurring inside *yields*.

Symmetrically, we also assume that when A *is* appears as a conjunct at the right of turnstile it can be replaced by any of its consequences. And finally we assume of course that A *is* is a consequence of A *is* itself, that is $A \vdash A$.

So, summing up, the rules involving turnstile are simply:

identity $\qquad A \vdash A$

composition

on the left $\dfrac{\Gamma \vdash A \quad \Gamma' \vdash \Delta}{\Gamma'(\Gamma/A) \vdash \Delta}$ $\qquad$ *on the right* $\dfrac{\Gamma \vdash \Delta' \quad A \vdash \Delta}{\Gamma \vdash \Delta'(\Delta/A)}$

We here write $\Gamma'(\Gamma/A)$ for the replacement of (one occurrence of) A by Γ in Γ'.

This setting is not yet sufficient for our purpose. Actually, it is not sufficient also for any sequent calculus, because a third form of assertion besides *and* and *yields* is necessary in order to understand the meaning of an inference rule, say $\Gamma \vdash A / \Gamma \vdash A \vee B$. Even if it does not appear explicitly, one must realize that Γ, A, B are parameters, that is, that the symbols Γ and A, B can be replaced with any specific conjunction of atomic assertions and with any atomic assertions, respectively. So to understand the above rule means to know that *forall* Γ *forall* A *forall* B((Γ *yields* A *is*) *yields* (Γ *yields* $A \vee B$ *is*)) where of course one must be careful to keep the meta-level particle *forall* well distinct from an object-level universal quantification. Any definitional equation must include some parameters; those considered here, moreover, have *forall* only as the outermost links[7].

We are finally ready to put all the previous remarks at work and show that each of the six connectives of basic logic is obtained via the reflection principle from one of the only two links *and* and *yields*, but in different environments. The arguments to solve the definitional equations are all extremely elementary, and actually many of them have already variously appeared in the literature (an early reference is [27], a recent one is [21]). Here they are explained as elements of one conceptual structure, connecting language with metalanguage. It allows to conjecture that no other definitional equations are solvable.

We adopt the general principle of denoting a connective always by the same sign, in all extensions of basic logic. The motivation of such choice overlaps with the motivation of basic logic itself, which is the search for a conceptual unity of different logics. Moreover, we choose signs in such a way that the resulting connective in the case of the extension corresponding to linear logic will be exactly the sign adopted by Girard [23]. The only exception is implication, for which we prefer the standard sign $\rightarrow$ rather than $\multimap$, which is kept for implication as defined in terms of other connectives.

The first connective we consider is the same given as unknown in the example at the beginning of this section, that is multiplicative conjunction $\otimes$, read "times". We analyse

[7] It can be shown that also the usual quantifiers $\forall$ and $\exists$ can be defined according to the principle of reflection, but then $\forall$ must appear also inside the definitional equation, at the right of $\vdash$ for $\forall$ and at the left for $\exists$, see [31].

it in detail, both to introduce terminology and to check that we use only the properties of metalanguage as specified above. The definitional equation is properly written as

$$forall\ \Delta, \qquad A \otimes B \vdash \Delta \ \ if\ and\ only\ if \ \ A, B \vdash \Delta$$

where the symbols are now read as shorthands as explained above (and of course the link *if and only if* is a shorthand for *yields* in both directions). Its two directions are called $\otimes$-formation and implicit $\otimes$-reflection, respectively:

$\otimes$-formation $$\frac{A, B \vdash \Delta}{A \otimes B \vdash \Delta}$$

implicit $\otimes$-reflection $$\frac{A \otimes B \vdash \Delta}{A, B \vdash \Delta}$$

The choice of names should be clear. Formation says when the compound proposition $A \otimes B$ can be formed and asserted; here it specifies that the compound assertion pushed down from the meta-level to object-level has a link *and* at the left position. Reflection says that the assertion of $A \otimes B$ can be reflected back to the meta-level situation from which it was born. While $\otimes$-formation as it stands is a perfectly good formal rule, implicit $\otimes$-reflection is still the statement of a desideratum, which contributes to specifying the meaning of $\otimes$ only in an implicit way. To characterize the meaning of $\otimes$ without vicious circles, we must find rules, called rules of *explicit $\otimes$-reflection*, which are equivalent to implicit $\otimes$-reflection, but which do not assume $A \otimes B$ to be already known, that is in the premises. By so doing, we solve the definitional equation.

First, by making its premiss trivial, i.e. taking Δ to be $A \otimes B\ is$, implicit $\otimes$-reflection is transformed into an equivalent axiom:

axiom of $\otimes$-reflection $$A, B \vdash A \otimes B$$

To recover implicit $\otimes$-reflection, one application of composition is sufficient:

$$\frac{A, B \vdash A \otimes B \quad A \otimes B \vdash \Delta}{A, B \vdash \Delta}$$

In this way the proposition $A \otimes B$ now appears on the opposite side of that in which it was born. The solution is then reached by transforming the $\otimes$-axiom again into an equivalent rule, but now one which acts on A and B separately; this is obtained by replacing A and B with arbitrary lists. Indeed, assuming that A and B are produced from Γ and Γ' respectively, i.e. $\Gamma \vdash A$ and $\Gamma' \vdash B$, we apply composition on A and B separately, that is

$$\frac{\Gamma' \vdash B \qquad \dfrac{\Gamma \vdash A \quad A, B \vdash A \otimes B}{\Gamma, B \vdash A \otimes B}}{\Gamma, \Gamma' \vdash A \otimes B}$$

and so from the $\otimes$-axiom we obtain

explicit $\otimes$ -reflection $$\frac{\Gamma \vdash A \quad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \otimes B}$$

The converse is immediate, since $\otimes$-axiom is just the trivial instance of implicit $\otimes$-reflection with Γ, Γ' equal to A, B respectively.

The definitional equation for $\otimes$ is thus completely solved, and the connective $\otimes$ is characterized by the rules of $\otimes$-formation and explicit $\otimes$-reflection. We say that $\otimes$ reflects *and* at the left of turnstile.

Incidentally, it might be interesting to note also that the rule

combined $\otimes$-reflection $$\frac{\Gamma \vdash A \quad \Gamma' \vdash B \quad A \otimes B \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta}$$

combines both the previous two rules and the axiom of $\otimes$-reflection. In fact, it is immediately derivable by applying explicit $\otimes$-reflection to the first two premisses, and then composition with the third premiss. Conversely, implicit $\otimes$-reflection is obtained as a special case by trivializing two premisses with $\Gamma \equiv A$ and $\Gamma' \equiv B$, explicit $\otimes$-reflection by similarly trivializing the premiss $A \otimes B \vdash \Delta$, and finally $\otimes$-axiom by trivializing all premisses.

The rules for $\otimes$ in linear logic are

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'}$$

that are equal to the above rule for $\otimes$ except for the presence of contexts. By modifying the arguments above and by using unrestricted composition, or full cut (as formulated on page 14), it is easy to check that Girard's $\otimes$ is the connective obeying the definitional equation

$$forall\ \Gamma\ forall\ \Delta, \qquad \Gamma, A \otimes B \vdash \Delta\ if\ and\ only\ if\ \Gamma, A, B \vdash \Delta$$

This shows why the corresponding connective in basic logic is conceptually simpler: it is defined by a simpler equation, with only one free parameter Δ. In more detail, the difference between the rules for $\otimes$ in basic logic and in linear logic is that in the former the active formulae A, B and $A \otimes B$ always appear on one side of $\vdash$ with no extra context; we say that they are *visible*.

Now the symmetry of the framework (that is the fact that we have not assumed anything on the left side of $\vdash$ which makes it different from the right side) allows us to say that the equation symmetric to that defining $\otimes$ will also define a connective. Such a connective is the so-called multiplicative disjunction $\parr$, read "par". The definitional equation is:

$$forall\ \Gamma, \qquad \Gamma \vdash A \parr B\ \ if\ and\ only\ if\ \ \Gamma \vdash A, B$$

Its solution is exactly as that for $\otimes$, but "on the other side", and it leads to the rules

$\parr$-*formation*
$$\frac{\Gamma \vdash B, A}{\Gamma \vdash B \parr A}\,\parr R$$

explicit $\parr$-*reflection*
$$\frac{B \vdash \Delta' \quad A \vdash \Delta}{B \parr A \vdash \Delta', \Delta}\,\parr L$$

Summing up, $\otimes$ reflects *and* at the left of turnstile and $\parr$ reflects *and* at the right[8]; in both cases, the principal sign is $\vdash$, and *and* is under its scope. What happens if *and* is outside $\vdash$? That is, can we push down *and* into a connective when it occurs in a situation like $\Gamma \vdash A$ *and* $\Delta \vdash B$? It is easy to see that any definitional equation with such a situation at the right side would make little sense, that is, it would not be solvable. It is indeed solvable when $\Gamma = \Delta$, and what we obtain is the additive conjunction &. The definitional equation is:

$$forall\ \Gamma, \qquad \Gamma \vdash B \& A\ \ if\ and\ only\ if\ \ \Gamma \vdash B\ and\ \Gamma \vdash A$$

Its solution follows the same pattern as that for $\otimes$ and $\parr$, except that we now take into consideration also the obvious properties of *and* since it is the outermost link. As before, one direction is ($\Gamma \vdash B$ *and* $\Gamma \vdash A$) *yields* $\Gamma \vdash B \& A$ which, when written as usual with *and* replaced by empty space and with *yields* replaced by the horizontal bar, gives:

&-*formation*
$$\frac{\Gamma \vdash B \quad \Gamma \vdash A}{\Gamma \vdash B \& A}\,\& R$$

The other direction is $\Gamma \vdash B \& A$ *yields* $\Gamma \vdash B$ *and* $\Gamma \vdash A$ which, written as usual as the conjunction of two rules, gives:

implicit &-reflection
$$\frac{\Gamma \vdash B \& A}{\Gamma \vdash B} \qquad \frac{\Gamma \vdash B \& A}{\Gamma \vdash A}$$

[8] This shows that the meaning of $\parr$ is that of a multiplicative disjunction only if full contexts are available, as in linear logic. In basic logic, nothing but compositions is assumed on *and* inside $\vdash$, and that is why its reflection produces multiplicative connectives.

We can trivialize it by taking $\Gamma = B\&A$ and obtain

axiom of &-reflection $\qquad B\&A \vdash B \qquad B\&A \vdash A$

from which implicit &-reflection follows immediately by composition. Now, from the &-axiom by using only pure composition we immediately have

explicit &-reflection $\qquad \dfrac{B \vdash \Delta}{B\&A \vdash \Delta} \qquad \dfrac{A \vdash \Delta}{B\&A \vdash \Delta}\,\&L$

which in turn gives the axiom back when the premiss is trivialized. We have thereby reached the usual rules for conjunction, additive conjunction "with" in Girard's terminology. Here too, as for all other connectives, a combined form of &-reflection is immediately found by joining the premisses of implicit and of explicit &-reflection.

The same idea leading from $\otimes$ to $\parr$ now leads from & to additive disjunction $\oplus$. Its definitional equation is:

$$forall\ \ \Delta, \qquad A \oplus B \vdash \Delta\ if\ and\ only\ if\ A \vdash \Delta\ and\ B \vdash \Delta$$

The solution is exactly the same, symmetry apart, as that for &, and it leads to the rules

$\oplus$-*formation* $\qquad \dfrac{A \vdash \Delta \quad B \vdash \Delta}{A \oplus B \vdash \Delta}\,\oplus L$

explicit $\oplus$*-reflection* $\qquad \dfrac{\Gamma \vdash A}{\Gamma \vdash A \oplus B} \qquad \dfrac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\,\oplus R$

So here too no disjunctive link on assertions is necessary to characterize the connective $\oplus$ of additive disjunction, and $\oplus$ is just the symmetric of &, obtained from & by interchanging the roles of assumptions and conclusions. Such symmetry was noticed by Gentzen, but only at the formal level of inference rules; what we add here is the symmetry at the level of semantics, that is symmetry of definitional equations. This is why it is just natural to conceive of $\oplus$ as formed (or introduced) at the left, on assumptions, and reflected (or eliminated) at the right[9].

So, summing up, the four connectives so far introduced all reflect a link *and*; two of them, i.e. $\otimes$ and $\oplus$, reflect *and* at the left, and two of them, i.e. $\parr$ and &, at the right. Moreover, the distinction between multiplicatives $\otimes$, $\parr$ and additives $\oplus$, & acquires a new motivation in terms of the reflection principle: additives reflect a link *and* which is principal, i.e. not under the scope of *yields*, or "outside" a sequent, while multiplicatives reflect a link *and* which is under the scope of *yields*, or "inside" a sequent.

The principle of reflection leads to the same four propositional constants as in linear logic. The first two constants, namely 1 and $\bot$, reflect the empty assertion at the left and right of $\vdash$, respectively. So the definitional equation for 1 is

$$forall\ \Delta, \qquad 1 \vdash \Delta\ \ if\ and\ only\ if\ \ \vdash \Delta$$

The formation rule is thus

1-*formation* $\qquad \dfrac{\vdash \Delta}{1 \vdash \Delta}\,1L$

while the reflection rule is

1-reflection $\qquad \dfrac{1 \vdash \Delta}{\vdash \Delta}$

from which we derive, for $\Delta = 1$, the axiom

1-*axiom* $\qquad \vdash 1\ \ 1R$

[9] This explains also why we have preferred to adopt the new terminology formation-reflection, since forcing the old one introduction-elimination to the new conceptual understanding would have been quite confusing.

which immediately gives 1-reflection, by composition. Since 1 has no components, there can be no composition producing 1, and thus explicit 1-reflection is the same as 1-axiom. Symmetrically, the constant $\bot$ reflects the empty assertion at the right

$$forall\ \Gamma, \qquad \Gamma \vdash \bot \ \ if\ and\ only\ if\ \ \Gamma \vdash$$

and its rules will be

$$\bot\text{-}formation \qquad \frac{\Gamma \vdash}{\Gamma \vdash \bot}\ \bot R$$

$$\bot\text{-}axiom \qquad \bot \vdash \quad \bot L$$

The constants 1 and $\bot$ are similar but not identical to those of linear logic; for instance, they are not neuter elements for $\otimes$ and $\parr$ respectively, because $A \otimes 1 \vdash A$ and $A \vdash A \parr \bot$ have no cut-free derivation. To obtain this property, one must conceive 1 as the solution of a definitional equation with two parameters, that is: $forall\ \Gamma, forall\ \Delta, \qquad \Gamma, 1 \vdash \Delta$ *if and only if* $\Gamma \vdash \Delta$. It produces as a solution the rules for 1 of linear logic (that is, the rules as above, except that $1L$ has a context Γ at the left).

In basic logic, rather than neuter elements for $\otimes$, one can describe 1 as the least derivable proposition: 1 is derivable, i.e. $\vdash 1$ by 1-axiom, and if A is derivable, i.e. $\vdash A$, then $1 \vdash A$ by 1-formation. Similarly, we say that A is refutable if $A \vdash$ and then $\bot$ is the greatest refutable proposition.

The rules for the constants 0 and $\top$ arise from the definitional equations

$$forall\ \Delta, \qquad A \vdash \Delta\ and\ 0 \vdash \Delta\ \ if\ and\ only\ if\ \ A \vdash \Delta$$

$$forall\ \Gamma, \qquad \Gamma \vdash B\ and\ \Gamma \vdash \top\ \ if\ and\ only\ if\ \ \Gamma \vdash B$$

which say that $0 \vdash \Delta$ and $\Gamma \vdash \top$ are the trivial assertions with respect to a link *and* in outermost position. From such equations we have the following formation rules:

$$0\text{-}formation \qquad 0 \vdash \Delta\ \ 0L$$

$$\top\text{-}formation \qquad \Gamma \vdash \top\ \ \top R$$

From them, it is easy to prove that also in basic logic 0 and $\top$ are neuter elements for $\oplus$ and & respectively, that is $A \oplus 0 = A$ and $A \& \top = A$ for any A. The reflection rules, in the case of 0 and $\top$, do not exist, since they correspond to the "only if" directions of the above equivalences, which are trivial.

We now show how the reflection principle leads also to the definition of a primitive implication $\rightarrow$ (and its symmetric $\leftarrow$). The peculiarity of implication is that it reflects a link *yields*, that is the turnstile sign $\vdash$ itself, and this is what makes it different from other connectives[10]. So to see that implication follows the same conceptual pattern as all other connectives, a richer metalanguage is needed, in which the link *yields* to be reflected appears inside the scope of another link *yields*. In terms of the shorthand notation, also nested occurrences of $\vdash$ must be considered. Then the definitional equation for $\rightarrow$ is simply

$$forall\ \Gamma, \qquad \Gamma \vdash A \rightarrow B\ \ if\ and\ only\ if\ \ \Gamma \vdash (A \vdash B)$$

[10]Such peculiarity is witnessed by traditional terminology, in which two different words are used for *yields* and $\rightarrow$, for instance "deduction" and "implication", while there is no specific word for *and* (and here it is called just "conjunction", as &). This is one of the reasons why some resistance has to be forced upon ourselves to accept the idea that $\oplus$ reflects *and*. In fact, following common terminology it would be read as "disjunction reflects conjunction" which of course is rightly felt as nonsense. Our *and* and *yields* correspond to what in display logic is denoted with punctuation signs like , ; * etc. and called "structural connectives". However now it should also be clear that adopting such terminology, rather than "metalinguistic links" as we did, would jeopardize all the effort to clarify the role of metalanguage.

It can be solved following the same pattern as other connectives, but it needs two new forms of composition, that is composition of formulae inside two occurrences of $\vdash$. Then one reaches the rules

$$\frac{\Gamma \vdash (A \vdash B)}{\Gamma \vdash A{\rightarrow}B} \qquad \frac{\Gamma \vdash A \quad B \vdash \Delta}{A{\rightarrow}B \vdash (\Gamma \vdash \Delta)}$$

which however cannot be expressed in the traditional shape of sequent calculus, where nested occurrences of $\vdash$ are not considered. For more on this, we refer to [8].

Here we prefer to follow a more traditional course, and look for the rules on $\rightarrow$ which can be expressed in a usual calculus of sequents. The $\rightarrow$-formation rule we adopt in basic logic is thus obtained from the above by taking Γ to be empty:

$$\rightarrow\textit{-formation} \qquad \frac{A \vdash B}{\vdash A{\rightarrow}B}\rightarrow R$$

and similarly for reflection

$$\textit{explicit}\rightarrow\textit{-reflection} \qquad \frac{\vdash A \quad B \vdash \Delta}{A{\rightarrow}B \vdash \Delta}\rightarrow L$$

Although a rigorous derivation of such rules from the definitional equation remains accessible only with nested $\vdash$, we can give at least the flavour of it by applying the reflection principle for $\rightarrow$ in case of contexts liberalized on the left (that is, in the logic **BL** as defined in the next section). The definitional equation therefore becomes

$$\textit{forall } \Gamma, \qquad \Gamma \vdash A{\rightarrow}B \;\; \textit{if and only if} \;\; \Gamma, A \vdash B$$

The rule of reflection in this case is

$$\text{implicit } \rightarrow\text{-reflection} \qquad \frac{\Gamma \vdash A{\rightarrow}B}{\Gamma, A \vdash B}$$

For $\Gamma = A{\rightarrow}B$, it gives

$$\text{axiom of } \rightarrow\text{-reflection} \qquad A{\rightarrow}B, A \vdash B$$

and conversely by composition. By composing the $\rightarrow$-axiom with $\Gamma \vdash A$ and $B \vdash \Delta$, we obtain

$$\text{explicit } \rightarrow\text{-reflection} \qquad \frac{\Gamma' \vdash A \quad B \vdash \Delta}{A{\rightarrow}B, \Gamma' \vdash \Delta}$$

and conversely by trivializing both premisses.

The combined rule

$$\text{combined } \rightarrow\text{-reflection} \qquad \frac{\Gamma \vdash A{\rightarrow}B \quad \Gamma' \vdash A \quad B \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta}$$

is immediately derivable from explicit $\rightarrow$-reflection by composition, and conversely it gives back the above rules and axiom as special cases, by trivializing some of the premisses as was true for other connectives. Note that all such equivalences (given also in [6]) are proved assuming only composition on the left, that is over basic logic, even if none of the above rules hold in it, by an argument based on cut-elimination. In other terms, the usual deduction theorem $\Gamma, A \vdash B$ *if and only if* $\Gamma \vdash A \rightarrow B$ does not hold in **B**.

In the case of implication, combined reflection has a further source of interest, since by specializing it to the case in which $\Delta = B$, gives

$$\frac{\Gamma \vdash A{\rightarrow}B \quad \Gamma' \vdash A}{\Gamma, \Gamma' \vdash B}$$

which is the traditional Modus Ponens, or $\rightarrow$-elimination in natural deduction. When $\Gamma' = A$, Modus Ponens gives $\rightarrow$-reflection as a special case, and hence it is also equivalent to all other rules.

When a system with nested $\vdash$ is available, all the above conditions remain equivalent when comma at the left is replaced by $\vdash$ (see [8]). Thus the two rules we have chosen for $\rightarrow$ in basic logic are justified; they are not enough, however, since for instance the derivation

$$\dfrac{\dfrac{A \vdash B \quad C \vdash D}{B \rightarrow C \vdash (A \vdash D)}}{B \rightarrow C \vdash A \rightarrow D}$$

produces a sequent $B \rightarrow C \vdash A \rightarrow D$ (with just one $\vdash$) which is valid if the sequents $A \vdash B$ and $C \vdash D$ are valid. Thus we must also add the rule

$$\rightarrow\textit{-unified} \qquad \dfrac{A \vdash B \quad C \vdash D}{B \rightarrow C \vdash A \rightarrow D} \rightarrow U$$

which has also the side-effect of allowing replacement of equivalent formulae.

The correctness of the choice of rules for $\rightarrow$ is also witnessed by the cut-elimination theorem, to be proved in section 4; from it, we can immediately see that

$$\text{for any } A \text{ and } B, \ \vdash A \rightarrow B \text{ if and only if } A \vdash B$$

holds, which is all that can be expressed of the original definitional equation for $\rightarrow$ when no nested $\vdash$ are available.

To keep the symmetry of the resulting calculus, we add a second connective moving formulae from one side to the other of $\vdash$, and denote it by $\leftarrow$ (to be read "exclusion"). Its definitional equation is just the symmetric of that for $\rightarrow$, that is:

$$\textit{forall } \Delta, \qquad A \leftarrow B \vdash \Delta \ \textit{ if and only if } \ (A \vdash B) \vdash \Delta$$

and, as for all previous pairs of symmetric connectives, it leads to rules which are symmetric to those for $\rightarrow$, that is

$$\leftarrow\textit{-formation} \qquad \dfrac{B \vdash A}{B \leftarrow A \vdash} \leftarrow L$$

$$\textit{explicit} \leftarrow\textit{-reflection} \qquad \dfrac{\Gamma \vdash B \quad A \vdash}{\Gamma \vdash B \leftarrow A} \leftarrow R$$

$$\leftarrow\textit{-unified} \qquad \dfrac{D \vdash C \quad B \vdash A}{D \leftarrow A \vdash C \leftarrow B} \leftarrow U$$

The meaning of exclusion $\leftarrow$ is the dual of that of implication $\rightarrow$, just like the rules for $\leftarrow$ are symmetric to those for $\rightarrow$. The intuitive interpretation of $\rightarrow R$ is that to make $A \rightarrow B$ true, i.e. to prove $\vdash A \rightarrow B$, we need to know how to pass from the truth of A to the truth of B, i.e. $A \vdash B$. Now suppose that not only do we read $A \vdash B$ as A *true yields* B *true*, but also as B *false yields* A *false* (cf. [4]). Then $\rightarrow L$ says that to make $A \rightarrow B$ *false*, i.e. $A \rightarrow B \vdash$, we need to know that A is true and B is false, i.e. $\vdash A$ and $B \vdash$. Symmetrically, the intuitive interpretation of $\leftarrow L$ is that to make $B \leftarrow A$ false, i.e. to prove $B \leftarrow A \vdash$, we need to know how to pass from the falsity of A to the falsity of B, i.e. $B \vdash A$. And $\leftarrow R$ says that to make $B \leftarrow A$ true, i.e. $\vdash B \leftarrow A$, we need to know that B is true and A is false, i.e. $\vdash B$ and $A \vdash$. Apart from a justification of its meaning, the introduction of $\leftarrow$ is fully justified simply because it allows arguments "by symmetry", and this is an essential feature of basic logic.

We have now justified all the rules of basic logic except one, that is the single structural rule we are going to assume, namely exchange. Note that all arguments up to this point have not used it and hence non-commutative basic logic is quite possible[11]. Still, we add exchange somewhat artificially, to concentrate our mind on the novelties of basic logic, which should thus become a bit more transparent.

We can finally formally say that basic logic is the propositional logic with connectives $\otimes$, $\parr$, $\&$, $\oplus$, $\rightarrow$, $\leftarrow$ and constants 1, $\bot$, 0, $\top$, and characterized by the sequent calculus **B** with

Table 1: Basic sequent calculus **B**

Axioms

$$A \vdash A$$

Structural rules

$$\frac{\Gamma, \Sigma, \Pi, \Gamma' \vdash \Delta}{\Gamma, \Pi, \Sigma, \Gamma' \vdash \Delta}\ exchL \qquad exchR\ \frac{\Gamma \vdash \Delta, \Pi, \Sigma, \Delta'}{\Gamma \vdash \Delta, \Sigma, \Pi, \Delta'}$$

Operational rules

Multiplicatives

formation	$\dfrac{B, A \vdash \Delta}{B \otimes A \vdash \Delta}\ \otimes L$	$\parr R\ \dfrac{\Gamma \vdash A, B}{\Gamma \vdash A \parr B}$
reflection	$\dfrac{B \vdash \Delta_1 \quad A \vdash \Delta_2}{B \parr A \vdash \Delta_1, \Delta_2}\ \parr L$	$\otimes R\ \dfrac{\Gamma_2 \vdash A \quad \Gamma_1 \vdash B}{\Gamma_2, \Gamma_1 \vdash A \otimes B}$
formation	$\dfrac{\vdash \Delta}{1 \vdash \Delta}\ 1L$	$\bot\ R\ \dfrac{\Gamma \vdash}{\Gamma \vdash \bot}$
reflection	$\bot\vdash\ \bot\ L$	$1R\ \ \vdash 1$

Additives

formation	$\dfrac{B \vdash \Delta \quad A \vdash \Delta}{B \oplus A \vdash \Delta}\ \oplus L$	$\&R\ \dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}$
reflection	$\dfrac{B \vdash \Delta}{B \& A \vdash \Delta} \quad \dfrac{A \vdash \Delta}{B \& A \vdash \Delta}\ \&L$	$\oplus R\ \dfrac{\Gamma \vdash A}{\Gamma \vdash A \oplus B} \quad \dfrac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}$
formation	$0 \vdash \Delta\ \ 0L$	$\top R\ \ \Gamma \vdash \top$

Implications

formation	$\dfrac{B \vdash A}{B \leftarrow A \vdash}\ \leftarrow L$	$\rightarrow R\ \dfrac{A \vdash B}{\vdash A \rightarrow B}$
reflection	$\dfrac{\vdash B \quad A \vdash \Delta}{B \rightarrow A \vdash \Delta}\ \rightarrow L$	$\leftarrow R\ \dfrac{\Gamma \vdash A \quad B \vdash}{\Gamma \vdash A \leftarrow B}$
order	$\dfrac{D \vdash C \quad B \vdash A}{D \leftarrow A \vdash C \leftarrow B}\ \leftarrow U$	$\rightarrow U\ \dfrac{A \vdash B \quad C \vdash D}{B \rightarrow C \vdash A \rightarrow D}$

inference rules given in the table on page 13, where as usual $A, B, C, D, \ldots$ are formulae and $\Gamma, \Gamma', \ldots, \Delta, \Delta', \ldots$ are finite lists of formulae.

To complete the justification of basic logic, we have to show the validity of the rules we have used, namely identity and composition. Identity is imported into the formal system as it stands. Composition is just what is usually called cut. To show that it is valid in the calculus **B**, we have to show that its conclusion is derivable in **B** when so are its premises. Exactly this is expressed by the theorem of elimination of cuts, which we prove in section 4.

Why not importing into the formal system also cut as uch? If we had added cut as a formal rule, we would not have accomplished our task, which was to characterize each connective by its definitional equation, that is exclusively by the rules directly concerning it. In fact, as we alrady noticed, composition contains implicit information on all connectives, since it says that all derivations can be composed. Using composition up to now, has been a sort of *desideratum*. Almost all the arguments used to solve definitional equations involve composition, and they would have little value if composition were not valid. In this sense, cut elimination is not an option.

It is a remarkable fact that a common formulation of cut, i.e the rule which we call full cut

$$\frac{\Gamma \vdash \Delta \quad \Gamma' \vdash \Delta'}{\Gamma'(\Gamma/A) \vdash \Delta'(\Delta/A)}\ full\ cut$$

is not valid in **B**, and in fact we have not used it. The proof rests on the following property, exploiting visibility:

Proposition 2.1 *If $\Gamma \vdash \Delta$ is provable in* **B** *without cuts, then either Γ or Δ contains at most one formula.*

Proof. Immediate induction on cut-free derivations: axioms enjoy the property, and by visibility all rules preserve it. □

Assuming full cut, a derivation like the following would be possible:

$$\frac{\dfrac{B \vdash B \quad E \vdash E}{B \parr E \vdash B, E}\ \parr L \quad \dfrac{B \vdash B \quad D \vdash D}{B, D \vdash B \otimes D}\ \otimes R}{B \parr E, D \vdash E, B \otimes D}\ full\ cut$$

The sequent $B \parr E, D \vdash E, B \otimes D$ however has no cut-free derivation in **B**, by the above proposition, since it has two formulae on both sides. Since **B** does admit elimination of $cut L$ and $cut R$, this tells that full cut would really bring out of provable sequents, and thus it is by no means valid.

It is quite easy to check that all the connectives behave as expected with respect to the usual ordering on formulae given by $\vdash$. In fact, assume that $A \vdash B$ and $C \vdash D$ are provable; by applying the (explicit) reflection rule followed by the formation rule, we obtain

$$\frac{A \vdash B \quad C \vdash D}{A \circ B \vdash C \circ D}$$

in case $\circ$ is any of the connectives $\otimes$, $\parr$, $\oplus$, $\&$, which shows that all of them are monotonic in both arguments, both at the left and at the right.

The same pattern would hold also for the two connectives $\leftarrow$ and $\rightarrow$ in the system with nested $\vdash$; the same task, i.e. reflection followed by formation, is played in **B** by the rules $\leftarrow U$ and $\rightarrow U$, which say that both implications are monotonic in one argument and antimonotonic in the other. Actually, it is easy to see that the rule $\rightarrow U$ is equivalent to the two separate rules

$$\frac{A \vdash B}{C \rightarrow A \vdash C \rightarrow B} \qquad \frac{A \vdash B}{B \rightarrow C \vdash A \rightarrow C}$$

and similarly for $\leftarrow$.

Denoting by $A = B$ the equivalence with respect to provability, i.e. $A \vdash B$ and $B \vdash A$, as an immediate consequence we have:

[11] The main difference, as for non-commutative linear logic, would be the presence, at least in extensions of basic logic, of two implications $\rightarrow_1$ and $\rightarrow_2$ in place of $\rightarrow$, and then by symmetry also two exclusions $\leftarrow_1$ and $\leftarrow_2$ in place of $\leftarrow$.

Proposition 2.2 *a. For any formulae A, A', B, B' and any connective $\circ$, if $A = B$ and $A' = B'$, then $A \circ B = A' \circ B'$, and hence the property of replacement of equivalents holds; b. for any formula A, the sequent $A \vdash A$ is provable assuming only axioms of the form $p \vdash p$ for some propositional variable p.*

We conclude the section with some remarks on negation. A unary connective of negation is definable in basic logic by putting, as usual in intuitionistic logic,

$$\neg A \equiv A \rightarrow \bot$$

Actually, the approach via the reflection principle allows us to see why this is a convenient way to treat negation. The intuitive idea is that the assertion of the negation of A must be equivalent to the assertion of A on the other side of $\vdash$; that is, we would like $\neg$ to be introduced by the definitional equation

$$\vdash \neg A \ \ if and only if \ \ A \vdash$$

Such an equation, however, can not be solved as it stands, as happened with $\rightarrow$; actually, we can see it as a special case (for Γ empty) of the equation

$$forall\ \Gamma, \qquad \Gamma \vdash \neg A \ \ if and only if \ \ \Gamma \vdash (A \vdash)$$

which in turn can be seen as a special case of the definitional equation for $\rightarrow$, since $A \vdash$ if and only if $A \vdash \bot$. But then one can see that the adoption of the usual definition $\neg A \equiv A \rightarrow \bot$ is a more convenient and equivalent choice. In fact, since $A \vdash$ is equivalent to $A \vdash \bot$ by reflection for $\bot$, and $A \vdash \bot$ is equivalent to $\vdash A \rightarrow \bot$ by reflection of $\rightarrow$, the usual definition gives a solution of the first equation, as desired.

We can then show that the rules

$$\neg\text{-formation} \quad \frac{A \vdash}{\vdash \neg A} \qquad\qquad \neg\text{-reflection} \quad \frac{\vdash A}{\neg A \vdash}$$

are derivable by applying $\bot$ and $\rightarrow$-formation, and $\bot$ and $\rightarrow$-reflection, respectively. This shows how the usual intuitionistic definition of negation falls under a deep general scheme.

The symmetry of basic logic tells that also a symmetric negation $\sim$ is definable, as the solution of

$$\sim A \vdash \ \text{ if and only if } \ \vdash A$$

or equivalently as the symmetric of $A \rightarrow \bot$; we thus put $\sim A \equiv 1 \leftarrow A$ and obtain the rules

$$\sim\text{-formation} \quad \frac{\vdash A}{\sim A \vdash} \qquad\qquad \sim\text{-reflection} \quad \frac{A \vdash}{\vdash \sim A}$$

in a symmetric way to those for $\neg$. Finally note that antimonotonicity of both negations

$$\frac{A \vdash B}{\sim B \vdash \sim A} \qquad\qquad \frac{A \vdash B}{\neg B \vdash \neg A}$$

is immediately derivable by $\rightarrow U$ and $\leftarrow U$ respectively.

3 Symmetry and the cube of extensions

The main peculiarity of the rules of the calculus **B**, apart from the symmetry of the whole table, is that each active formula in any rule is visible, that is, the context at its side is empty. An easy way to obtain extensions of **B** is to relax such condition, that is to also allow contexts at the side of active formulae. So for any rule of **B** we introduce also its full form, that is the version with liberalized contexts on both sides. To include also intuitionistic-like logics, we consider also the form of a rule which has full context on the left only; and then, for the sake of symmetry, we consider also the form which is full on the right. We use fl, fr and f as exponent on the name of a rule to denote its full at the left, full at the right and full form, respectively.

Sometimes one of the two liberalizations has no effect. For instance we have:

$$\dfrac{\Gamma \vdash B, A}{\Gamma \vdash B \parr A}\ \parr R$$

$$\swarrow \qquad\qquad \searrow$$

$$\parr R^{fl} = \parr R \qquad\qquad \dfrac{\Gamma \vdash B, A, \Delta}{\Gamma \vdash B \parr A, \Delta}\ \parr R^{fr}$$

$$\searrow \qquad\qquad \swarrow$$

$$\parr R^{f} = \parr R^{fr}$$

In general, the full-on-the-left form of a rule on the right for additives or multiplicatives is just the same as the rule itself, since the context on the left is already free. So the full form is just one, and is denoted by the exponent f. By symmetry, this is also the case for the rules at the left. For convenience, we spell out the full form of rules for multiplicative and additive connectives:

$$\dfrac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}\ \otimes L^f \qquad \dfrac{\Gamma \vdash B, A, \Delta}{\Gamma \vdash B \parr A, \Delta}\ \parr R^f$$

$$\dfrac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \parr B \vdash \Delta, \Delta'}\ \parr L^f \qquad \dfrac{\Gamma' \vdash B, \Delta' \quad \Gamma \vdash A, \Delta}{\Gamma, \Gamma' \vdash B \otimes A, \Delta, \Delta'}\ \otimes R^f$$

$$\dfrac{\Gamma \vdash \Delta}{\Gamma, 1 \vdash \Delta}\ 1L^f \qquad \dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \bot, \Delta}\ \bot R^f$$

$$\dfrac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta}\ \oplus L^f \qquad \dfrac{\Gamma \vdash B, \Delta \quad \Gamma \vdash A, \Delta}{\Gamma \vdash B \& A, \Delta}\ \& R^f$$

$$\dfrac{\Gamma, A \vdash \Delta}{\Gamma, A \& B \vdash \Delta}\ \dfrac{\Gamma, B \vdash \Delta}{\Gamma, A \& B \vdash \Delta}\ \& L^f \qquad \dfrac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B, \Delta}\ \dfrac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta}\ \oplus R^f$$

$$\Gamma, 0 \vdash \Delta\ \ 0L^f \qquad \Gamma \vdash \top, \Delta\ \ \top R^f$$

Since the rules of movement, for $\rightarrow$ and $\leftarrow$, are bound to have empty context on both sides, they will have different full forms on the left and on the right. For instance, for $\rightarrow L$ we have:

$$\dfrac{\vdash A \quad B \vdash \Delta}{A \rightarrow B \vdash \Delta}\ \rightarrow L$$

$$\swarrow \qquad\qquad \searrow$$

$$\dfrac{\Gamma \vdash A \quad \Gamma', B \vdash \Delta}{\Gamma, \Gamma', A \rightarrow B \vdash \Delta}\ \rightarrow L^{fl} \qquad\qquad \dfrac{\vdash A, \Delta' \quad B \vdash \Delta}{A \rightarrow B \vdash \Delta', \Delta}\ \rightarrow L^{fr}$$

$$\searrow \qquad\qquad \swarrow$$

$$\dfrac{\Gamma \vdash A, \Delta' \quad \Gamma', B \vdash \Delta}{\Gamma, \Gamma', A \rightarrow B \vdash \Delta', \Delta}\ \rightarrow L^{f}$$

Quite similarly it happens for $\rightarrow R$ and $\rightarrow U$, and symmetrically for $\leftarrow$. Just to make sure that no misunderstanding is possible, we spell out also the other six rules obtained by liberalising contexts on $\rightarrow R$ and $\rightarrow U$ (and leave at least the remaining nine, for $\leftarrow$, to symmetry):

$$\dfrac{\Gamma, B \vdash A}{\Gamma \vdash B \rightarrow A}\ \rightarrow R^{fl} \qquad \dfrac{B \vdash A, \Delta}{\vdash B \rightarrow A, \Delta}\ \rightarrow R^{fr} \qquad \dfrac{\Gamma, B \vdash A, \Delta}{\Gamma \vdash B \rightarrow A, \Delta}\ \rightarrow R^{f}$$

$$\dfrac{\Gamma, A \vdash B \quad \Gamma', C \vdash D}{\Gamma, \Gamma', B \rightarrow C \vdash A \rightarrow D}\ \rightarrow U^{fl} \quad \dfrac{A \vdash B, \Delta \quad C \vdash D, \Delta'}{B \rightarrow C \vdash A \rightarrow D, \Delta, \Delta'}\ \rightarrow U^{fr} \quad \dfrac{\Gamma, A \vdash B, \Delta \quad \Gamma', C \vdash D, \Delta'}{\Gamma, \Gamma', B \rightarrow C \vdash A \rightarrow D, \Delta, \Delta'}\ \rightarrow U^{f}$$

It is now easy to define extensions for **B**. For any sequent calculus **X**, we call **XL** the version of **X** liberalized at the left, i.e. the sequent calculus obtained by taking the full at the left form of the operational rules of **X**. More explicitely, for instance, **BL** has the same axioms and structural rules of **B**, and the rules:

$$
\begin{array}{llllll}
L^f & \parr L^f & 1L^f & \oplus L^f & \& L^f & 0L^f \\
\parr R & \otimes R & \bot\ R & \& R & \oplus R & \top R \\
\rightarrow L^{fl} & \leftarrow L^{fl} & \rightarrow U^{fl} & \leftarrow U^{fl} & & \\
\leftarrow R & \rightarrow R & & & &
\end{array}
$$

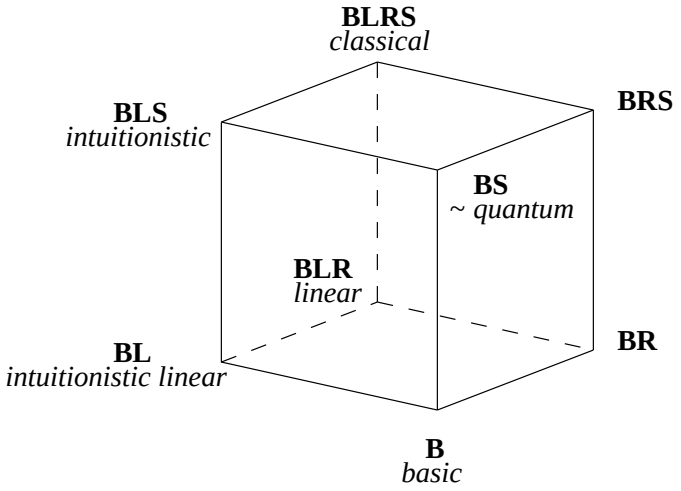
Note that the liberalized rules are 13, compared with 11 which remain equal to those of **B**, since the rules $\rightarrow U$ and $\leftarrow U$ are self-symmetric; also note that $1R$ and $\bot\ L$ remain untouched, since they are reflection axioms, and thus have no context.

For any calculus **X**, we define **XR** symmetrically, liberalising on the right. Since liberalising on the left rules which are already full on the right just gives full rules, the calculus **BLR** (which is the same as **BRL**) is obtained, so to say, by exponentiating to f all operational rules. Note that by liberalising $cut R$ on the left one obtains cut in its full form, or full cut, while liberalising $cut L$ on the left gives no effect. Symmetrically for $cut L$.

The logics **B**, **BL**, **BR**, **BLR** are all linear-like, in the sense that they all lack the traditional structural rules of weakening and contraction:

$$
\begin{array}{lll}
\textit{weakening} & \dfrac{\Gamma, \Gamma' \vdash \Delta}{\Gamma, \Sigma, \Gamma' \vdash \Delta}\ wL & \dfrac{\Gamma \vdash \Delta', \Delta}{\Gamma \vdash \Delta', \Sigma, \Delta}\ wR \\[2ex]
\textit{contraction} & \dfrac{\Gamma, \Sigma, \Sigma, \Gamma' \vdash \Delta}{\Gamma, \Sigma, \Gamma' \vdash \Delta}\ cL & \dfrac{\Gamma \vdash \Delta', \Sigma, \Sigma, \Delta}{\Gamma \vdash \Delta', \Sigma, \Delta}\ cR
\end{array}
$$

We write **XW** for the calculus obtained from the calculus **X** by adding both the weakening rules, and similarly **XC** for contraction. To keep the number of combinations under easier control, we also write **XS** for the calculus **XWC** (which is just the same as **XCW**). So, since the order in which the extensions by **L**, **R** and **S** are considered is irrelevant (and since repeating them has trivially no effect), we obtain a cube of sequent calculi extending **B**.



We anticipate that many of them are equivalent formulations of well known logics, as we have shown in the picture. We will return to it in the last section, when we can look at it also with the tools of symmetry and cut-elimination.

The symmetry of basic logic rests on the symmetry between the left side (assumptions, or inputs) and the right side (conclusions, or outputs) of a sequent. Whatever action is taken on the right can also be performed on the left, and vice versa. Accordingly, when applying the principle of reflection to justify the rules of basic logic, if a definitional equation was solved for a metalinguistic link appearing on the right, then its symmetric form was also solved with the same link appearing on the left. This is why all connectives and rules of basic logic come in symmetric pairs. This evident symmetry can now be turned into some formal statements, showing how aesthetics is turned into a useful tool to prove new results.

In basic logic, the symmetry between the left and the right side in a sequent does not mean that inputs and outputs are identified, as in linear logic. By keeping symmetry as a

meta-property, inputs and outputs here remain clearly distinct and not interdefinable. In this way it is possible to extend symmetry in a natural way from formulae to derivations and logics themselves (for example, see the two symmetric ways of strengthening basic logic which in the previous section we have denoted by **L** and **R**).

For any formula A, its symmetric formula A^s is defined by the clauses:

i) $p^s \equiv p$ for any propositional variable p

ii) $1^s \equiv \bot$, $\bot^s \equiv 1$; $0^s \equiv \top$, $\top^s \equiv 0$

iii) $(A \circ B)^s \equiv B^s \circ^s A^s$, where the symmetric $\circ^s$ of any connective $\circ$ is defined as in the table:

$$\otimes^s \equiv \parr \qquad \parr^s \equiv \otimes$$

$$\oplus^s \equiv \& \qquad \&^s \equiv \oplus$$

$$\leftarrow^s \equiv \rightarrow \qquad \rightarrow^s \equiv \leftarrow$$

We can extend the above definition to include negation by putting $\neg^s \equiv \sim$ and $\sim^s \equiv \neg$, which of course is justified by $(A \rightarrow \bot)^s \equiv 1 \leftarrow A^s$ and $(1 \leftarrow A)^s \equiv A^s \rightarrow \bot$.

By the definition (including negations, if wished), it is immediately evident that:

Proposition 3.1 *For any formula A, A^{ss} coincides with A.*

Note that the equality of A^{ss} with A is just identity of formulae, qua strings of symbols.

The symmetric of a list of formulae is defined by putting $\Gamma^s \equiv C_n^s, \ldots, C_1^s$ if $\Gamma = C_1, \ldots, C_n$, and similarly for Δ. Then we say that the sequent $\Delta^s \vdash \Gamma^s$ is symmetric to the sequent $\Gamma \vdash \Delta$ and proposition 3.1 is immediately extended to sequents.

Now also the symmetric of a rule J can be formally defined as the rule J^s which leads exactly from the symmetric of the premises of J to the symmetric of the conclusion of J. For example, the symmetric of $\rightarrow L$ is $\leftarrow R$, since by definition it is $(A \rightarrow B)^s \equiv B^s \leftarrow A^s$. Informally speaking, J^s does the same job as J, but on the other side of the sequent. Obviously, J^{ss} is exactly the same as J.

This definition can be applied to any rule in the language of basic logic. So we can extend symmetry to logics and say that a sequent calculus L has a symmetric calculus L^s, which by definition is formed by all sequents $\Delta^s \vdash \Gamma^s$ which are the symmetric of some axiom $\Gamma \vdash \Delta$ of L (note that in particular the symmetric of $A \vdash A$ is $A^s \vdash A^s$, which is again an instance of identity), and by all the rules J^s which are the symmetric of some rule J of L. As expected, L^{ss} is exactly the same as L. According to such definition, the calculus **BR** is obviously the symmetric of **BL**, and **BRS** the symmetric of **BLS**. On the other hand, **B**, **BS**, **BLR** and **BLRS** coincide with their symmetric, and thus are called (self-)symmetric. This shows how symmetry is present also in the cube of extensions of **B**.

Now the notion of symmetric for a sequent calculus also allows us to figure the notion of symmetric derivation, and put such a notion into formal terms. Every proof Π in a sequent calculus L has a symmetric proof Π^s in the symmetric sequent calculus L^s, obtained from Π by replacing every assumption of an axiom with an assumption of its symmetric axiom and every application of a rule J with an application of its symmetric rule J^s. The formal definition of Π^s is obtained by induction on the generation of Π. If Π is obtained in L from Π_1 (and Π_2) by applying rule J, then Π^s is defined to be obtained in L^s from ($\Pi_2{}^s$ and) $\Pi_1{}^s$ by applying rule J^s. Such an inductive definition is precisely sufficient to prove the following theorem:

Theorem 3.2 *For any two lists of formulae Γ, Δ, any sequent calculus L in the language of basic logic and any derivation Π in it,*

$$\Pi \textit{ is a proof of } \Gamma \vdash \Delta \textit{ in } L \textit{ if and only if } \Pi^s \textit{ is a proof of } \Delta^s \vdash \Gamma^s \textit{ in } L^s.$$

This theorem is the *raison d'etre* of symmetry. A visual image can help to make the intuitions concrete: we can think of Π^s as obtained from Π by rotating it by 180° through a vertical axis placed on the sign $\vdash$ of the conclusion, as a hand from palm to back, and by simultaneously swapping any application of J with one of J^s, and by turning any sequent

$\Gamma \vdash \Delta$ into $\Delta^s \vdash \Gamma^s$ (that is, everything is rotated, or dualized, except the sign $\vdash$). Based on such an image, the rotation applied twice is obviously the same as identity, and hence the interchange of Π and Π^s is called a swap of derivations. In the case of derivations which belong to symmetric systems, by definition, the swap produces derivations which still belong to the same system. The technique of substituting a derivation with its symmetric has been applied in [19], [17], [20] in order to obtain cut-elimination proofs.

As is evident in the image of the cube, the self-symmetry of logics **B** and **BS** is deeper than that of linear classical logic **BLR** and of classical logic **BLRS** respectively: it is obtained by looking also underneath (linear) intuitionistic logic, by considering the common structure of **BL** and **BR**, (or of **BLS** and **BLRS**, respectively) rather than their union, and thus recovering the symmetry which was lost in the step from classical to intuitionistic logic. This also allows us to treat in a symmetric way implication, i.e. the only connective which keeps communication between inputs and outputs (and which is essentially lost in linear logic without exponentials).

4 Visibility and elimination of cuts

Visibility is strictly linked with cut-elimination. As we recalled in the introduction, one of the first motivations leading to the development of basic logic was the wish to find a common denominator of the various refinements of classical logic, among which is orthologic, one of the main quantum logics. This logic is non-distributive, and such a property is obtained in a sequent calculus by imposing restrictions on the context of those rules which are needed to prove it, that is $\oplus$ on the left (and hence also & on the right by symmetry), negation and implication. Such restrictions, which had to be inherited by basic logic, make the proof of cut-elimination for orthologic extremely difficult[12]. In fact, consider the following derivation, where the left premiss of cut is obtained by a restricted introduction of $\oplus$ and in the right premiss the last rule applied is the usual & introduction at the left:

$$\dfrac{\dfrac{A \vdash C\&D \quad B \vdash C\&D}{A \oplus B \vdash C\&D}\oplus L \quad \dfrac{\Gamma, C \vdash \Delta}{\Gamma, C\&D \vdash \Delta}}{\Gamma, A \oplus B \vdash \Delta}\, cut L$$

Since the cut-formula is principal in the right premiss, Gentzen's procedure should lift the cut along the left branch, to reduce the left rank:

$$\dfrac{\dfrac{A \vdash C\&D \quad \Gamma, C\&D \vdash \Delta}{\Gamma, A \vdash \Delta} \qquad \dfrac{B \vdash C\&D \quad \Gamma, C\&D \vdash \Delta}{\Gamma, B \vdash \Delta}}{-}\ ?$$

Now, unless Γ is empty (the $\oplus L$ rule must have empty context on the left), it is not allowed to apply $\oplus L$ to obtain the conclusion.

The strategy leading to the cut-free calculus has been a change in perspective. Rather than thinking of a system of rules where some of the rules make life difficult, because of their restrictions, we reverse the point of view. We have a system in which some rules are more attractive: active formulae stand up as pure, well *visible*. And we make this into a general property of the calculus, namely visibility. The result is a uniform procedure to give the rules of basic logic, in which the rules $\oplus L$ and for $\rightarrow$ are no longer exceptions, but rather part of a global picture. What was previously the constraint of some rules, now becomes the strength of the system (cf.[16]). We show below how visibility is immediately transformed into elimination of cuts.

4.1 Cut elimination in basic logic B and structured basic logic BS

Visibility deeply determines the form and structure of derivations (as expressed by the two lemmas of substitution and history, see below) and the procedure of cut-elimination itself.

[12] As witnessed by the intricacies of the literature, as briefly exposed in [17].

Connected with visibility is also the presence of two forms of cut (which as we have already seen is not an ad hoc choice), a fact which will have a remarkable relevance.

So suppose that we are given an application of $cutL$ ($cutR$ is treated in a symmetric way) and that we want to lift it up until it operates on axioms or on principal formulae. As we shall shortly see, what happens is that we are forced to persistently lift the left premiss of the cut in the right branch until the cut-formula becomes principal in the right premiss, and only *then* lift the cut in the left premiss. Indeed, the last rule applied to obtain $\Sigma \vdash C$ in the left premiss must be a $\circ L$ rule, except when the cut formula is principal in the left premiss $\Sigma \vdash C$, because otherwise a $\circ R$ rule would introduce a formula occurrence on the right-hand side of the sequent, which would be the cut formula itself.

Now, if we wanted to lift the cut in the left branch keeping $\Gamma, C \vdash \Delta$ as the right premiss, the context Γ would remain on the left after the cut, which would block the application of the rule $\circ L$ needed in order to obtain the conclusion $\Sigma, \Gamma \vdash \Delta$.

On the other hand, the rules applied above the right premiss of the cut, but under the rule introducing the cut-formula, must all be $\circ R$ rules, which have free context on the left-hand side of the sequents. When we arrive at an application of a $\circ L$ rule, by visibility such rule must be the rule introducing the cut formula and we can be sure that no context appears at its side. Now the difficulty mentioned above has disappeared and lifting the cut in the left branch presents no problem.

After pointing out the delicate aspects of cut-elimination, the content and purpose of the lemmas we are going to show should be clear.

We assume knowledge of standard terminology, as in [32] or [24]. To deal properly with substitutions, however, the notion of linked occurrences is useful. Two occurrences of the same formula in two consecutive sequents are said to be *linked* when they are in the same place in lists of formulae which in the description of the rules are denoted by the same letter in premisses and conclusion (e.g. Γ_1 in the conclusion and Γ_1 in the premisses). We explicitly note that as a consequence: *i.* a formula occurrence which is introduced by a rule or by an axiom, is not linked to any formula occurrence above it; *ii.* in additive rules with two premisses the formula occurrences in Γ in the conclusion are linked to those in the same place in Γ of both premisses; *iii.* contraction identifies two occurrences of A into one, which is linked to both occurrences from which it comes.

All rules are insensitive to the content of the passive context: by this we mean that any application of a rule remains an application of the same rule and with the same active formulae if two linked occurrences of a formula C are replaced by any list of formulae Σ. Insensitivity of rules to substitutions of course extends immediately to proof-tree trunks. We write $\Gamma[\Sigma/C]$ for the result of replacing C with Σ in Γ, $\Pi[\Sigma/C]$ for the result of replacing Σ for all linked occurrences of C in the proof-tree trunk Π. Also, from now on, we say 'at the left' as an abbreviation of 'at the left-side of all the sequents in the proof-tree trunk under consideration'. Then we have:

Lemma 4.1 (Substitution of a formula occurrence) *Let Π be a proof-tree trunk in which the only rules applied are either rules with passive context on the left side, or structural rules*[13] *or cuts. Let C be any formula occurrence at the left. Then, for any list of formulae Σ, the result of the substitution Σ/C in Π is again a proof-tree trunk. If Π has conclusion $\Gamma \vdash \Delta$ (hence it must be $C \in \Gamma$), then $\Pi[\Sigma/C]$ has conclusion $\Gamma[\Sigma/C] \vdash \Delta$.*

A dual statement holds when C is at the right.

The *history* of a formula occurrence C in a derivation Π is the least proof-tree trunk Π^C which contains all the linked occurrences of C. Intuitively, considering the history Π^C amounts to climbing up Π until C is introduced. Then the following lemma prepares the ground for the substitution of C.

Lemma 4.2 (History of a formula occurrence) *If Π is a derivation in* **B** *or* **BS** *and if C is a formula occurrence at the left in the conclusion then:*

[13] Note that one of the reasons for choosing a form of weakening and contraction in which lists (rather than formulae as usual) are considered, is to be able to keep the statement of this lemma as simple as possible.

(I) the history Π^C of C in Π consists only of operational rules with no restrictions on the left context, cuts and structural rules (only exchange in **B**, *also weakening and contraction in* **BS**);

(II) any occurrence of C in the leaves of Π^C is:
(a) visible: $C \vdash \Delta'$ for some Δ', or
(b) part of the passive context of $\top R$: $\Gamma' \vdash \top$ for some Γ' containing C, or
(c) (only in **BS**) *introduced by wL: $\Gamma', \Sigma \vdash \Delta'$ with C in Σ.*

A dual statement holds at the right.

Proof. (I) Since C is present at the left, by visibility it is not possible to apply any rule with active formulae on the left (unless on C itself, but then that application has C as the principal formula and is not in the history of C).
(II) When C is introduced by a rule, it has no context since it is an active formula. If C is introduced by an axiom, $\top R$ is the only axiom with context at the left. □

It is now convenient to extend the definition of principal formula to include both formula occurrences in an axiom $A \vdash A$, the formulae 1 and $\bot$ introduced by $1R$ and $\bot L$ and the formulae $\top$ and 0 introduced by $\top R$ and $0L$ (while the formulae introduced in a passive context in $\Gamma \vdash \top$ or $0 \vdash \Delta$ are not principal). From now on, we include in case (a) of the above lemma only those sequents not falling under cases (b) or (c); so (a) consists of those cases in which C occurs as the principal formula.

The theorem of cut-elimination for **B** is obtained in the usual way from the following, which is proved by *induction on the degree only*:

Proposition 4.3 *A derivation Π in* **B** *with an application of cut as the last inference, and no other applications of cut, can be transformed into a derivation with the same conclusion and with no cuts.*

Such a derivation Π in **B** is of the form:

$$\dfrac{\begin{array}{cc}\vdots\,\Pi_1 & \vdots\,\Pi_2\\ \Sigma \vdash C & \Gamma, C \vdash \Delta\end{array}}{\Gamma, \Sigma \vdash \Delta}\ cutL \qquad or \qquad \dfrac{\begin{array}{cc}\vdots\,\Pi_1 & \vdots\,\Pi_2\\ \Sigma \vdash C, \Lambda & C \vdash \Delta\end{array}}{\Sigma \vdash \Delta, \Lambda}\ cutR$$

The *cut-elimination procedure in* **B** consists of the application of the following two steps. First the derivation is transformed, as described in **(i)** below, into one (with the same conclusion) in which cuts are always on principal formulae, and thus of the form:

$$\dfrac{\Sigma' \vdash C \quad C \vdash \Delta'}{\Sigma' \vdash \Delta'}\ cut$$

Then, cuts are reduced to cuts of lower degree, according to **(ii)**. Now the proposition is proved by the inductive hypothesis.

(i) We describe the case of $cutL$; by symmetry, this describes also the way to treat $cutR$. The idea is to lift (in one step) the cut up to where C is principal. This is obtained easily, by exploiting the history of the cut formula and the lemma of substitution.

A derivation with conclusion $\Gamma, \Sigma \vdash \Delta$ and in which the cut rule is applied only on principal formulae is obtained by operating on Π_2 as follows. Consider the history ${\Pi_2}^C$ of C in Π_2, and suppose that $C \vdash \Delta_1, \ldots C \vdash \Delta_m$ are all the leaves in which C is principal. Let Φ_j be the sub-derivation of Π_2 with conclusion $C \vdash \Delta_j$. Similarly, suppose $\Sigma_1 \vdash C, \ldots \Sigma_n \vdash C$ are all the leaves of ${\Pi_1}^C$ where C is principal, and let $\Psi_i, i \leq m$ be their derivations. Replacing Σ for C in ${\Pi_2}^C$ gives $\Gamma, \Sigma \vdash \Delta$ as conclusion, as required, and $\Sigma \vdash \Delta_1, \ldots, \Sigma \vdash \Delta_n$ as new leaves (by lemma 4.2, the other leaves in which C occurred were of the form $\Gamma' \vdash \top$ and they remain axioms after the substitution; about the leaves where C does not occur at all, we need not worry). Each $\Sigma \vdash \Delta_j$, $j \leq n$ is obtained as the conclusion of ${\Pi_1}^C[\Delta/C]$, which becomes a derivation when its leaves $\Sigma_i \vdash \Delta_j$ are obtained as the conclusion of a cut

$$\dfrac{\begin{array}{cc}\vdots\,\Psi_i & \vdots\,\Phi_j\\ \Sigma_i \vdash C & C \vdash \Delta_j\end{array}}{\Sigma_i \vdash \Delta_j}\ cut$$

where C is principal on both sides.
(ii) In (1) we show that if any one of the premisses of cut is an axiom, then we eliminate that application of cut; in (2) we show that if both premisses of the cut are the result of an introduction rule, then the derivation is transformed into one of lower degree (with the same conclusion).

(1) The axiom is the left premiss. Since the rank is 2, the cut formula C has rank 1 also in the right premiss of cut, and hence it is either a principal formula or an axiom. (Notice that the case in which the cut formula belongs to the passive context of $\top R$ or $0L$ has already been treated.)

(a) $C \vdash C$:

$$\dfrac{C \vdash C \quad \begin{array}{c}\vdots\\ C, \Gamma \vdash \Delta\end{array}}{C, \Gamma \vdash \Delta}\, cutL \qquad \rightsquigarrow \qquad \begin{array}{c}\vdots\\ C, \Gamma \vdash \Delta\end{array}$$

(b) $\vdash 1$: 1 can be introduced in the right premiss only by $1 \vdash 1$ (and then it is symmetric to (a)), or $1L$:

$$\dfrac{\vdash 1 \quad \dfrac{\begin{array}{c}\vdots\\ \vdash \Delta\end{array}}{1 \vdash \Delta}\, 1L}{\vdash \Delta}\, cutL \qquad \rightsquigarrow \qquad \begin{array}{c}\vdots\\ \vdash \Delta.\end{array}$$

(c) $\Gamma \vdash \top$: $\top$ can be introduced in the right premiss only by $\top \vdash \top$ (symmetric to (a)).

If the axiom is the right premiss of cut, the argument is absolutely symmetric.

(2) The derivation has the form

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma \vdash C\end{array}}{\Gamma \vdash C}\, \circ R \quad \dfrac{\begin{array}{c}\vdots\\ C \vdash \Delta\end{array}}{C \vdash \Delta}\, \circ L}{\Gamma \vdash \Delta}\, cut$$

where C is principal in both premisses of cut; we lower the degree by the following reductions.

connective $\otimes$:

$$\dfrac{\dfrac{\Gamma \vdash A \quad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \otimes B}\, \otimes R \quad \dfrac{A, B \vdash \Delta}{A \otimes B \vdash \Delta}\, \otimes L}{\Gamma, \Gamma' \vdash \Delta}\, cut \qquad \rightsquigarrow \qquad \dfrac{\Gamma' \vdash B \quad \dfrac{\Gamma \vdash A \quad A, B \vdash \Delta}{\Gamma, B \vdash \Delta}\, cutL}{\Gamma, \Gamma' \vdash \Delta}\, cutL$$

connective $\parr$: symmetric of $\otimes$

connective $\&$:

$$\dfrac{\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}\, \& R \quad \dfrac{A \vdash \Delta}{A \& B \vdash \Delta}\, \& L}{\Gamma \vdash \Delta}\, cut \qquad \rightsquigarrow \qquad \dfrac{\Gamma \vdash A \quad A \vdash \Delta}{\Gamma \vdash \Delta}\, cut$$

connective $\oplus$: symmetric of $\&$

connective $\multimap$: four cases are possible:

$(\multimap Uni\text{-}\multimap Uni)$

$$\dfrac{\dfrac{A \vdash B \quad C \vdash D}{B \multimap C \vdash A \multimap D}\, \multimap U \quad \dfrac{E \vdash A \quad D \vdash F}{A \multimap D \vdash E \multimap F}\, \multimap U}{B \multimap C \vdash E \multimap F}\, cut \qquad \rightsquigarrow \qquad \dfrac{\dfrac{E \vdash A \quad A \vdash B}{E \vdash B}\, cut \quad \dfrac{C \vdash D \quad D \vdash F}{C \vdash F}\, cut}{B \multimap C \vdash E \multimap F}\, \multimap U$$

$(\rightarrow Uni\text{-}\rightarrow L)$

$$\dfrac{\dfrac{A\vdash B \quad C\vdash D}{B\rightarrow C\vdash A\rightarrow D}\rightarrow U \quad \dfrac{\vdash A \quad D\vdash \Delta}{A\rightarrow D\vdash \Delta}\rightarrow L}{B\rightarrow C\vdash \Delta}\,cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash A \quad A\vdash B}{\vdash B}\,cut \quad \dfrac{C\vdash D \quad D\vdash \Delta}{C\vdash \Delta}\,cut}{B\rightarrow C\vdash \Delta}\rightarrow L$$

$(\rightarrow R\text{-}\rightarrow Uni)$

$$\dfrac{\dfrac{B\vdash C}{\vdash B\rightarrow C}\rightarrow R \quad \dfrac{A\vdash B \quad C\vdash D}{B\rightarrow C\vdash A\rightarrow D}\rightarrow U}{\vdash A\rightarrow D}\,cut \quad\rightsquigarrow\quad \dfrac{\dfrac{A\vdash B \quad B\vdash C}{A\vdash C}\,cut \quad C\vdash D}{\dfrac{A\vdash D}{\vdash A\rightarrow D}\rightarrow R}\,cut$$

$(\rightarrow R\text{-}\rightarrow L)$

$$\dfrac{\dfrac{A\vdash B}{\vdash A\rightarrow B}\rightarrow R \quad \dfrac{\vdash A \quad B\vdash \Delta}{A\rightarrow B\vdash \Delta}\rightarrow L}{\vdash \Delta}\,cut \quad\rightsquigarrow\quad \dfrac{\dfrac{\vdash A \quad A\vdash B}{\vdash B}\,cut \quad B\vdash \Delta}{\vdash \Delta}\,cut$$

connective $\leftarrow$: symmetric of $\rightarrow$

The above procedure can be easily modified to prove cut-elimination for structured basic logic **BS**. Indeed, by lemma 4.2, only case (c) for the leaves of the history of C remains to be considered. Of course, we must substitute an application of weakening introducing C with one introducing Σ. Now, also the case of weakening and contraction is (automatically) considered. In fact, let us note that the substitution Σ/C in Π, now means that a weakening introducing C becomes a weakening introducing Σ and the same applies for contraction.

4.2 Cut-elimination in the extensions of basic logic

We now show how the above procedure of cut-elimination can be adjusted to obtain cut-elimination also for the extensions of **B**. We do not consider **BLR** and **BLRS** since they are exactly the usual two-sided formulation of linear and classical logic, respectively, and thus also with the usual procedure of cut-elimination. Of the remaining four extensions, it is enough by symmetry to treat only the extensions of **B** obtained by liberalizing the contexts on the left, that is **BL** and **BLS**.

Before proceeding, we need the definition of rank, which will be the second parameter in the proof of cut-elimination. The *rank* of a formula occurrence C in a sequent of a derivation is the height of the history of C. The right rank $R\text{-}\rho$ (resp. left rank $L\text{-}\rho$) of a cut is, as usual, the rank of the cut-formula in the right (resp. left) premise of the cut. The rank ρ of a cut is the sum of its right and left rank.

We are now ready for cut-elimination in **BL** and **BLS**. At the right, visibility is preserved in both logics. So at the right the structure of axioms and rules, and hence derivations, remains the same as that in **B** and **BS**; in particular, the lemma of history of a formula occurrence at the right continues to hold, exactly as for **B** and **BS**. Also the lemma of substitution holds; even better, it can be strengthened by extending substitution to include a context at the left, as follows. By writing $\Sigma\text{-}\Lambda/C$ we mean the substitution which replaces the formula occurrence C with Λ and adds Σ at the left of the sequent: $\Gamma\vdash\Delta[C]$ becomes $\Gamma,\Sigma\vdash\Delta[\Lambda]$.

Lemma 4.4 (Substitution of a formula occurrence in BL and BLS) *Let Π be a proof-tree trunk in which the only rules applied are either rules for connectives which do not operate on the right or structural rules or cuts. Let C be any formula occurrence at the right. Then, for any list of formulae Λ and Σ, the result of the substitution $\Sigma\text{-}\Lambda/C$ in Π is again a proof-tree trunk. If Π has conclusion $\Gamma\vdash\Delta$, then $\Pi[\Sigma\text{-}\Lambda/C])$ has conclusion $\Gamma,\Sigma\vdash\Delta[\Lambda/C]$).*

Now we can see that **BL** and **BLS** admit elimination of full cut. Just as in **B** and **BS** the task of eliminating cuts was reduced to the elimination of cuts without contexts, now in the extensions at the left **BL** and **BLS** the key step is to reduce the problem to the elimination of cuts with context at the left, i.e. $cutL$. Indeed, by history and substitution of a formula occurrence at the right, we have:

Lemma 4.5 (Reduction of left rank of cut to 1 in BL and BLS) *A derivation* Π *in* **BL** *and* **BLS** *with cut as last rule, and with no other cuts, can be immediately transformed into a derivation with the same conclusion, but in which the application of cut has the form*

$$\frac{\Gamma' \vdash C \quad \Sigma, C \vdash \Lambda}{\Gamma', \Sigma \vdash \Lambda}\ cutL$$

where C *of the left premiss is principal.*

The next proposition now follows easily:

Proposition 4.6 *A derivation in* **BL** *and* **BLS** *with cut as last rule, and with no other cuts, can be transformed into a derivation with the same conclusion and without cuts.*

Proof (Sketch). The proof is by induction on the degree and (right) rank. If ρ is the rank, we distinguish the two cases:
($\rho = \mathbf{2}$) If one of the premisses of cut is an axiom, the cut is eliminated; if both premisses of cut are the result of a rule for connectives, by reducing the degree, cut is eliminated by induction.
($\rho > \mathbf{2}$) First, if the left rank is $L\text{-}\rho > 1$, the derivation is transformed into one with the same conclusion and $L\text{-}\rho = 1$, by the reduction just seen in the preceding lemma. Now cuts to be eliminated are only $cutL$, with $L\text{-}\rho = 1$. Assuming that cut-elimination holds for derivations with the same degree but lower rank, the derivation is transformed into one with the same conclusion but with lower right rank. We proceed in the usual way (Gentzen style), by lifting $\Gamma' \vdash C$, the left premiss of cut, along the right branch: cut is lifted, and the rule is applied afterwards. This is always possible, since lifting $cutL$ up to the premisses of the rule leaves the right side unaltered, and the effect is a substitution Γ'/C acting only at the left, where the rules have liberalized context. □

5 Equations depending on the control of contexts

We now analyse the question of which of the most common properties fail in basic logic because of the control of contexts and in which way such properties are recovered in the extensions. Basic logic has a richer structure and is therefore able of finer distinctions than its extensions. By going upwards towards classical logic more equations hold and some connectives are identified or become definable.

In the following result, we find an equation equivalent to the presence of the context beside the active formula for each formation rule, except those of implications. We need to consider rules where the context beside the active formula consists of only one formula. Moreover, we need to associate a formula $\otimes\Gamma$ to any list of formulae Γ, and, symmetrically, a formula $\parr\Delta$ to any list of formulae Δ. $\otimes\Gamma$ and $\parr\Delta$ are obtained by replacing commas at the right or at the left with $\otimes$ or $\parr$, respectively; the specific order in which $\otimes$ and $\parr$ are applied to define $\otimes\Gamma$ and $\parr\Delta$ is irrelevant, as long as it is fixed. Whatever the choice is, the sequents $\Gamma \vdash \otimes\Gamma$ and $\parr\Delta \vdash \Delta$ will be derivable in **B** by applying $\otimes R$ and $\parr L$, respectively, in the suitable order.

Proposition 5.1 *a) In any extension of* **B**, *the inequalities*

1 *is neuter for* $\otimes$*:*	$C \otimes 1 \vdash C$
$\otimes$ *distributes over* $\oplus$*:*	$C \otimes (A \oplus B) \vdash (C \otimes A) \oplus (C \otimes B)$
0 *nullifies* $\otimes$*:*	$C \otimes 0 \vdash 0$

are equivalent to the following rules, respectively

$$\frac{C \vdash \Delta}{C, 1 \vdash \Delta} \qquad \frac{C, A \vdash \Delta \quad C, B \vdash \Delta}{C, A \oplus B \vdash \Delta} \qquad C, 0 \vdash \Delta$$

Symmetrically for the inequalities

$\bot$ *is neuter for* $\parr$*:*	$D \vdash D \parr \bot$
distributivity of $\parr$ *over* $\&$*:*	$(D \parr A) \& (D \parr B) \vdash (A \& B) \parr D$
$\top$ *maximizes* $\&$*:*	$(D \parr A) \& (D \parr B) \vdash (A \& B) \parr D$

b) The following are equivalent over $\mathbf{B}$*:*

i) Associativity of $\otimes$ *:* $C \otimes (A \otimes B) \vdash (C \otimes A) \otimes B$

ii) Strong reflection for $\otimes$*:*
for any Γ, Γ', Δ, $\quad \Gamma', \otimes\Gamma \vdash \Delta$ *if and only if* $\Gamma', \Gamma \vdash \Delta$

iii) The full rule $\otimes L^f$*.*

iv) The following form of $\otimes L$*:*

$$\frac{C, A, B \vdash \Delta}{C, A \otimes B \vdash \Delta}$$

Symmetrically for $\parr$*.*

When associativity of $\otimes$ *holds, any rule full at the left is equivalent to its form in which the context consists of only one formula; symmetrically for* $\parr$ *and right.*

Proof. *a*) Easy to see via cut.
b) - *i*) $\Rightarrow$ *ii*) Let us see that, for every Δ, one has

$$\otimes\Gamma \vdash \Delta \quad \text{if and only if} \quad \Gamma \vdash \Delta$$

In fact, if $\otimes\Gamma \vdash \Delta$ then, since $\Gamma \vdash \otimes\Gamma$, also $\Gamma \vdash \Delta$ by cut. Conversely, when $\Gamma \vdash \Delta$ is derivable, also $\otimes\Gamma \vdash \Delta$ is derivable by suitably applying $\otimes L$, as can be seen by induction on the derivation. The only non trivial case is that of a derivation which ends with $\otimes R$, which is transformed as follows:

$$\frac{\begin{array}{cc}\vdots\,\Pi_1 & \vdots\,\Pi_2 \\ \Gamma_1 \vdash A & \Gamma_2 \vdash B\end{array}}{\Gamma_1, \Gamma_2 \vdash A \otimes B}\otimes R \qquad \text{becomes} \qquad \frac{\dfrac{\begin{array}{cc}\vdots\,\Pi_1^{\otimes} & \vdots\,\Pi_2^{\otimes} \\ \otimes\Gamma_1 \vdash A & \otimes\Gamma_2 \vdash B\end{array}}{\otimes\Gamma_1, \otimes\Gamma_2 \vdash A \otimes B}\otimes R}{\otimes\Gamma_1 \otimes \otimes\Gamma_2 \vdash A \otimes B}\otimes L$$

where $\Pi_1^{\otimes}$ and $\Pi_2^{\otimes}$ are the proofs given by the inductive hypothesis and where $\otimes\Gamma_1 \otimes \otimes\Gamma_2$ is equal to $\otimes(\Gamma_1, \Gamma_2)$ by associativity. We can now complete our argument: let us assume $\Gamma', \otimes\Gamma \vdash \Delta$; then $\Gamma', \Gamma \vdash \Delta$ follows by cut since $\Gamma \vdash \otimes\Gamma$. Conversely, from $\Gamma', \Gamma \vdash \Delta$ it follows (by the above equivalence) that $\otimes(\Gamma', \Gamma) \vdash \Delta$, hence by associativity $\otimes\Gamma' \otimes \otimes\Gamma \vdash \Delta$, from which $\otimes\Gamma', \otimes\Gamma \vdash \Delta$ by reflection and finally, by cut with $\Gamma' \vdash \otimes\Gamma'$, also $\Gamma', \otimes\Gamma \vdash \Delta$.
ii) $\Rightarrow$ *iii*) Let us assume strong reflection for $\otimes$; then we have in particular

$$\Gamma, A \otimes B \vdash \Delta \quad \text{if and only if} \quad \Gamma, A, B \vdash \Delta$$

one direction of which is the full rule $\otimes L^f$. Obviously, the full rule implies its form in which the context is given by one formula only.
iv) $\Rightarrow$ *i*) Associativity is derivable as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{C \vdash C \quad A \vdash A}{C, A \vdash C \otimes A} \quad B \vdash B}{C, A, B \vdash (C \otimes A) \otimes B}}{C, A \otimes B \vdash (C \otimes A) \otimes B}}{C \otimes (A \otimes B) \vdash (C \otimes A) \otimes B}$$

Finally, note that strong reflection for $\otimes$ makes it equivalent to consider a single formula rather than a list of formulae as left context of any rule. □

The formation rules full at the left considered above are equivalent to the *equalities* $C \otimes 1 = C$, $C \otimes (A \oplus B) = (C \otimes A) \oplus (C \otimes B)$, $C \otimes 0 = 0$ and $C \otimes (A \otimes B) = (C \otimes A) \otimes B$, respectively. In fact, as it is easy to control, the sequents $C \vdash C \otimes 1$, $(C \otimes A) \oplus (C \otimes B) \vdash C \otimes (A \oplus B)$, and $0 \vdash C \otimes 0$, are derivable in $\mathbf{B}$, while the schema $(C \otimes A) \otimes B \vdash C \otimes (A \otimes B)$ is equivalent to its converse, since $\otimes$ is commutative. Symmetrically for rules full at the right and equalities with $\parr$.

As a consequence of the above result we obtain that: the formation rules $\otimes L^f$, $1L^f$, $\oplus L^f$, $0L^f$ are not derivable in $\mathbf{B}$ and in $\mathbf{BR}$, as well as their symmetric $\parr R^f$, $\bot\, R^f$, $\& R^f$

and $\top R^f$ are not derivable in **B** and in **BL**. That is, contexts on the side of active formulae cannot be added and the basic form of such four rules is strictly weaker than their full form.

The above remark applies also to the two remaining formation rules with active formulae on the left, namely $\leftarrow L^{fl}$ and $\rightarrow R^{fl}$, as well as to their symmetric, namely $\rightarrow R^{fr}$ and $\leftarrow L^{fr}$. In this case the remark extends to logics with **S** too. In fact, the inequalities $C \otimes (A \leftarrow C \otimes A) \vdash$ and $C \vdash A \rightarrow (C \otimes A)$ are derivable in **BL**, while they are not derivable in **BRS**, as one can see by cut-elimination. Symmetrically the inequalities $\vdash (A \parr D \rightarrow A) \parr D$ and $(A \parr D) \leftarrow A \vdash D$ are derivable in **BL** and not derivable in **BLS**.

Summing up, we have proved that:

Proposition 5.2 *For all the formation rules of* **B**, *the form full either at the left or at the right is not derivable in* **B**. *Similarly, the form full at the right of all the formation rules of* **BL** *is not derivable in* **BL** *(and symmetrically for* **BR**). *Moreover, the formation rules* $\leftarrow L^{fl}$ *and* $\rightarrow R^{fl}$ *are not derivable in* **BRS**, *and symmetrically,* $\rightarrow R^{fr}$ *and* $\leftarrow L^{fr}$ *are not derivable in* **BLS**. *Thus the full form, either on the left or on the right, of the formation rules for implications, are underivable in* **BS**.

This proposition makes explicit what has always been evident thus far, namely that the control of contexts makes basic logic intrinsically different from other usual logics. This applies not only to the structure of proofs but, by the previous propositions, also to the set of provable formulae.

The behaviour of reflection rules is quite different from that of formation rules. In fact, they often admit the presence of contexts, as specified in the following statement.

Proposition 5.3 *In* **B** *and all its extensions,* $\oplus R$ *is equivalent to* $\oplus R^f$ *and symmetrically* $\& L$ *is equivalent to* $\& L^f$; *moreover,* $\rightarrow L$ *is equivalent to a rule with premises* $\vdash A$ *and* $\Gamma \vdash B$ *and conclusion* $\Gamma, A \rightarrow B \vdash \Delta$, *and analogously for* $\leftarrow R$.

In any of the extensions of **B** *where full cut holds (that is, all extensions except* **BS**), $\otimes R$ *is equivalent to* $\otimes R^f$, *and symmetrically* $\parr L$ *is equivalent to* $\parr L^f$; *moreover,* $\rightarrow L^{fl}$ *is equivalent to* $\rightarrow L^f$.

It is remarkable that, on the contrary no context can be added freely in the left premise of $\rightarrow L$ in **B** and **BR**, that is the rule

$$\frac{\Gamma \vdash A \quad B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta}$$

is not derivable in **B** and **BR**. In fact, as seen in section 2, this rule is equivalent to the "if" direction in the deduction theorem $\Gamma, A \vdash B$ *if and only if* $\Gamma \vdash A \rightarrow B$, which fails. Moreover, by proposition 5.2, the "only if" direction" also fails in **BR** and in **B**, and symmetrically for $\leftarrow$ in **B** and **BL**. So in **B** no movement between the left and the right side of a sequent is possible in presence of a context. Observe that such strict control of movement cannot be separated from visibility in general: loosening the former would also destroy the latter. For instance, it is well-known that allowing the deduction theorem for $\rightarrow$, distributivity of $\otimes$ over $\oplus$ is derivable, and the full form of $\oplus L$ with it.

The diversity of primitive connectives present in basic logic is lost as long as the control of contexts loosens. However, the identification of some connectives is present already in **B** itself, but only at the meta-level and in absence of contexts.

In the sequel, we use the signs $\multimap$ and $\circ\!\!-$ for the "classical" definitions of implications, namely we put

$$A \multimap B \equiv {\sim}A \parr B \text{ and } B \circ\!\!- A \equiv B \otimes \neg A$$

Proposition 5.4 *a) The following equivalences hold:*

i) In **B**, $\vdash A \& B$ *is provable if and only if* $\vdash A \otimes B$ *is provable; symmetrically* $B \oplus A \vdash$ *is provable if and only if* $B \parr A \vdash$ *is provable.*

ii) In **B**, $\vdash \neg A$ *is provable if and only if* $\vdash {\sim} A$ *is provable; symmetrically* ${\sim}A \vdash$ *is provable if and only if* $\neg A \vdash$ *is provable.*

iii) *In* **BR** $\vdash A \rightarrow B$ *is provable if and only if* $\vdash A \multimap B$ *is provable; symmetrically, in* **BL** $B \leftarrow A \vdash$ *is provable if and only if* $B \circ\!\!- A \vdash$ *is provable.*

b) *In* **B**, *the sequents* $A \otimes B \vdash A \& B$ *and* $A \oplus B \vdash A \parr B$ *are derivable via weakening and the sequents* $A \& B \vdash A \otimes B$ *and* $A \parr B \vdash A \oplus B$ *are derivable via contraction. So in* **BS** *additive and multiplicative connectives are identified.*

In **BL**, *the sequents* $A \multimap B \vdash A \rightarrow B$ *and* $B \circ\!\!- A \vdash B \leftarrow A$ *are derivable; symmetrically, in* **BR** *one has* $A \rightarrow B \vdash A \multimap B$ *and* $B \leftarrow A \vdash B \circ\!\!- A$. *So in* **BLR** *it is* $A \rightarrow B = A \multimap B$ *and* $B \leftarrow A = B \circ\!\!- A$.

Finally, in **BL** *and in* **BR**, *both* $\sim A \vdash \neg A$ *and* $\neg A \vdash \sim A$ *are derivable, that is* $\neg A = \sim A$.

Proof. *a*) The proofs of the three items follow essentially a uniform schema, that consists of finding an assertion which is equivalent to the sequent at the left by the reflection principle, and is equivalent to the sequent at the right by a reflection rule and because of cut-elimination. Here are the details in the case *iii*).
By the reflection principle, $\vdash A \rightarrow B$ is derivable if and only if $A \vdash B$ is derivable. Then, from $\vdash 1$ and $A \vdash B$ one derives $\vdash \sim A, B$ by the reflection rule $\leftarrow R^{fr}$, from which $\vdash A \multimap B$; conversely, assuming that $\vdash A \multimap B$ is derivable, then also $\vdash \sim A, B$ is derivable, but then, by cut-elimination, $A \vdash B$ is derivable.
b) The sequent $\sim A \parr B \vdash A \rightarrow B$ is derivable in **BL** from non-contradiction $A, \sim A \vdash$ (which also holds in **BL**!) and from the axiom $B \vdash B$ by $\parr L^{fl}$ and then by $\rightarrow R^{fl}$; while the sequent $B \otimes \neg A \vdash B \leftarrow A$ is derivable from the axiom $B \vdash B$ and from the other non-contradiction $\neg A, A \vdash$, by $\leftarrow R^{fl}$ and then by $\otimes L$.

Finally, observe that $\neg A \vdash \sim A$ follows in **BL** from $B \otimes \neg A \vdash B \leftarrow A$ when $B = 1$, but symmetrically also in **BR** from $A \rightarrow B \vdash \sim A \parr B$ when $B = \bot$. In **BL**, from $A \vdash A$ one derives $A, 1 \vdash A$, hence $A, \sim A \vdash$ by $\leftarrow L^{fl}$ and then $\sim A \vdash \neg A$ by $\bot\ R$ and $\rightarrow R^{fl}$. In **BR**, the symmetric derivation proves the same sequent $\sim A \vdash \neg A$, since such sequent is self-symmetric. □

So, when adding weakening and contraction, that is when the distinction between the multiplicative and additive form of the rules (that is called control of contexts) is lost, the two conjunctions and the two disjunctions are identified. Similarly, when adding contexts beside the active formulae, that is when visibility is lost, the two implications and the two exclusions are identified. These two facts are parallel and independent. In this sense, basic logic is the logic which operates a strict and uniform control of contexts. Its extensions are obtained when such control is relaxed.

Loosening the control of contexts is necessary in order to derive the above equalities. In fact, assuming $A \otimes B = A \& B$, one can derive weakening and contraction. We leave the proof, even if it requires some more care with respect to the case of linear logic. In case of the equalities involving implications, the converse to the above proposition holds in the weak sense that the equality $B \leftarrow A = B \circ\!\!- A$ is unprovable in any of the extensions of **B** where contexts at the right are controlled, and symmetrically for the equality $A \multimap B = A \rightarrow B$ where contexts are controlled at the left. This amounts to prove that:

Proposition 5.5 *The sequent* $B \leftarrow A \vdash B \circ\!\!- A$ *is underivable in* **BLS**; *symmetrically, the sequent* $A \multimap B \vdash A \rightarrow B$ *is underivable in* **BRS**.

Proof. Observe that $\circ\!\!-$ satisfies in **BL** (and hence a fortiori in **BLS**) the rule

$$\frac{\Gamma, B \vdash A, \Delta}{\Gamma, B \circ\!\!- A \vdash \Delta}$$

In fact, since in **BL** the rule $\rightarrow L^f$ is derivable, by proposition 5.3, from the premiss $\Gamma, B \vdash A, \Delta$ and from the axiom $\bot \vdash$ one can derive $\Gamma, B, \neg A \vdash \Delta$, from which the conclusion $\Gamma, B \circ\!\!- A \vdash \Delta$ by $\otimes L^f$. Therefore the sequent $B \leftarrow A \vdash B \circ\!\!- A$ is not derivable, otherwise it would be derivable by cut also the rule $\leftarrow L^f$, contrary to lemma 5.2. □

This means that in **BLS** (and **BL**) $\leftarrow$ and $\circ\!\!-$ are different, even if $\circ\!\!-$ obeys the inference rules for $\leftarrow$. In fact, we have already seen above that $\circ\!\!- L^{fl}$ is a derivable rule

(actually, also $\circ\!\!- L^f$); in addition $\circ\!\!- R^{fl}$ is easily seen to be derivable. So it happens that a connective is not uniquely characterized by its rules.

We conclude with some brief remarks on the asymmetric extensions of **B**.

Let us consider **BL**, which probably has a more familiar aspect than its symmetric **BR**. We have seen that $\rightarrow$ in **BL** satisfies the deduction theorem, while $\leftarrow$ does not; moreover, inequalities with $\otimes$ hold while their symmetric with $\parr$ fail. Thus **BL** is a cut-free calculus for linear intuitionistic logic, which includes a non-associative and non-distributive "par" connective $\parr$, and an additional implication connective $\leftarrow$ (see [3]).

The calculus **BLS** is a formulation of intuitionistic logic that, besides contexts at the right as in [15], includes an extra connective $\leftarrow$. In fact, since multiplicatives and additives are identified because of $\mathcal{S}$, both conjunction and disjunction satisfy associativity and distributivity, since the rules $\parr R^f$ and $\& R^f$ are equivalent to the rules $\oplus R^f$ and $\otimes R^f$, which are derivable in **BL** by proposition 5.3. The results above show that $\leftarrow$ is not definable in intuitionistic logic. So, emerging back to the surface of intuitionistic logic after the exploration of the dark depths of basic logic has left us with the reward of a new connective, which we suggest to be the intuitionistic way to deal with negative notions in a primitive way.

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Dipartimento di Matematica Pura ed Applicata
Università di Padova
Via Belzoni 7, I–35131 Padova, Italy
e-mail: sambin,giulia,claudia@math.unipd.it