A predicative characterization of quantum states and Matte Blanco's bi-logic*

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Abstract. We show a correspondence between a predicative characterization of quantum states, we have recently introduced, and bi-logic, a logical setting proposed by the Chilean psychoanalist I. Matte Blanco in order to describe the logic of the unconscious. The unconscious deals with "infinite" objects for which the "symmetric mode" is characterized, where negation and implication do not exist. In our model it is possible to define a class of first order domains, termed virtual singletons, that are uncountable, and that allow a generalization of the notion of duality, termed symmetry. Symmetry makes negation and logical consequence collapse, in favour of different links between judgements, that are due to quantum correlations.

Introduction

The relation between quantum physics and consciousness is a very controversial and challeging topic in quantum interaction. Even though its experimental basis is not stabilized yet, it is considered reasonable by several authors to investigate formal approaches connecting quantum physics and mind. Such approaches aim, as a first instance, to establish applications of quantum structures to psychological and cognitive fields, and, more specifically, to search isomorphisms which are compatible with the existence of a relation and could support it. There is an increasing number of proposals in such directions, witnessed in the proceedings [QI07], [QI08], [QI09], [QI11], [QI12] and the references therein; moreover we quote the authors [Ae], [ABGS], [AGS], [AS], [AS2], [BPFT], [BB].

The present note would like to contribute to the development of a formal approach to the problem of consciousness, describing an isomorphism between a logical model, recently proposed by the author in the field of quantum computational logics in [Ba], then developed in [Ba2] and [Ba3], and *bi-logic*, namely, the logical system proposed in the '70s by the Chilean psychoanalyst Ignacio Matte Blanco [MB], to describe two logical sides of the human thinking, the rational thinking and the logic of the unconscious.

The proposal of Matte Blanco is performed in a completely descriptive, not formalized way, since it comes out as a synthesis of the logical common traits he had been observing for thirty years in his clinical experience. However, his logical

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approach, inspired by set theory, is very effectively synthetized in the ideas of *infinite sets*, and of *symmetry*. This makes it possible to discuss a comparison with a logical model, formally developed.

The framework in which the quantum computational model here considered has been developed is proof-theoretical, in the quite peculiar approach that was proposed in basic logic [SBF]. It is based on *definitory equations*, which define logical connectives and derive their rules in sequent calculus from metalinguistic links between logical judgements. The predicative interpretation of quantum states here considered results from the application of such a method to the case of some judgements in quantum physics, considering the definitory equations of quantifiers introduced in [MS] and the definition of the equality predicate given by Maietti (see [Ma]).

We think that the approach we have chosen is particularly suited to the case of a logic arising from psychoanalysis, for it is based on a direct analiyisis of judgements, which can distinguish between the metalevel and the object level. Moreover, basic logic can discuss the notion of symmetry itself, in terms of logical consequence, represented, in sequent calculus, by the sequent sign $\vdash$. This allows a direct comparison with Matte Blanco's notion of symmetry, in terms of logical derivations and connectives.

In the following, we refer to results and definitions contained in [Ba2] and [Ba3] for the first section, and in [Ba3] for the second section.

1 Matte Blanco's Infinite Sets in quantum terms

Matte Blanco characterizes the objects of the unconscious thinking as "infinite" sets, for, he says, the unconscious treats the part as the whole thing. This means, in particular, that a bijection between a subset and the whole set is possible, namely the set one is considering is infinite.

We start assuming the following perpective in order to approach the infinite sets. It consists of a way of reading the well known logical distinction between propositional formulae on closed terms and quantifiers. Let us consider any set D. We remind the following result, that links membership relation, quantifiers and equality relation:

Proposition 1. *The equivalence between the proposition $z \in D$ and the proposition $(\exists x \in D)x = z$ is provable from the definitory equations of the existential quantifier and of the equality relation. If $D = \{t_1, \ldots, t_n\}$, the sequent $z = t_1 \vee \ldots \vee z = t_n \vdash z \in D$ is provable from the defiinition of the additive disjunction $\vee$.*

The converse sequent $z \in D \vdash z = t_1 \vee \ldots \vee z = t_n$ is not provable. We adopt the intuitionistic interpretation of disjunction. With respect to it, one can characterize a particular class of finite sets:

Definition 1. *A finite set $D = \{t_1, \ldots, t_n\}$ is* focused *by a certain logical system if and only if the sequent $z \in D \vdash z = t_1 \vee \ldots \vee z = t_n$ is true in that system, it is* unfocused *otherwise.*

A set $D = \{t_1, \ldots, t_n\}$, which is recognized as finite in the metalanguage, is not recognized as such inside a logical system, if it is unfocused in it, since its elements can be counted by the system only if, picking up a generic element $z \in D$, it can be recognized that $z = t_i$ for some index i.
One has the following characterization of focused sets, that is very significant for our model:

Proposition 2. *Given a finite set* $D = \{t_1, \ldots t_n\}$*, the schema* $(\forall x \in D)A(x)$ *and the schema* $A(t_1)\& \ldots \&A(t_n)$ *are equivalent if and only if the domain* D *is focused.*

Such a setting is suited to introduce a predicative representation of quantum states, as follows. Let us consider a particle $\mathcal{A}$ and fix an observable. Then one has a random variable Z associated to the measurement of the particle with respect to that observable. One has the set

$$D_Z = \{(s(z), p\{Z = s(z)\})\}$$

of the outcomes of measurement with their probabilities; moreover one has the proposition

$A(z)$: "The particle $\mathcal{A}$ is in state $s(z)$ with probability $p\{Z = s(z)\}$"

describeing the eventual state of the particle after measurement, in the first order variable z. If the measurement hypothesis, concerning the preparation, are denoted by Γ, the resulting metalinguistic assertion is:

Γ *yield* $A(z)$ *forall* $z \in D_Z$,

translated into "*forall* $z \in D_Z$, $\Gamma \vdash A(z)$", furtherly converted into the unique sequent $\Gamma, z \in D_Z \vdash A(z)$, where the sequent sign $\vdash$ translates the metalinguistic consequence *yield* and $z \in D$ is put as a premise of the sequent (see [MS]; notice that the premises in Γ do not depend on the variable z, since the measurement hypothesis cannot depend on its eventual outcome). Then one considers the definition of $\forall$:

$$\Gamma \vdash (\forall x \in D)A(x) \quad \textit{if and only if} \quad \Gamma, z \in D \vdash A(z)$$

for all formulae $A(z)$ where z is a free variable in a domain D. It allows to our mind to attribute a state to the particle by the proposition $(\forall x \in D_Z)A(x)$, that hence represents $\mathcal{A}$ in its state.

On the contrary, one can see that the propositional formula $A(t_1)\& \ldots \&A(t_n)$, on the closed terms t_i describes the mixed state obtained after measurement, rather than the pure state. This follows from the definition of the propositional conjunction &. For, it is true that the preparation Γ *yields* $A(t_i)$ for every i: in logic, every sequent $\Gamma \vdash A(t_i)$ is derived from the sequent $\Gamma, z \in D_Z \vdash A(z)$ by the substitution t_i/z in the sequent, then cutting the true assumption $t_i \in D_Z$. Then, we say that substituting represents a measurement in logic.

Characterization 2 shows that the nature, pure or mixed, of states, can be told in terms of sets, since the set $D_Z = \{t_1, \ldots t_n\}$ of the eventual outcomes of measurement has a finite description, by means of the propositional intuitionistic disjunction $\vee$, after quantum superposition has collapsed. So, quantum superposition corresponds to the infinite character of the set D_Z.

2 Virtual singletons: converting duality into symmetry

Following Matte Blanco, the logical mode of the unconscious is symmetric. This means, primarily, that the unconscious treats every relation as if it were symmetric. The "symmetric mode" and the "infinite mode" are strongly related in Matte Blanco: one has an infinite cardinality since the unconscious considers the part as the whole thing, since the inclusion relation is treated as if it were symmetric. Notice that the unique sets for which this happens, for nonempty subsets, are singletons. Moreover, the same sets characterize the original requirement by Matte Blanco, since a set has only symmetric relations if and only if it is a singleton. So, in order to treat the symmetric mode, we need to introduce infinite sets acting as singletons: then "normal" singletons will be the finite shadow of them.

By definition, singletons are sets V for which there is an $u \in V$ such that, if $z \in V$, then $z = u$. Then we write $V = \{u\}$. It is quite natural to assume the sequent $z \in V \vdash z = u$ (where u is a closed term denoting the same element), by extensionality.

Then, in a normal logical setting, singletons are focused. However, they are quite close to unfocused sets, since they are not splitted by a disjunction. We assume that the particular odd logical behaviour of singletons is due to their borderline situation and we claim the existence of *virtual singletons*, namely unfocused sets characterized as singletons. Since unfocused sets are characterized as first order domains of quantifiers, we need to displace our characterization of singletons from sets to domains of predicative formulae. So we consider the following:

$(\forall x \in V)A(x) = (\exists x \in V)A(x)$ is true for every A, if and only if V is a (virtual) singleton.

Namely quantification is performed by a unique connective rather than by the couple ($\forall$, $\exists$) of dual connectives. In our quantum model, this produces other logical features discussed by Matte Blanco, such as paraconsistency, absence of negation and of logical consequence.

We need to show non trivial virtual singletons in a predicative model of quantum states. We have shown that non trivial virtual singletons are conceivable only if the logic we are considering does not admit the substitution rule for them. Since in the model substituting means measuring, virtual singletons can exist only prior to measurement. The specific model for this is obtained considering, as an observable, the spin, that has no classical equivalent. Let us fix an axis, say the z axis. The measurement of the spin observable, on particles in the sharp states $\uparrow$ and $\downarrow$, determines, as domains, two singletons. The formulae quantified on such domains are equivalent to propositional formulae, since their domains are singletons. They are interpretable as pairs of opposites: one puts a duality $\perp$ that can switch $\uparrow$ and $\downarrow$. It is the logical translation of the application of the Pauli matrix σ_X, namely the NOT gate of computation. As is well known, a duality between couple of opposite literals can be inductively extended to all formulae, in order to obtain a negation (Girard's negation), which satisfies the equivalence $A \vdash B$ if and only if $B^\perp \vdash A^\perp$, for every formulae A, B.

On the other side, in the same measurement context, the dual states (written $|+\rangle = 1/\sqrt{2}|\downarrow\rangle + 1/\sqrt{2}|\uparrow\rangle$ and $|-\rangle = 1/\sqrt{2}|\downarrow\rangle - 1/\sqrt{2}|\uparrow\rangle$, respectively, as vectors in the ket notation) are switched by the Pauli matrix σ_Z . On the contrary, they are eigenvectors for σ_X. In turn, we can translate all this into logic. Then, in logic, particles in the dual states correspond to predicative formulae which are fixed points for the duality $\perp$. Then such formulae satisfy the equivalence $A \vdash B$ if and only if $B \vdash A$. This eliminates the direction of logical consequence, represented by $\vdash$. The set associated to the measurement of $|+\rangle$ and $|-\rangle$ contains the two pieces of information "up" and "down", one of which can be considered the negation of the other, as just seen. Changing the measurement context and measuring the spin w.r.t the x axis, $|+\rangle$ and $|-\rangle$ would produce an objective property, since the domains would be, provably, singletons. The corresponding formulae would be equivalent to propositional formulae that would be switched by the duality corresponding to σ_Z. However, different spin observables are incompatible, hence, once the measurement context has been fixed, the domain of particles in the dual states is a virtual singleton: for it is unfocused and one can see, by means of the duality originated by σ_Z, that the equality $(\forall x \in V)A(x) = (\exists x \in V)A(x)$ is true for the first order domains originated by $|+\rangle$ and $|-\rangle$.

So we find the logical framework expected by bi-logic: an asymmetric mode where negation is meaningful, a symmetric mode where negation is meaningless. The second is given by virtual singletons, that are infinite sets. In the virtual singletons of the spin model just seen, opposite pieces of information coexist. In the symmetric mode, negation is meaningless since the opposite coexist. This is one of the features of the unconscious thinking, termed condensation, that, as pointed out by Matte Blanco, is related to the absence of negation.

A further feature of the unconscious thinking is displacement. Following Matte Blanco, it also goes back to the symmetric mode, since two different subclasses of the same class are treated as identical by the unconscious, due to symmetry: both subclasses are identified with the larger class and then treated as identical. In such a case, symmetry is applied at the second order. In the predicative model, one can widen the action of virtual singletons to the second order, considering "virtual singletons of indexes of formulae" that allow to identify formulae whose index is in the same virtual singleton. This allows, in particular, to represent the quantum correlations of the Bell's states, since the correlation takes place when the same variable is displaced elsewhere, considering another index. We think that the same kind of identification at the second order, namely at the level of formulae, could be exploited in order to justify displacement. Matte Blanco discusses the possibility of different logical links to be discovered for the logic of the unconscious. A further anaysis of the consequences of virtual singletons of indexes of formulae in logic is in progress. However, we consider it very intriguing to find that what hides logical consequence supplies, at the same time, a different way to link judgements, by means of correlations of which we cannot be aware.

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