QUANTUM SEQUENTS

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Boston - IQSA 2010

We aim to put assertions from quantum mechanics in terms of sequents.

A sequent is an object of the form

$$A_1, \dots A_n \vdash B_1 \dots B_m$$

(summing up $\Gamma \vdash \Delta$)
where $\vdash$ represents a consequence relation. A sequent represents an assertion.

A sequent calculus derives assertions and is given by rules on sequents.

We adopt the view of basic logic, developed as a common platform for sequent calculi of extensional logics.

One derives the rules of logical connectives putting definitory equations of the form

$$\Gamma \vdash A \circ B \quad \equiv \quad \Gamma \vdash A \approx B$$

where $\circ$ is the connective defined in terms of the metalinguistic link $\approx$.

We consider a preparation of a quantum system. The preparation and all the measurement hypothesis are described in a the set of premises Γ.
We represent by the sequent

$$\Gamma \vdash A_1, \ldots, A_n$$

the information $A_1, \ldots, A_n$ one can achieve from the preparation by a quantum measurement.

Quantum measurements enables us to distinguish three logical levels:

- quantum states prior to measurement: predicative level
- density operators: propositional level with probabilities
- classical states: propositional level

The measurement process of a quantum state w.r.t. an observable is a random variable.
Its outcomes are associated to elements of an orthonormal basis of the Hilbert space associated to the system.

Let Z be the random variable produced by a measurement of a certain particle in a certain state. This defines a set

$$D_Z \equiv \{z = (\xi, p\{Z = \xi\}) : \xi \text{ state of the outcome}\}$$

where $p\{Z = \xi\} > 0$.

We shall term D_Z *random first order domain*.

We consider a particle $\mathcal{A}$ and an observable producing a random variable Z and hence a r.f.o.d. D_Z.

We obtain the assertion:

"In the measurement hypothesis Γ, the state of the outcome is ξ with probability $p\{Z = \xi\}$ for all pairs $z = (\xi, p\{Z = \xi\}) \in D_Z$".

More formally, we write this assertion

"*forall* $z \in D_Z$, $\Gamma \vdash A(z)$"

and finally we summarize it in the sequent:

$$\Gamma, z \in D_Z \vdash A(z)$$

(where Γ does not depend on z, since the measurement hypothesis does not depend on the outcome.)

We put the following equivalence:

$$\Gamma \vdash (\forall x \in D_Z)A(x) \;\equiv\; \Gamma, z \in D_Z \vdash A(z)$$

which summarizes the assertion by means of the quantifier $\forall$.

The first order variable z (asociated to the random variable Z) is used as a logical glue for the different outcomes.
In this sense we claim that the proposition

$$(\forall x \in D_Z)A(x)$$

represents the superposed state of the particle.

For example, a particle represented in the Hilbert space C^2, with orthonormal basis $\{|0\rangle, |1\rangle\}$. The state is represented by the vector:

$$\alpha|0\rangle + \beta|1\rangle$$

$(\alpha, \beta \in C, |\alpha|^2 = a, |\beta|^2 = b)$

The random first order domain is

$$D_Z = \{(|0\rangle, a), (|1\rangle, b)\}$$

and the state is represented by the following proposition

$$(\forall x \in \{(|0\rangle, a), (|1\rangle, b)\})A(x)$$

When $a = 0$ or $b = 0$ the r.f.o.d. is a singleton, for example $D_1 = \{(|1\rangle, 1)\}$.

When $a = b = 1/2$ (uniform distribution) the r.f.o.d. is $D_U = \{(|0\rangle, 1/2), (|1\rangle, 1/2)\}$.

Performing a quantum measurement determines a collapse.

In our terms we consider the collapse of the variable due to a substitution by a closed term.

We consider the provable sequent

$$(\forall x \in D_Z)A(x), z \in D_Z \vdash A(z)$$

The substitution z/t yields

$$(\forall x \in D_Z)A(x), t \in D_Z \vdash A(t)$$

from which

$$(\forall x \in D_Z)A(x) \vdash A(t)$$

If $t_1, \ldots, t_n$ denote the n elements of D_Z, one obtains (by & rule):

$$(\forall x \in D_Z)A(x) \vdash A(t_1)\& \ldots \& A(t_n)$$

The proposition $A(t_1)\& \ldots \& A(t_n)$ represents a mixed state.

We have represented a non selective quantum measurement.

To represent a selective measurement, yielding a pure state:

We consider a substitution which “forgets” the probability and gives probability 1 to the result:

$$(\forall x \in D_Z)A(x) \vdash A_f(s)$$

where s is a term denoting the state $|b\rangle$ after the measurement. $s = (|b\rangle, 1)$.

For every formula $A(x)$, we put the axiom

$$A(s) \vdash (\forall x \in \{(|b\rangle, 1)\})A(x)$$

Since it is also $(\forall x \in \{(|b\rangle, 1)\})A(x) \vdash A(s)$, one has the equality

$$(\forall x \in \{(|b\rangle, 1)\})A(x) = A(s)$$

(in particular, it allows to interpret the outcome $A_f(s)$ of the measurement of any state as a sharp state).

Sharp states can be identified with propositional formulae.

But, for $n > 1$

$$(\forall x \in D_Z)A(x) \neq A(t_1)\& \ldots \& A(t_n)$$

For, the sequent $A(t_1)\& \ldots \& A(t_n) \vdash (\forall x \in D_Z)A(x)$ is not derivable.
It is equivalent to $A(t_1)\& \ldots \& A(t_n), z \in D_Z \vdash A(z)$, that, since it is
$z \in D_Z \;\Leftrightarrow\; z = t_1 \vee \cdots \vee z = t_n$
is equivalent to
$A(t_1)\& \ldots \& A(t_n), z = t_i \vdash A(z)$ for all i.

This implies that an equality

$$z = t_i$$

should be definable *in a uniform way* on the set D_Z. This implies to choose a unique phase factor.
True if and only if the domain is a singleton!!!

We now consider a set of compatible observables, giving random variables Z_i, $i = 1 \dots m$.

We obtain sequents of the form

$$\Gamma, z_1 \in D_{Z_1}, \dots z_m \in D_{Z_m} \vdash A_1(z_1), \dots, A_m(z_m)$$

After measurement, we have the sequent

$$\Gamma \vdash \Delta_Z$$

where $\Delta_Z = A_1(s_1), \dots, A_m(s_m)$ are the values obtained.

Incompatible observables cannot be determined. Then "nothing incompatible" can be added to make the list Δ_Z longer.

In basic logic we say this exploiting the definition of the costant $\perp$ (multiplicative falsum of linear logic):

$$\Gamma \vdash \Delta_Z, \perp_Z \qquad \equiv \qquad \Gamma \vdash \Delta_Z$$

The maximum of the uncertainty corresponds to the uniform distribution for the values of the measurement.

In our terms the uniform distribution (on a finite set $\{1 \ldots n\}$), with respect to an observable O, is represented by a proposition of the form

$$\perp_O = A(u_1)\& \ldots A(u_n)$$

where u_i denotes $(|b_i\rangle, 1/n)$.

Hence the measurement of a group of compatible observables gives

$$\Gamma \vdash \Delta_Z, \perp_{O_1}, \perp_{O_2} \ldots$$

where the O_i are incompatible.

If we consider more than one particle, and consider an observable, we may obtain again an assertion of the form $\Gamma \vdash A_1, \ldots, A_n$, $n > 1$.

For example we have a couple of particles, $\mathcal{A}$ and $\mathcal{A}'$.
If the two particles are separated, that is, if the measurement result on the first is independent from the measurement on the second, we obtain two different independent random variables, Z and Z'.

So we define two distinct domains D_Z and D'_Z and describe the measurement of the compound system by the sequent:

$$\Gamma, z \in D_Z, z' \in D_{Z'} \vdash A(z), A'(z')$$

that is converted into

$$\Gamma \vdash (\forall x \in D_Z)A(x) * (\forall x \in D_{Z'})A'(x)$$

(where $*$ is the multiplicative disjunction of linear logic).

Example: the separated state

$$(1/\sqrt{2}|0\rangle + 1/\sqrt{2}|1\rangle) \otimes (1/\sqrt{2}|0\rangle + 1/\sqrt{2}|1\rangle)$$

The state of the sistem is represented by the *compound* proposition

$$(\forall x \in D_U)A(x) * (\forall x \in D_U)A'(x) = (\forall x \in D_U)(\forall y \in D_U)A(x) * A'(y)$$

(two different occurrences of the same first order domain, independent variables).

The case of entangled particles is different. In such case one does not have independent measurements and variables.

, one can define a generalized n-ary quantifier, denoted $\bowtie^n$ (in particular, $\bowtie^1$ is $\forall$).

It is defined in order to represent entangled states.
The proposition

$$\bowtie^2_{x \in D_Z} (A_1; A_2)$$

represents the entangled state of 2 particles “sharing” the same random variable Z, and hence the same r.f.o.d. D_Z.

It comes from the following definition:

$$\Gamma \vdash \bowtie^2_{x \in D_Z} (A_1; A_2) \quad \equiv \quad \Gamma, z \in D_Z \vdash A_1(z) ,_Z A_2(z)$$

where A_1 and A_2 depend on the same variable z and the indexed comma $,_Z$ indicates the correlation between the two particles.

Example: the Bell's states in $C^2 \otimes C^2$:

$$1/\sqrt{2}|00\rangle \pm 1/\sqrt{2}|11\rangle \qquad 1/\sqrt{2}|01\rangle \pm 1/\sqrt{2}|10\rangle$$

A measurement of one of the two particles determines the symultaeous identical (or opposite) result on the other, and we describe this by the assertion:

$$\Gamma, z \in D_Z \vdash A_1(z),_Z A_2(z)$$

Their representation as proposition has the form

$$\bowtie^2_{x \in D_U} (A_1(x); A_2(x))$$

where $D_U = \{(|0\rangle, 1/2), (|1\rangle, 1/2)\}$.

Note that the domain D_U is "simpler" than the state, since it is the same domain of a particle of C^2. Two particles share the same domain.

Let $D_Z = \{(\xi, p\{Z = \xi\})\}$ a domain where $\xi \in \{|0\rangle, |1\rangle\}$.

We put

$$D_Z^{\perp} \equiv \{(\xi^{\perp}, p\{Z = \xi\})\}$$

where the state $\xi^{\perp}$ is the *NOT* of ξ.

$D_Z^{\perp}$ is the *dual domain* of D_Z.

The proposition with the dual domain

$$(\forall x \in D_Z^{\perp})A(x)$$

denotes the *NOT* of the state denoted by $(\forall x \in D_Z)A(x)$.

1. In which terms can the definition of dual domain extend the usual duality?
2. In which terms is the proposition $(\forall x \in D_Z^{\perp})A(x)$ to be considered a logical negation?

Definitory equations can be put in *symmetric* pairs, as follows:

$$\Gamma \vdash A \circ B \qquad \equiv \qquad \Gamma \vdash A \approx B$$

and

$$A \circ^s B \vdash \Delta \qquad \equiv \qquad A \approx B \vdash \Delta$$

so that logical connectives come out in symmetric pairs $(\circ, \circ^s)$, each pair corresponding to the same metalinguistic link $\approx$: $(\&, \vee)$, $(*, \otimes)$, ... $(\forall, \exists)$.

Then, formally, we have a *symmetric representation* of the state, by the existential quantifier:

$$(\exists x \in D_Z)A(x)$$

Symmetric equations are solved in a symmetric way, finding couples of rules "mirroring each other". So, one finds symmetric sequent calculi (or couples of symmetric sequent calculi) and a *symmetry theorem*:

$$\Pi \text{ proves } \ \Gamma \vdash \Delta \qquad \text{iff} \qquad \Pi^s \text{ proves } \ \Delta^s \vdash \Gamma^s$$

where $p = p^s$ on literals and Π^s has the right/left rule for $\circ^s$ where Π has the left/right rule for $\circ$.

In logic, the symmetry theorem becomes real when it is applied considering a duality $(-)^\perp$:

$$\Gamma \vdash \Delta \qquad \text{iff} \qquad \Delta^\perp \vdash \Gamma^\perp$$

where $p^\perp$ is the negation of p (Girard's duality) and everything else is as for symmetry. Symmetry acts as a real duality on connectives!

If we put:

$A(z)^{\perp} \equiv A(z)$ (where z has its values in $D^{\perp}$ in $A(z)^{\perp}$!)

$A(z/t)^{\perp} \equiv A(z/t^{\perp})$ (where $t^{\perp}$ denotes the element obtained as the *NOT* of the element denoted by t)

$(z \in D)^{\perp} \equiv z \in D^{\perp}$

the dual representation of $(\forall x \in D_Z)A(x)$ is $(\exists x \in D_Z^{\perp})A(x)$.

We have to see that:

$(\forall x \in D_Z^{\perp})A(x)$ is the negation of $(\forall x \in D_Z)A(x)$ (it is consistent with the usual negation)

our position extends the usual propositional duality.

The idea is that the quantum gate *NOT*, applied to sharp states, behaves as the gate *NOT* of a classical computer.

The dual domain of the singleton $\{(|b\rangle, 1)\}$, denoted by s, is the singleton $\{(NOT|b\rangle, 1)\}$. If $s^{\perp}$ denotes its element, the dual of the state $A(s)$ is $A(s^{\perp})$.

The propositions $A(s)$ and $A(s^{\perp})$ are like a couple of propositional literals: p_y and p_n, that can be interpreted as a couple of opposites.

We obtain a primitive negation.

If we put $p_y^{\perp} = p_n$ and conversely, we obtain a consistent extension of the duality theorem.

On the other side, a random first order domain coincides with its opposite when the domain corresponds to an eigenstate of the *NOT* gate.

In C^2, the domain $D_U = \{(|0\rangle, 1/2), (|1\rangle, 1/2)\}$ is equal to its dual.

In $C^2 \otimes C^2$ the Bell's states are representable by means of an entanglement quantifier which has the same domain D_U.

So, in our setting, we have formulae which coincide with their own negation. We can consider them another kind of primitive literals and label them by capital letters U. We term them "uniform literals". It is $U_y \equiv U_n$.

Uniform literals aren't propositional formulae and do not coincide with their symmetric, with the existential quantifier. We must distinguish universal literals $U_\forall$ and existential literals $U_\exists$, where the symmetric of $U_\forall$ is $U_\exists$ and conversely.

On uniform literals duality coincides with symmetry: the dual of $U_\forall$ is $U_\exists$.

The role of symmetry and duality is exchanged for uniform literals!

Since $U_y \equiv U_n$, any U can be considered as asserted and as rejected at the same time.
(This does not mean that the sequent $U \vdash U'$ coincides with the opposite sequent $U' \vdash U$, since from $U \vdash U'$ one has $U'^s \vdash U^s$. They can be distinguished w.r.t. the turnstyle $\vdash$ by symmetry).

Literals U are maximal with respect to this, since any other component of a proof is obtained as a composition of elements which admit a dual different from themselves.

Then it is important to gather as much information as possible in literals U.

In this terms we represent massive quantum parallelism when the computation is a computation of assertions, namely a logical proof.

Thank you for your attention!

For those who like quantum theories of mind:

Psychoanalist Matte Blanco (The Unconscious as infinite sets):

There is the “bivalent mode” for the conscious thinking

There is the “indivisible/symmetric mode” for the unconscious thinking, where “...the opposites merge to sameness”.

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- Sambin G., Battilotti G., Faggian F., (2000) Basic logic: reflection, symmetry, visibility, *The Journal of Symbolic Logic 65*, 979-1013.