Basic logic
and the cube of its extensions

Giulia Battilotti
Dipartimento di Matematica, Università di Siena
Via del Capitano, 15 - 53100 Siena, Italy
Giovanni Sambin
Dipartimento di Matematica P. e A., Università di Padova
Via Belzoni 7, 35131 Padova, Italy
e-mail: giulia,sambin@math.unipd.it

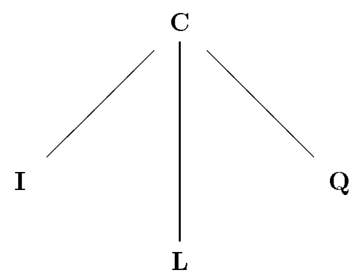
Abstract

A basic logic **B** *is introduced, which is weaker than intuitionistic, quantum and linear logic. Moreover, three independent properties* **C**, **D** *and* **S** *are determined:* **C** *consists of double negation axioms for basic negation,* **D** *consists of usual properties of implication, that is the deduction theorem and the link with usual negation,* **S** *consists of structural rules of weakening and contraction, plus the identification of two constants expressing falsum. The eight possible combinations of properties* **C**, **D** *and* **S** *produce a cube of logics. In particular, adding* **C** *and* **D** *to* **B** *gives linear logic, adding* **D** *and* **S** *gives intuitionistic logic, adding* **C** *and* **S** *gives orthologic and finally adding all of* **C**, **D** *and* **S** *gives classical logic. On the other hand, adding* **C** *to* **B** *produces a new logic, which is the common part of linear and orthologic and thus could be of interest for theoretical physics.*

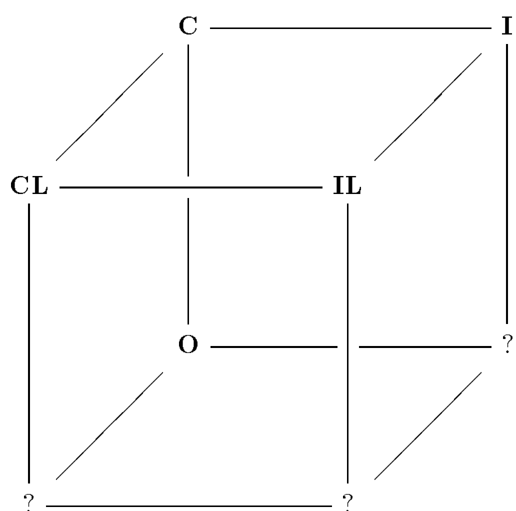
1 Introduction

Up to the end of last century, the only logic was classical logic (and possibly its extensions by modalities). Later some "weakenings" of classical logic were introduced, with the aim of expressing also at the level of logical propositions some distinctions which hold in a specific scientific context but are ignored by classical logic. The first example arises from intuitionism, which points out the distinction, when dealing with infinity, between constructive proofs and proofs based on reductio-ad-absurdum; intuitionistic logic, by rejecting the law of double negation, allows to express such a distinction. In the thirties, it was realized that ortholattices (or orthomodular lattices), rather than boolean algebras, were the convenient algebraic structures to deal with quantum mechanics; thus in orthologic, as well as in ortholattices to which it corresponds, the classical equation given by distributivity of conjunction with disjunction fails. Finally, various motivations lead to the third, more recent "weakening" of classical logic. The philosophical aim of overcoming paradoxes of classical implication produced relevant logics and, later, proof-theoretical motivations and the search for a logic well suited for theoretical computer science, produced linear logic; the common technical aspect is the rejection of one or more structural rules, which results for instance in the distinction made by linear logic between multiplicative and additive conjunction.

Summing up, classical logic has been weakened in three different, fully independent ways. Thus a picture could be:



where **C** is classical logic, **I** intuitionistic logic, **O** orthologic and **CL** Girard's linear logic. Furtherly, a combination of two such weakenings, namely intuitionistic linear logic **IL**, has already been studied. It is then natural to wonder whether also all other combinations, including that with all the three weakenings, produce a "logic", and possibly to determine it. The same question can be expressed as: what should be put for the question marks in the following cube?



We here describe a new logic, which we call basic logic **B**, since it lies at the bottom of the above cube. We adopt the tool of sequent calculus, which has revealed powerful enough to express all the weakness, or, better, the subtleties, of basic logic. Moreover, we specify three properties, namely **C** for classical, **D** for deductive (or distributive) and **S** for structural, such that the cube above coincides with the cube of logics obtained from **B** by adding all combinations of **C**, **D** and **S**, and thus in particular characterize the two remaining question marks. Writing **XY** for the logic obtained by adding property **X** to the logic **Y**, our results can be summarized by the following cube (where $\mathbf{Y} \cong \mathbf{Z}$ means that **Y** and **Z** characterize the same logic):

$$\begin{tikzcd}
& \mathbf{BCDS} \cong \mathbf{LK} \arrow[rr, dash] \arrow[dl, dash] \arrow[dd, dash] & & \mathbf{BDS} \cong \mathbf{LJ} \arrow[dl, dash] \arrow[dd, dash] \\
\mathbf{BCD} \cong \mathbf{CLL_0} \arrow[rr, dash] \arrow[dd, dash] & & \mathbf{BD} \cong \mathbf{ILL_0} \arrow[dd, dash] & \\
& \mathbf{BCS} \cong \mathbf{GO_1} \arrow[rr, dash] \arrow[dl, dash] & & \mathbf{BS} \arrow[dl, dash] \\
\mathbf{BC} \arrow[rr, dash] & & \mathbf{B} &
\end{tikzcd}$$

To understand better what the cube means, assume that we have proved the equivalences written in the above picture. Then it follows that all edges are proper; in fact, it is enough to realize that the top three edges of the cube are proper, that is, that $\mathbf{BCDS} \not\cong \mathbf{BDS}$, $\mathbf{BCDS} \not\cong \mathbf{BCD}$ and $\mathbf{BCDS} \not\cong \mathbf{BCS}$. In fact, from $\mathbf{BCDS} \not\cong \mathbf{BDS}$, for instance, we can conclude also that $\mathbf{BC} \not\cong \mathbf{B}$, $\mathbf{BCS} \not\cong \mathbf{BS}$ and $\mathbf{BCD} \not\cong \mathbf{BD}$ since otherwise by adding one or two properties we would obtain $\mathbf{BCDS} \cong \mathbf{BDS}$, and obviously the same argument applies to all other cases. Now $\mathbf{BCDS} \not\cong \mathbf{BDS}$ is clear, since **BCDS** is equivalent to classical logic **LK** and **BDS** to intuitionistic logic **LJ**, and certainly **LK** and **LJ** are not equivalent. Similarly, $\mathbf{BCDS} \not\cong \mathbf{BCD}$ because **BCD** is equivalent to classical linear logic $\mathbf{CLL_0}$, which certainly is not equivalent to **LK**. Finally, we will prove in the final section that $\mathbf{BCDS} \not\cong \mathbf{BCS}$ making use of the fact that the fragment $\mathbf{BCS}^-$ is equivalent to orthologic $\mathbf{GO^1}$ and orthologic is not equivalent to classical logic. In fact, as it is well known, orthologic was conceived as "the logic of ortholattices", whereas a semantics for classical logic is given by boolean algebras. Moreover, since five out of eight logics of the cube enjoys an algebraic semantics (besides boolean algebras for **C** and ortholattices for **O** one has boolean quantales for **CL**, quantales for **IL** and frames for **I**), a natural question is then whether there is also a notion of basic structure, which should give an algebraic semantics for **B** and produce the well known structures as particular cases, but such idea is not developed here[1].

[1]Actually, the cube itself was first conceived in algebraic terms, our first idea was to

A two-sided sequent calculus for **B** is presented in the first section of the paper. In the subsequent sections we find out what the properties **C**, **D** and **S** should consist of so that the claims contained in the above pictured cube can be proved. The leading principle is that going upwards, i.e. adding one of the three properties, two distinct formulas or rules are turned into equivalent formulas or rules, and thus it may happen also that some distinct connectives are identified. For example, the face of the cube where **C** holds is formed by "classical" logics, which are characterized by the principle of double negation, and each of them is over a corresponding logic in the "intuitionistic" face, where the double negation of a formula A is distinct from A. Similarly, the upper face where **D** holds is formed by "deductive" logics, where a formula can be moved from the assumptions to the conclusions (as the antecedent of an implication) and conversely, and it is above the face of "quantum" logics, where there is no communication between assumptions and conclusions; this, as we will see, will go together with the identification of the primitive negation $A^\star$, defined in basic logic, with negation defined, as usual, $A \rightarrow \perp$. Finally, the face where **S** holds is that of "structural" logics, where structural rules of weakening and contraction hold, and hence only one connective for conjunction is present, is opposed to the face of "linear" logics, where one must distinguish two connectives for conjunction.

The paper is meant as a contribution to the comprehension of propositional extensional logics. By extensional logics we mean here those logics in which there is only one way of asserting a proposition, as opposed to intensional logics, where the assertion "A is true" is accompanied by an assertion like "A is necessarily true" or "A will be true", etc., which at the level of propositions is expressed by means of modalities, $\Box A$, FA, etc. A major philosophical concern should be to characterize the concept of proposition and of proof in a given extensional logic so clearly that the inference rules specific of that logic can be obtained as a consequence. At the moment, this has been achieved satisfactorily only for classical logic, where a proposition is simply a way to denote one of the two truth values, and intuitionistic logic, where the meaning of a proposition is given by its proofs; it seems plausible that propositions of linear logic apply to a less abstract reality, where resources are taken into account (and in fact the interpretations which have been proposed range from chemistry to games, plugs, recipes, computers and restaurant menus), but a clearcut philosophical "definition" is still lacking.

Though we make no step in this direction here, we have it in mind when we show that many different logics can be explained, at least from a proof-theoretical point of view, by means of a few natural properties; then, it is on such properties that further philosophical investigations can concentrate. Moreover,

find a common frame to the representation of quantales via pretopologies (cf. [1]) and the representation of ortholattices via polarities (cf. [2], [3]), since the representation of complete boolean algebras given in [12] seems to be a particular case of both. We don't know yet whether the problem has a simple solution. A talk on this was given in Monselice (Italy), during the 3rd Linear Logic Italian Workshop, October 14-15, 1994, where the cube was presented for the first time.

we propose two new logics, basic logic **B** and its classical counterpart **BC**, which are linear and quantic at the same time, to such a conceptual analysis, hoping that adding consideration of resources to usual quantum logic can be of use in the study of quantum mechanics. In any case, **B** and **BC** seem to meet a desideratum variously expressed in the recent literature (cf. [M], where quantales, later shown to be a complete semantics for linear logic, are proposed as an alternative structure to orthomodular lattices in the study of quantum mechanics, cf. [4] and [6], where a common sublogic of quantum logic and linear logic is described, cf. finally [10], in which linear logic plays the role of a dynamic quantum logic).

A final question is whether weaker logics can be strengthened by adding modalities in such a way that stronger logics can be interpreted in them. It is known, for instance, that linear logic, since its birth, has been equipped with a modality ! which allows to interpret classical and intuitionistic logic. We have been able to find how to interpret each logic of the cube into each weaker one, augmented with suitable modalities. However, the study of interpretations is yet to be completed (for instance we would wish to make all interpretations commute with each other) and will appear in a sequel to the present paper.

1.1 Acknowledgements

We thank John L. Bell for pointing the representation of ortholattices and quantum logic to our attention, Silvio Valentini for useful conversations and suggestions. We are indebted to Marisa Dalla Chiara for encouragement and for inviting us to Florence were on December 19, 1994 we gave the first exposition of the contents of this paper.

2 A sequent calculus B for basic logic

As mentioned in the introduction, basic logic **B** is, so to say, the common denominator of intuitionistic, linear and orthologic. So a sequent calculus for **B** (see the table of rules below) will sum up the peculiarities of a sequent formulation of such three non-classical logics.

Like in intuitionistic logic, the context Δ on the right hand side of a sequent $\Gamma, A \vdash B, \Delta$ is strictly under control: it must be empty (actually, it is enough that Δ is empty only for some of the inference rules, but this is inessential at the moment). For convenience, we choose a formulation in which in addition there is always exactly one formula at the right, thus disregarding sequents like $\Gamma \vdash$ and $\vdash$.

Like in linear logic, in place of the classical conjunction there will be two connectives, characterized by the way in which contexts on the left side are handled; adopting Girard's terminology and notation in [7], they are the multiplicative conjunction $\otimes$ ("times"), characterized by an introductory rule with different contexts in the premises which are just put one aside the other in the conclusion, and the additive conjunction & ("with"), where contexts in the premises and the conclusion are equal. By using only $\otimes$-rules and cut, one can easily check that $\Gamma, A, B \vdash C$ is derivable iff $\Gamma, A \otimes B \vdash C$ is derivable, which means that, like in linear logic, the meaning of the comma "," at the left is $\otimes$, and not & as one is accustomed from classical logic. Like in the so called intuitionistic linear logic (see for instance [11] or [13] for an exposition), it is not possible to introduce multiplicative disjunction, which is defined in classical linear logic either as the dual of $\otimes$ or by allowing two formulas at the right; in fact **B** does not have free contexts on the right and does not satisfy the double negation principle.

The constant atomic formulas are exactly the same as in linear logic, and they are $0, \bot, 1$ and $\top$. The rules for 0 and $\top$ say that they are the minimum and the maximum respectively in the derivability order given by $\vdash$; in particular, $0 \vdash \bot$ and $1 \vdash \top$ hold, while, as we will see, some other relations involving constants which are derivable in linear logic fail in **B**.

Note anyway that **B** is substantially different from (intuitionistic) linear logic, since, contrary to linear logic but like orthologic, it lacks the usual rules for implication and for negation, which allow to move formulas from one side of a sequent to the other. This, however, does not prevent **B** from having a connective $\rightarrow$ for implication and a connective $\star$ for negation that, contrary to usual logics, are not related at all in basic logic. **B** includes a rule for $\rightarrow$ (denoted $\rightarrow U^{\emptyset}$) and one for $\star$ (denoted $\star PI^{\emptyset}$) which impose the behaviour usually expected from any implication and any negation for what concerns the order between formulas given by $\vdash$. The rules $\rightarrow U^{\emptyset}$ and $\star PI^{\emptyset}$ are enough to derive antimonotonicity in the antecedent and monotonicity in the consequent for $\rightarrow$, and "pre-involution" properties for $\star$. Hence they make $\rightarrow$ and $\star$ real connectives, in the sense that a replacement theorem holds or equivalently that the Lindenbaum algebra construction is possible. The common characteristic of $\rightarrow U^{\emptyset}$ and $\star PI^{\emptyset}$ is their limitation to the cases of empty context to the left (in our notation, such limitation is transcribed by the apices "$\emptyset$").

The requirement of empty context on the left is present also in other rules of **B** (namely $\vee L^{\emptyset}$ and $0L^{\emptyset}$) and it is peculiar of any sequent calculus for logics related to quantum mechanics (cf. e.g. [9], [5]). As we shall see, dropping the restriction on left context for $\rightarrow U^{\emptyset}$ or $\star PI^{\emptyset}$ (in our notation, dropping the apex $\emptyset$, getting $\rightarrow U$ or $\star PI$), is enough, if one adds some basic axioms, to get the usual intuitionistic or classical linear implication, respectively. Note finally that $\star$ enjoys also a rule of $\star$-introduction to the left, like in orthologic (cf. [9], [5]).

Summing up, the language of (propositional) basic logic **B** contains four constants for atomic formulas $0, \bot, \top, 1$, one sign for unary connective $\star$, four signs for binary connectives $\otimes, \&, \vee, \rightarrow$. A sequent formulation for **B** is given by the following axioms and rules of inference:

Axioms

$$A \vdash A$$

Rules of inference

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ exchange \qquad \frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\ cut$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ \otimes R \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ \otimes L$$

$$\vdash 1 \ \ 1R \qquad \frac{\Gamma \vdash C}{\Gamma, 1 \vdash C}\ 1L$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}\ \& R \qquad \frac{\Gamma, A \vdash C}{\Gamma, A \& B \vdash C} \quad \frac{\Gamma, B \vdash C}{\Gamma, A \& B \vdash C}\ \& L$$

$$\Gamma \vdash \top \ \ \top R \qquad 0 \vdash C \ \ 0L^{\emptyset}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \vee R \qquad \frac{A \vdash C \quad B \vdash C}{A \vee B \vdash C}\ \vee L^{\emptyset}$$

$$\frac{A \vdash B \quad C \vdash D}{B \rightarrow C \vdash A \rightarrow D}\ \rightarrow U^{\emptyset}$$

$$\frac{A \vdash B^{\star}}{B \vdash A^{\star}}\ \star PI^{\emptyset}$$

$$- - - - \qquad \frac{\Gamma \vdash A}{\Gamma, A^{\star} \vdash \perp}\ \star L$$

It is possible to add also[2] the rules introducing $\rightarrow$ to the left and to the right, as long as the context is kept empty:

$$\frac{A \vdash B}{\vdash A \rightarrow B}\ \rightarrow R^{\emptyset} \qquad \frac{\vdash A \quad B \vdash C}{A \rightarrow B \vdash C}\ \rightarrow L^{\emptyset}$$

Moreover, in the same spirit it is possible to add also[3]:

$$\frac{A \vdash \perp}{\vdash A^{\star}}\ \star R^{\emptyset}$$

We call $\mathbf{B}'$ the calculus obtained from B by adding $\rightarrow R^{\emptyset}, \rightarrow L^{\emptyset}$ and $\star R^{\emptyset}$.

To grasp better the meaning of some of the rules, we propose below some possible alternative formulations of some rules of **B**. The content of the rule $\rightarrow U$ is to impose to the connective $\rightarrow$ the typical behaviour of the implication with respect to the order given by the derivation $\vdash$, as it is clarified by :

[2] Which is the course followed in the actual exposition in Florence at LMPS'95.
[3] Which is the course followed in a lecture given in Göteborg, November 1995.

Proposition 2.1 *The single rule* $\to U^{\emptyset}$ *is equivalent to the following pair of rules, called monotonicity and antimonotonicity for* $\to$*:*

$$\frac{C \vdash D}{A \to C \vdash A \to D}\ \to M^{\emptyset} \qquad \frac{A \vdash B}{B \to C \vdash A \to C}\ \to AM^{\emptyset}$$

Proof. Assume $\to U^{\emptyset}$ holds; then

$$\frac{A \vdash A \qquad C \vdash D}{A \to C \vdash A \to D}\ \to U^{\emptyset} \quad \text{and} \quad \frac{A \vdash B \qquad C \vdash C}{B \to C \vdash A \to C}\ \to U^{\emptyset}$$

are the derivations of antimonotonicity and monotonicity, respectively (note that, in the first derivation, the occurrence of A on the left in the premise becomes the occurrence of A on the right in the conclusion).

Conversely, assume $\to M^{\emptyset}$ and $\to AM^{\emptyset}$ hold. Then we have the following derivation of $\to U^{\emptyset}$:

$$\frac{\dfrac{A \vdash B}{B \to C \vdash A \to C}\ \to AM^{\emptyset} \quad \dfrac{C \vdash D}{A \to C \vdash A \to D}\ \to M^{\emptyset}}{B \to C \vdash A \to D}\ cut$$

□

We will later need also a version of the above proposition in which rules appear with full context; as the reader can easily check, the above proof is immediately extended to a proof of the fact that the rule

$$\frac{\Gamma, A \vdash B \quad \Delta, C \vdash D}{\Gamma, \Delta, B \to C \vdash A \to D}\ \to U$$

is equivalent to the pair of rules

$$\frac{\Gamma, C \vdash D}{\Gamma, A \to C \vdash A \to D}\ \to M \qquad \frac{\Gamma, A \vdash B}{\Delta, B \to C \vdash A \to C}\ \to AM$$

Using only the relation between $\to$ and the order $\vdash$ which is expressed by the rule $\to U^{\emptyset}$, it is possible to show that the rules $\to R^{\emptyset}$ and $\to L^{\emptyset}$ can be expressed as axioms (we will see the analogue of this for $\star$ in proposition 2.4):

Proposition 2.2 *Assuming B, one can prove that:*

a. *The rule* $\to R^{\emptyset}$ *is equivalent to the axiom* $\to RAx$*:* $\vdash A \to A$

b. *The rule* $\to L^{\emptyset}$ *is equivalent to the axiom* $\to LAx$*:* $1 \to C \vdash C$

Proof. a. Applying $\to R^{\emptyset}$ to the basic axiom $A \vdash A$ one obtains $\to RAx$; conversely, by the following derivation:

$$\frac{\vdash A \to A \quad \dfrac{A \vdash B}{A \to A \vdash A \to B}\ \to M^{\emptyset}}{\vdash A \to B}\ cut$$

b. Applying $\rightarrow L^{\emptyset}$ to the basic axioms $\vdash 1$ and $C \vdash C$, one obtains $1 \rightarrow C \vdash C$; conversely, by the following derivation:

$$\dfrac{\dfrac{\dfrac{\vdash A}{1 \vdash A} \quad B \vdash C}{A \rightarrow B \vdash 1 \rightarrow C} \rightarrow U^{\emptyset} \quad 1 \rightarrow C \vdash C}{A \rightarrow B \vdash C}$$

□

Hence both axioms on $\rightarrow$ hold in $\mathbf{B}'$. We shall see in proposition 4.6 how the above equivalences are extended to the case of full contexts.

The unary connective $\star$ may be interpreted as a weak primitive negation. The behaviour of such negation is illustrated by the following facts. The first concerns the behaviour of $\star$ with respect to the order given by $\vdash$ (it is the analogue of prop. 2.1). The crucial rule is $\star PI^{\emptyset}$, where PI stands for pre-involution[4]; to clarify its meaning, let us consider the condition of antimonotonicity for $\star$, given by the rule:

$$\frac{A \vdash B}{B^{\star} \vdash A^{\star}} \star AM^{\emptyset}$$

Then we have:

Proposition 2.3 *The rule $\star PI^{\emptyset}$ is equivalent to the rule $\star AM^{\emptyset}$ together with the axioms $A \vdash A^{\star\star}$.*

Proof. If $\star PI^{\emptyset}$ holds, we get the sequents $A \vdash A^{\star\star}$ by the deduction:

$$\frac{A^{\star} \vdash A^{\star}}{A \vdash A^{\star\star}} \star PI^{\emptyset}$$

and hence we have the following derivation of $\star AM^{\emptyset}$:

$$\dfrac{\dfrac{A \vdash B \quad B \vdash B^{\star\star}}{A \vdash B^{\star\star}} cut}{B^{\star} \vdash A^{\star}} \star PI^{\emptyset}$$

Conversely, we have the derivation:

$$\dfrac{B \vdash B^{\star\star} \quad \dfrac{A \vdash B^{\star}}{B^{\star\star} \vdash A^{\star}} \star AM^{\emptyset}}{B \vdash A^{\star}} cut$$

□

Like 2.1, also proposition 2.3 holds in the case of full-context rules.

As for $\rightarrow$, we can show that the rules for $\star$ with empty context are equivalent to axioms:

[4] A unary operation $\star$ on a lattice is usually said to be an involution if it satisfies $a \leq b \Rightarrow b^{\star} \leq a^{\star}$ and $a^{\star\star} = a$. The rule $\star PI^{\emptyset}$ corresponds to the requirement $a \leq b^{\star} \Rightarrow b \leq a^{\star}$, which is a weaker condition, that we name indeed "pre-involution".

Proposition 2.4 *Assuming* B*, one can prove that:*

a. *The rule* $\star R^{\emptyset}$ *is equivalent to the axiom* $1 \vdash \perp^{\star}$*, or equivalently* $\perp \vdash 1^{\star}$ *(*$\star RAx$*);*

b. *The rule* $\star L^{\emptyset}$ *is equivalent to the axiom* $1^{\star} \vdash \perp$ *(*$\star LAx$*).*

Proof. a. Applying $\star R^{\emptyset}$ to the axiom $\perp \vdash \perp$ one obtains $\vdash \perp^{\star}$, hence $1 \vdash \perp^{\star}$, from which $\perp \vdash 1^{\star}$ by $\star PI^{\emptyset}$. Conversely, by the following derivation:

$$\dfrac{\vdash 1 \qquad \dfrac{1 \vdash \perp^{\star} \qquad \dfrac{A \vdash \perp}{\perp^{\star} \vdash A^{\star}} \star AM^{\emptyset}}{1 \vdash A^{\star}}}{\vdash A^{\star}}$$

b. Applying $\star L^{\emptyset}$ to the axiom $\vdash 1$ one obtains $1^{\star} \vdash \perp$. Conversely, by the following derivation:

$$\dfrac{\dfrac{\dfrac{\vdash A}{1 \vdash A}}{A^{\star} \vdash 1^{\star}} \star AM^{\emptyset} \qquad 1^{\star} \vdash \perp}{A^{\star} \vdash \perp}$$

□

Hence the axiom $1^{\star} \vdash \perp$ holds in **B** and both axioms hold in **B**$'$.

Since in B we have the full context rule $\star L$, we see immediately some of its equivalents:

Proposition 2.5 *In* B*,* $\star L$ *can be replaced by one of the following equivalents:*

NC $\quad \dfrac{\Gamma \vdash A^{\star}}{\Gamma, A \vdash \perp} \qquad NCAx \quad A, A^{\star} \vdash \perp$

In particular, any of the above rules holds in **B***.*

Proof. $\star L$ is equivalent to NCAx because:

$$\dfrac{A \vdash A}{A, A^{\star} \vdash \perp} \star L \quad \text{and} \quad \dfrac{\Gamma \vdash A \qquad A, A^{\star} \vdash \perp}{\Gamma, A^{\star} \vdash \perp} cut$$

Similarly, NC is equivalent to NCAx, by the derivations:

$$\dfrac{A^{\star} \vdash A^{\star}}{A^{\star}, A \vdash \perp} NC \quad \text{and} \quad \dfrac{\Gamma \vdash A^{\star} \qquad A, A^{\star} \vdash \perp}{\Gamma, A \vdash \perp} cut$$

For the sake of completeness, it is possible to prove the equivalence between NC and $\star L$ in a direct way, via $\star PI^{\emptyset}$. We leave it to the reader. □

The absence of a full context $\star R$ in the basic calculus will be explained in proposition 4.10, and its underivability will be proved in proposition 4.7.

3 The classical face

The distinction between a sequent calculus for classical logic, like Gentzen's LK, and for intuitionistic logic, like LJ, is that in the former all rules are formulated with an arbitrary context at the right hand side of any sequent. However, adding a right context to the rules of **B** would be irrelevant to the aim of getting a classical version of **B**. In fact, to have an interpretation of comma on the right hand side of a sequent, one cannot rely on the usual interpretation of the left context allowed by the $\otimes L$ rule, as exemplified in the derivation

$$\cfrac{\cfrac{\cfrac{\Gamma \vdash A, B, \Delta}{\Gamma, A^\star, B^\star \vdash \Delta}\,\star L}{\Gamma, A^\star \otimes B^\star \vdash \Delta}\,\otimes L}{\Gamma \vdash (A^\star \otimes B^\star)^\star, \Delta}\,\star R$$

In fact, **B** lacks any form of $\star$-introduction to the right and we will see in the third section that adding it would strenghten the system up to reach classical linear logic.

Neither one can rely on the usual interpretation allowed by the $\vee R$-rule and by contraction on the right:

$$\cfrac{\cfrac{\cfrac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, B, \Delta}\,\vee R}{\Gamma \vdash A \vee B, A \vee B, \Delta}\,\vee R}{\Gamma \vdash A \vee B, \Delta}\,contr$$

since **B** lacks structural rules. So, allowing context on the right, one should introduce a new primitive connective "par", which would not be linked to $\otimes$ in the usual way of linear logic, because such link is derivable only in presence of both $\star L$ and $\star R$. In any case, adding a full context on the right would not solve, because of the absence of $\star R$, the problem of deriving the double negation principle, i.e. sequents $A^{\star\star} \vdash A$ [5]. Indeed, we shall see soon how adding a $\star R$ rule or structural rules makes the system collapse into other well known systems.[6] So, the axioms $A^{\star\star} \vdash A$ are necessary to get a classical system. Hence here we adopt, as the simplest, the solution of having a double negation principle in a calculus with restriction on the right hand side, and we give the following definition:

Definition 3.1 *We say that a sequent calculus satisfies property* **C** *when the double negation principle holds, i.e.* $A^{\star\star} \vdash A$ *is derivable for any* A. *If* **L** *is any calculus, we write* **CL** *for the calculus obtained by adding* $A^{\star\star} \vdash A$ *as axiom for any* A. *In particular,* **BC** *and* **CB**$'$ *are the classical basic sequent calculi defined*

[5] Cf. [9], [5], where the double negation principle is assumed, even if no problem of interpretation of the context on the right arises, because those systems enjoy structural rules.

[6] To be pedantic, in absence of $\star R$ it is even possible to conceive sequent calculus systems that are structural on the left and linear on the right side or viceversa, but we are not interested at the moment in such solutions.

by all axioms and rules of $\mathbf{B}$ *and* $\mathbf{B}'$*, respectively, and in addition the axioms* $A^{\star\star} \vdash A$.

A proof of the fact that the double negation axioms are not derivable in $\mathbf{B}$ will be provided after proposition 4.4 (while we leave it to be done for $\mathbf{B}'$).

By analogy with proposition 2.3, an alternative to double-negation axioms is given by the following rule $\star I^{\emptyset}$, where I stands for involution:

$$\frac{A^{\star} \vdash B}{B^{\star} \vdash A} \star I^{\emptyset}$$

The next lemma explains why we can not assume $\star I^{\emptyset}$ to hold in B. In fact:

Lemma 3.2 *The rule* $\star I^{\emptyset}$ *is equivalent to the rule* $\star AM^{\emptyset}$ *together with double-negation axioms* $A^{\star\star} \vdash A$.

Proof. Applying $\star I^{\emptyset}$ to the axiom $A^{\star} \vdash A^{\star}$ one obtains $A^{\star\star} \vdash A$; conversely, by the derivation

$$\cfrac{\cfrac{A^{\star\star} \vdash A \quad A \vdash B}{A^{\star\star} \vdash B}}{B^{\star} \vdash A^{\star}}$$

$\square$

Proposition 3.3 *All the following assumptions on the connective* $\star$ *give equivalent formulations of* $\mathbf{BC}$*:*

1. $\star PI^{\emptyset}$ *together with* $A^{\star\star} \vdash A$*, i.e.* $\mathbf{BC}$*;*
2. $\star AM^{\emptyset}$ *together with* $A = A^{\star\star}$*;*
3. $\star I^{\emptyset}$ *together with* $A \vdash A^{\star\star}$*;*
4. $\star AM^{\emptyset}$ *together with the rules*

$$\frac{A \vdash B}{A^{\star\star} \vdash B} \quad \text{and} \quad \frac{A \vdash B^{\star\star}}{A \vdash B}$$

Proof. (1) is equivalent to (2) by proposition 2.3, and (2) is equivalent to (3) by the above lemma. Finally, equivalence of (4) with (2) is obtained easily by cut. $\square$

As usual, the same statements can be proved substituting rules $\star PI^{\emptyset}$ and $\star I^{\emptyset}$ with their full-context versions $\star PI$ and $\star I$.

4 The deductive face

In the basic sequent calculus **B**, and also in its classical version **BC**, the connective $\rightarrow$ does not satisfy the usual characterization of implication, namely the statement of the deduction theorem:

$$\Gamma, A \vdash B \text{ iff } \Gamma \vdash A \rightarrow B \qquad \mathcal{DT}$$

A proof of the fact that the basic system **B** does not enjoy $\mathcal{DT}$ will be provided later in this section, see proposition 4.6. We see now which conditions can be added to **B** or **BC** to obtain the usual rules for implication and thus usual deductive logics. In a sequent calculus, the two rules introducing $\rightarrow$ to the right and to the left are usually of the form

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \rightarrow R$$

and

$$\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \rightarrow B \vdash C} \rightarrow L$$

both in intuitionistic (linear or not) and classical (linear or not) logic. It is now easy to see that $\rightarrow R$ and $\rightarrow L$ together are equivalent to the condition $\mathcal{DT}$ over basic logic. We first have to find some equivalents of $\rightarrow L$:

Proposition 4.1 *All the following assumptions are equivalent over* **B***:*

$\rightarrow L$: $\dfrac{\Gamma \vdash A \qquad B, \Delta \vdash C}{\Gamma, A \rightarrow B, \Delta \vdash C}$

$MPAx$: $A, A \rightarrow B \vdash B$ *(Modus Ponens as an axiom)*

MP: $\dfrac{\Gamma \vdash A \qquad \Delta \vdash A \rightarrow B}{\Gamma, \Delta \vdash B}$ *(Modus Ponens)*

ML: $\dfrac{\Gamma \vdash A \rightarrow B}{\Gamma, A \vdash B}$ *("Move Left")*

Proof. We first prove that $\rightarrow L$ is equivalent to $MPAx$; in fact, a derivation of $MPAx$ from $\rightarrow L$ is

$$\frac{A \vdash A \quad B \vdash B}{A, A \rightarrow B \vdash B} \rightarrow L$$

and conversely

$$\cfrac{\cfrac{\Gamma \vdash A \quad A, A \rightarrow B \vdash B}{\Gamma, A \rightarrow B \vdash B}\, cut \qquad \Delta, B \vdash C}{\Gamma, A \rightarrow B, \Delta \vdash C}\, cut$$

Now we see that MP is derivable from $MPAx$:

$$\cfrac{\Gamma \vdash A \qquad \cfrac{\Delta \vdash A \rightarrow B \quad A, A\rightarrow B \vdash B}{\Delta, A \vdash B}\ cut}{\Gamma, \Delta \vdash B}\ cut$$

and that ML is derivable from MP:

$$\frac{A \vdash A \quad \Gamma \vdash A\rightarrow B}{\Gamma, A \vdash B}\ MP$$

To complete the proof, it is enough to note that:

$$\frac{A\rightarrow B \vdash A\rightarrow B}{A\rightarrow B, A \vdash B}$$

is a derivation of $MPAx$ from ML. □

Now one can easily see that:

Proposition 4.2 *The following assumptions are equivalent over* **B***:*

The rules $\rightarrow R$ *and* $\rightarrow L$ *hold*

The characterization $\mathcal{DT}$ *holds.*

Proof. One direction of $\mathcal{DT}$ is exactly $\rightarrow R$, the other is exactly ML, which is equivalent to $\rightarrow L$ by the preceding proposition. □

A consequence of the above proposition 4.2 is that, if one added both any equivalent to $\rightarrow R$ and any equivalent to $\rightarrow L$ to **B**, one would obtain a calculus at least as strong as the linear commutative intuitionistic sequent calculus without exponentials $\mathbf{ILL_0}$, as formulated e.g. in [13]. On one hand, the only rules of $\mathbf{ILL_0}$ not appearing inside **B**, namely $\rightarrow R$, $\rightarrow L$, $\vee L$ and $0L$, hold in any extension of **B** satisfying condition $\mathcal{DT}$; in fact, $\rightarrow R$ and $\rightarrow L$ are derivable by proposition 4.2, and the full context rules $\vee L$ and $0L$ are derivable from their weak form $\vee L^{\emptyset}$ and $0L^{\emptyset}$, respectively, because by using $\rightarrow R$, $\rightarrow L$, $\otimes R$ and *cut*, the context can be moved from the left to the right and conversely, i.e. $\Gamma, A \vdash C$ iff $A \vdash \otimes\Gamma\rightarrow C$ holds. On the other hand, all rules of **B** and $\mathbf{B}'$, except those for $\rightarrow$ and $\star$, trivially hold in $\mathbf{ILL_0}$; since $\rightarrow R$ and $\rightarrow L$ are assumed in $\mathbf{ILL_0}$, $\rightarrow R^{\emptyset}$ and $\rightarrow L^{\emptyset}$ trivially hold, while $\rightarrow U^{\emptyset}$ is easily derivable, moreover, by interpreting the negation $A^{\star}$ as the linear negation $A\rightarrow \perp$, usually shorthanded as $A^{\perp}$, it is immediate to check that $\mathbf{ILL_0}$ enjoys $\star$ rules of **B**.

Hence, we adopt the following characterization of property **D**; here and in the sequel we write $A = B$ when A is equivalent to B, in the sense that $A \vdash B$ and $B \vdash A$ hold.

Definition 4.3 *We say that a sequent calculus is deductive, or distributive,*[7] *if it satisfies property* **D***, that is* $\rightarrow$ *satisfies* $\mathcal{DT}$ *and for any formula* A, $A^{\star} = A\rightarrow \perp$. *Moreover, we adopt for property* **D** *the same convention as for property* **C***; in particular,* **BD** *and* $\mathbf{DB}'$ *are the deductive basic calculi.*

[7] In fact, in a complete lattice the definability of a binary connective for implication is equivalent to distributivity.

Note anyway that adding **D** vanishes the distinction between **B** and **B′**, i.e. **BD** = **DB′**, in fact, not only $\rightarrow$ rules are available, but also $\star R$, because property **D** includes the fact that $\star$ is definable in terms of $\rightarrow$. What we have shown above definition 4.3 is:

Proposition 4.4 *The system* **BD** *is equivalent to the system of intuitionistic linear logic (without exponentials)* $\mathbf{ILL_0}$.

As a corollary, one can see that **B** is strictly weaker than **BC**; in fact, if one had a derivation of $A^{\star\star} \vdash A$ in **B**, one would derive also $A^{\perp\perp} \vdash A$ in $\mathbf{ILL_0}$. Another corollary is the following:

Proposition 4.5 *The system* **BCD** *is equivalent to the system of classical linear logic (without exponentials)* $\mathbf{CLL_0}$.

Proof. One can immediately see that **BCD** is equivalent to $\mathbf{ILL_0}$ augmented with the double negation axioms $A^{\perp\perp} \vdash A$. In fact, **BD** is equivalent to $\mathbf{ILL_0}$ by the previous proposition, while property C, in presence of D, is exactly $A^{\perp\perp} \vdash A$. To conclude, it is enough to notice that $\mathbf{ILL_0}$ added with the axioms $A^{\perp\perp} \vdash A$ is equivalent to $\mathbf{CLL_0}$; this is an exercise out of the scope of the present paper. □

Now one can see how the presence of a context in the rule $\rightarrow U$ is linked to deductivity:

Proposition 4.6 *The following are equivalent over* **B**:

The rule $\rightarrow U$ *with the axioms* $\rightarrow RAx$ *and* $\rightarrow LAx$.

The rules $\rightarrow R$ *and* $\rightarrow L$.

Proof. Assuming (1), one has the following derivation of $\rightarrow R$ from $\rightarrow M$ and $\rightarrow RAx$:

$$\dfrac{\vdash B\rightarrow B \qquad \dfrac{\Gamma, B \vdash C}{\Gamma, B\rightarrow B \vdash B\rightarrow C}\,\rightarrow M}{\Gamma \vdash B\rightarrow C}\,cut$$

and one has the following derivation of $\rightarrow L$ via $\rightarrow U$ and $\rightarrow LAx$:

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash A}{\Gamma, 1 \vdash A}\,1L \qquad \Delta, B \vdash C}{\Gamma, \Delta, A\rightarrow B \vdash 1\rightarrow C}\,\rightarrow U \qquad 1\rightarrow C \vdash C}{\Gamma, \Delta, A\rightarrow B \vdash C}\,cut$$

Conversely, assuming (2), one can derive the axioms by proposition 2.2 and the rule $\rightarrow U$ as follows:

$$\dfrac{\dfrac{\Gamma, A \vdash B \qquad \Delta, C \vdash D}{\Gamma, \Delta, A, B\rightarrow C \vdash D}\,\rightarrow L}{\Gamma, \Delta, B\rightarrow C \vdash A\rightarrow D}\,\rightarrow R$$

□

Now we see semantically why $\mathbf{B}'$ is a proper enrichment of $\mathbf{B}$. The facts that $\mathbf{B}'$ is strictly weaker than $\mathbf{BD}$ and $\mathbf{CB}'$ is strictly weaker than $\mathbf{DCB}'$ will be shown at the end of next section.

To obtain independence of $\rightarrow Rax$ and $\rightarrow Lax$ over $\mathbf{B}$, we consider an algebraic model of $\mathbf{B}$ in which they fail. Let $\widetilde{Z} = Z \cup \{-\infty, +\infty\}$ be the commutative boolean quantale obtained completing the set of integer numbers Z with respect to its order and then putting $+\infty + -\infty = 0$, $+\infty + n = +\infty$ and $-\infty + n = -\infty$, for every $n \in Z$. Let us consider the interpretation V of formulas of $\mathbf{B}$ into $\widetilde{Z}$ defined by the clauses (warning: of course, such interpretation is different from the interpretation of linear logic in a quantale):

$V(0) \equiv +\infty$, $V(\top) \equiv -\infty$, $V(1) \equiv 0$, $V(1^\star) \equiv z$, where z is an integer, and $V(\perp) \equiv n$, where n is a fixed integer $\leq z$;

$V(B \otimes C) \equiv V(B) + V(C)$;

$V(B \wedge C) \equiv max\{V(B), V(C)\}$;

$V(B \vee C) \equiv min\{V(B), V(C)\}$;

$V(B^\star) \equiv -V(B) + z$;

$V(B \rightarrow C) \equiv 1 + V(C) - V(B)$.

Similarly, let W be the interpretation defined by the clauses:

$W(0) \equiv -\infty$, $W(\top) \equiv +\infty$, $W(1) \equiv 0$, $W(1^\star) \equiv z$, where z is an integer, and $W(\perp) \equiv n$, where n is a fixed integer $\geq z$;

$W(B \otimes C) \equiv W(B) + W(C)$;

$W(B \wedge C) \equiv min\{W(B), W(C)\}$;

$W(B \vee C) \equiv max\{W(B), W(C)\}$;

$W(B^\star) \equiv -W(B) + z$;

$W(B \rightarrow C) \equiv 1 + W(C) - W(B)$.

Hence W differs from V in the valuation of the additive fragment, and possibly of $\perp$. Of course, if $\Gamma \equiv \{C_i\}_{i \in I}$, $V(\Gamma)$ and $W(\Gamma)$ are to be interpreted as $\Sigma_{i \in I} V(C_i)$ and $\Sigma_{i \in I} W(C_i)$. So, when I is empty, $V(\Gamma) = W(\Gamma) = 0$. By properties of ordered groups, it is easy to verify that, for every axiom $\Delta \vdash B$ of $\mathbf{B}$ one has $V(\Delta) \geq V(B)$ and $W(\Delta) \leq W(B)$, and that $\geq$ inequalities are preserved by rules of $\mathbf{B}$ when formulae are interpreted by valuation V, as well as $\leq$ inequalities are preserved by rules of $\mathbf{B}$ when formulae are interpreted by valuation W. Hence we have:

$$\Gamma \vdash C \Rightarrow V(\Gamma) \geq V(C) \qquad\qquad \Gamma \vdash C \Rightarrow W(\Gamma) \leq W(C)$$

for every theorem $\Gamma \vdash C$ of $\mathbf{B}$. Finally, note that, by definition of V and W, the double negation axioms $A^{\star\star} \vdash A$ are validated, and hence $\widetilde{Z}$ with interpretations V and W are models of $\mathbf{BC}$ too.

It is now easy to prove that:

Proposition 4.7 *None of the following is derivable in* **BC**, *and a fortiori in* B:

1. *the sequents* $\rightarrow RAx$ *and* $\rightarrow Lax$;
2. *the equivalence of* $A^\star$ *with* $A\rightarrow \perp$;
3. *the axiom* $\perp\vdash 1^\star$.

Proof. (1) If $\rightarrow RAx$ or $\rightarrow LAx$ were theorems of **B**, it would be $V(1) \equiv 0 \geq V(C\rightarrow C) = 1$ or $W(1\rightarrow C) = 1 + W(C) \leq W(C)$, respectively. (2) The equivalence of $A^\star$ with $A\rightarrow \perp$ would give $-V(A)+z = -V(A)+1+V(\perp)$, which is equivalent to $1 + V(\perp) = z$; hence it is enough to choose $V(\perp) \neq -1 + z$. (3) It is possible to choose V so that $V(\perp) < V(1^\star) \equiv z$. □

Hence **B** is weaker than **B**′ and **BC** than **CB**′, so a fortiori we have:

Proposition 4.8 *None of the conditions forming property* **D** *holds in* **BC** *or* **B**. *So* **B** *and* **BC** *are not deductive.*

A natural question is then whether, in the case of **BC** and other classical logics, a connective of implication is definable by means of other connectives, and whether it satisfies property **D**. Since no connective interpreting the comma at the right is available in **BC**, we are lead to define classical implication by means of negation and the conjunction interpreting comma at the left, namely $\otimes$. We thus put

$$A \supset B \equiv (A \otimes B^\star)^\star$$

which is exactly one of the usual characterizations of implication in classical linear logic. It is easy to show that $\supset$ is antimonotonic in the first argument and monotonic in the second argument, that is, by proposition 2.1, that it satisfies $\supset U^\emptyset$ in B; a derivation is:

$$\dfrac{\dfrac{A \vdash B \quad \dfrac{C \vdash D}{D^\star \vdash C^\star}\ \star AM^\emptyset}{A \otimes D^\star \vdash B \otimes C^\star}}{(B \otimes C^\star)^\star \vdash (A \otimes D^\star)^\star}\ \star AM^\emptyset$$

We thus may consider $\supset$ as the basic classical implication.

Contrary to what happens in classical linear logic, however, the system **BC** is not strong enough to prove that $\supset$ satisfies property D, namely the condition $(\mathcal{DT})$ for $\supset$:

$$\Gamma, A \vdash B \qquad \text{iff} \qquad \Gamma \vdash A \supset B$$

and the condition $A^\star = A \supset\perp$. As for this last condition, we have the following lemma:

Proposition 4.9 *In* **BC** *one can prove:*

a. $\star R^\emptyset$ *is equivalent to* $A \supset\perp\vdash A^\star$

b. $\star L^\emptyset$ *is equivalent to* $A^\star \vdash A \supset\perp$

Proof. Assuming $\star R^{\emptyset}$ and $\star L^{\emptyset}$, respectively, one has the following derivations:

$$\dfrac{\dfrac{A \vdash A \quad \dfrac{\perp\vdash\perp}{\vdash\perp^{\star}}\star R^{\emptyset}}{A \vdash A\otimes \perp^{\star}}\otimes R}{(A\otimes \perp^{\star})^{\star} \vdash A^{\star}}\star AM^{\emptyset} \qquad \dfrac{\dfrac{A \vdash A \quad \dfrac{\dfrac{\vdash 1}{1^{\star}\vdash\perp}}{\perp^{\star}\vdash 1}\star I^{\emptyset}}{A\otimes \perp^{\star}\vdash A}\otimes R}{A^{\star} \vdash (A\otimes \perp^{\star})^{\star}}\star AM^{\emptyset}$$

The two converse directions of a. and b. are obtained by putting $A = 1$ in the sequents $(A\otimes \perp^{\star})^{\star} \vdash A^{\star}$ and $A^{\star} \vdash (A\otimes \perp^{\star})^{\star}$ respectively, thus getting $\perp\vdash 1^{\star}$ and $1^{\star} \vdash\perp$, respectively, that are equivalent to $\star R^{\emptyset}$ and $\star L^{\emptyset}$ by proposition 2.4. $\square$

¿From this we have the following characterization of condition **D** for $\supset$:

Proposition 4.10 *In CB, the following are equivalent:*

1. $\supset R$ *and* $\star R^{\emptyset}$*;*
2. $\star R$*;*
3. *classical implication* $\supset$ *satisfies D.*

Proof. Assuming (1), from $\Gamma, A \vdash\perp$ one has $\Gamma \vdash A \supset\perp$ by $\supset R$, hence also $\Gamma \vdash A^{\star}$ since $A \supset\perp\vdash A^{\star}$ by the above proposition, so $\star R$ holds.

Assuming $\star R$, one has the following derivations for $(\mathcal{DT})$:

$$\dfrac{\dfrac{\dfrac{\Gamma, A \vdash B}{\Gamma, A, B^{\star} \vdash\perp}\star L}{\Gamma, A \otimes B^{\star} \vdash\perp}\otimes L}{\Gamma \vdash A \supset B}\star R \qquad \dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad B^{\star} \vdash B^{\star}}{A, B^{\star} \vdash A \otimes B^{\star}} \quad \dfrac{\Gamma \vdash A \supset B}{\Gamma, A \otimes B^{\star} \vdash\perp}\star NC}{\Gamma, A, B^{\star} \vdash\perp}cut}{\Gamma, A \vdash B^{\star\star}}\star R \quad B^{\star\star} \vdash B}{\Gamma, A \vdash B}cut$$

where $\star NC$ is an equivalent of $\star L$, as seen in 2.5. Moreover, $A^{\star} = A \supset\perp$ is an equivalent of $\star R^{\emptyset}$ and $\star L^{\emptyset}$ by the previous proposition, hence it holds in **BC** with $\star R$. Finally, assuming (3), (1) is obvious by definition of $\mathcal{DT}$ and proposition 4.9. $\square$

As a consequence of propositions 4.7 and 4.10, we have:

Proposition 4.11 *Property D fails for* $\supset$ *in* **BC**.

In the last section we will see that property **D** for $\supset$ fails also in **CB**$'$.

In **BC** there are two different connectives for implication, $\rightarrow$ and $\supset$, which have little to do with each other. In fact, it is easy to see that $A{\rightarrow}B \not\vdash A \supset B$ and $A \supset B \not\vdash A{\rightarrow}B$ by means of the interpretations W and V, respectively. This is possible since $\mathcal{DT}$ does not hold; in fact, it is easy to check that, in general, two connectives for implication coincide if they both satisfy $\mathcal{DT}$.

The following fact was implicit in proposition 4.5; here we give a direct proof:

Proposition 4.12 *In* **BCD**, $\supset$ *satisfies D, and hence it coincides with* $\rightarrow$.

Proof. The assumption $\Gamma, A \vdash \bot$ gives $\Gamma \vdash A \rightarrow \bot$ by $\rightarrow R$, but $A \rightarrow \bot = A^\star$ by D, hence $\Gamma \vdash A^\star$; so $\star R$ holds. Then, by proposition 4.10, $\supset$ too satisfies D. The last statement follows by the above remark. $\Box$

The following is analogous to 4.6 for the case of classical implication; it shows how a full context in $\star AM$ and possibly two axioms on $\star$ give a deductive classical logic.

Proposition 4.13 *In* **BC** *the following are equivalent:*

1. $\star AM$ *with* $\bot \vdash 1^\star$ *and* $1^\star \vdash \bot$*;*

2. $\star R$ *together with* $\star L$*.*

Proof. Assuming (1), and recalling that $\bot \vdash 1^\star$ if and only if $1 \vdash \bot^\star$ by $\star PI^\emptyset$ and if and only if $\vdash \bot^\star$ by cut, one has (2) by the following derivations:

$$\dfrac{\vdash \bot^\star \quad \dfrac{\Gamma, A \vdash \bot}{\Gamma, \bot^\star \vdash A^\star}\,\star AM}{\Gamma \vdash A^\star} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash A}{\Gamma, 1 \vdash A}}{\Gamma, A^\star \vdash 1^\star}\,\star AM \quad 1^\star \vdash \bot}{\Gamma, A^\star \vdash \bot}\,cut}{}$$

Conversely, one gets the axioms by proposition 2.4 and the rule $\star AM$ by the derivation:

$$\dfrac{\dfrac{\Gamma, A \vdash B}{\Gamma, A, B^\star \vdash \bot}\,\star L}{\Gamma, B^\star \vdash A^\star}\,\star R$$

$\Box$

5 The structural face

It is well known that adding structural rules to intuitionistic and classical linear logic, i.e. the systems $\mathbf{BD} \cong \mathbf{ILL_0}$ and $\mathbf{BCD} \cong \mathbf{CLL_0}$, produces intuitionistic and classical logic, respectively. Now we see which systems arise adding structural rules to B and **BC**.

The usual formulation of structural rules in an intuitionistic sequent calculus is the following

$$W: \ \dfrac{\Gamma \vdash C}{\Gamma, A \vdash C} \qquad \text{(weakening)}$$

$$C: \ \dfrac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C} \qquad \text{(contraction)}$$

In propositions 4.1 and 4.2, we showed that structural rules are not needed to prove the equivalence of different formulations of properties of implication; we now show the reciprocal fact, namely that the various formulations of structural rules are already equivalent in B, and thus in absence of usual implication.

Proposition 5.1 *The structural rule of weakening W is equivalent over B to any of the following axioms: $A \otimes B \vdash A\&B$; $A, B \vdash B$; $A \vdash 1$; $\top \vdash 1$. Similarly, the rule C is equivalent over B to any of the axioms: $A\&B \vdash A \otimes B$; $A \vdash A \otimes A$. Hence, the rules W and C together are equivalent over B to $A \otimes B = A\&B$.*

Proof. Assuming W, a derivation of $A \otimes B \vdash A\&B$ is

$$\dfrac{\dfrac{\dfrac{A \vdash A}{A, B \vdash A}\,W \quad \dfrac{B \vdash B}{A, B \vdash B}\,W}{A, B \vdash A\&B}}{A \otimes B \vdash A\&B}$$

From $A \otimes B \vdash A\&B$ and $A\&B \vdash B$, one obtains $A \otimes B \vdash B$ and hence the axiom $A, B \vdash B$. Then, by taking B to be 1 in $A, B \vdash B$, one obtains $A, 1 \vdash 1$ and hence, from $\vdash 1$ by cut, also $A \vdash 1$; $\top \vdash 1$ is then a special case. Finally, a proof of W from $\top \vdash 1$ is

$$\dfrac{\dfrac{A \vdash \top \quad \top \vdash 1}{A \vdash 1} \quad \dfrac{\Gamma \vdash C}{\Gamma, 1 \vdash C}}{\Gamma, A \vdash C}\,cut$$

Assuming C, we have

$$\dfrac{\dfrac{A\&B \vdash A \quad A\&B \vdash B}{A\&B, A\&B \vdash A \otimes B}\,\otimes R}{A\&B \vdash A \otimes B}\,C$$

and from $A\&B \vdash A \otimes B$ we have as a special case $A\&A \vdash A \otimes A$ from which $A \vdash A \otimes A$. Finally, assuming the axioms $A \vdash A \otimes A$,

$$\dfrac{A \vdash A \otimes A \quad \dfrac{\Gamma, A, A \vdash C}{\Gamma, A \otimes A \vdash C}\,\otimes L}{\Gamma, A \vdash C}\,cut$$

is a derivation of contraction. □

The structural rules of weakening and contraction do not exhaust the conditions we must require for the property S which characterizes structural logics. In fact, among structural logics there is intuitionistic logic which we expect to obtain by adding properties S and D to B. In intuitionistic logic, the negation of A, which is defined as usual by $\neg A \equiv A \rightarrow \bot$, must satisfy *ex falso quodlibet*, that is the axiom $A, \neg A \vdash C$ or, equivalently, $\bot \vdash C$; since the constant 0 is already present in the language of B, and with the rule $0 \vdash C$, to obtain *ex falso quodlibet*, it is enough to require $\bot \vdash 0$, or equivalently $\bot = 0$. We thus add $\bot = 0$ to the structural rules W and C to form property S.

Note that it is not possible to put $\bot = 0$ among the conditions defining property D. In fact, the rule of weakening is derivable over **BCD** from $\bot \vdash 0$ as follows: from $\bot \vdash 0$ and $0 \vdash \top^\star$ one has $\bot \vdash \top^\star$, hence $1^\star \vdash \top^\star$ since $1^\star \vdash \bot$ holds in B, and so finally $\top \vdash 1$, which is equivalent to weakening by 5.1. So, if

$\perp = 0$ were derivable from D, we would obtain that **BCD** satisfies weakening, contrary to the expectation that it is equivalent to classical linear logic.

To the same conclusion we would be lead by requiring that adding properties C and S to B we should obtain orthologic. In fact, in orthologic the axiom $A, A^\star \vdash 0$ must hold (since it corresponds to $a \wedge a^\star = 0$ which holds in an ortholattice) and it is obtained from $A, A^\star \vdash \perp$ by requiring $\perp = 0$. On the other hand, by the same reason as above, we could not put $\perp = 0$ among the conditions forming property C.

In conclusion, we put:

Definition 5.2 *We say that a sequent calculus satisfies the property* **S** *when it satisfies weakening, contraction and* $\perp = 0$. *We call structural any system satisfying* **S**, *in particular* **BS** *and* **SB**$'$ *are the basic structural systems.*

In **BS**, since $\perp = 0$ holds, the non-contradiction principle expressed by $NCAx$ $A, A^\star \vdash \perp$ becomes $A, A^\star \vdash C$, like in intuitionistic logic. Moreover, like in intuitionistic and classical logic, the conjunctions $\otimes$ and & coincide by proposition 5.1. In addition, the structure of constants becomes exactly that of intuitionistic logic, as shown by:

Proposition 5.3 *In* **BS**, *up to equivalence, there are only two constants,* $\perp$ *and* $\top$, *and they satisfy* $\perp^\star = \top$ *and* $\top^\star = \perp$.

Moreover, the condition $A^\star = A \supset \perp$ *of property* D *is derivable.*

Proof. By definition of S, $\perp = 0$. By proposition 5.1, weakening is equivalent to $\top \vdash 1$, while $1 \vdash \top$ holds by $\top R$, and hence $1 = \top$. The equivalence $\top = 0^\star$ hold also in B; in fact, $0^\star \vdash \top$ by $\top R$, and $\top^\star \vdash 0$ by $\star PI^\emptyset$ from $0 \vdash \top^\star$. Hence also $\perp^\star = \top$. To show $\top^\star = \perp$, it is enough to recall that $1^\star \vdash \perp$ holds in B, hence $\top^\star \vdash \perp$ holds in **BS**, while $\perp \vdash \top^\star$ by $0\,L$ and $0 = \perp$.

Finally, $A \supset \perp = A \supset 1^\star$ because $1^\star = \top^\star = \perp$, and $A \supset 1^\star \equiv (A \otimes 1^{\star\star})^\star = (A \otimes 1)^\star = A^\star$ because $1^{\star\star} = \top^{\star\star} = \perp^\star = \top = 1$. $\square$

As a consequence, one can see that **BS** can be characterized by the axioms $A \vdash A$, $\perp \vdash 0$ and the inference rules *exchange*, W, C, *cut*, $\&R$, $\&L$, $0L$, $\vee R$, $\vee L^\emptyset$, $\rightarrow U^\emptyset$, $\star PI^\emptyset$, $\star L$.

If one adds also property **D** to **BS**, one has, by $\mathcal{DT}$:

$$A\&B \vdash C \text{ if and only if } A \vdash B \rightarrow C,$$

so that $\rightarrow$ is the intuitionistic implication, and hence $A^\star = A \rightarrow \perp$ tells that $A^\star$ is the same as the intuitionistic negation. Then, after proposition 4.4, it is immediate that every rule of **BDS** is derivable in **LJ** (after putting $\top \equiv A \rightarrow A$, $0 \equiv \perp$ and $A^\star \equiv A \rightarrow \perp$), and conversely. So we have the statement:

Proposition 5.4 *The system* **BDS** *is equivalent to Gentzen's intuitionistic calculus* **LJ**.

Then it is also clear that:

Proposition 5.5 *The system* **BCDS** *is equivalent to Gentzen's classical calculus* **LK**.

To complete the proof of the claim pictured in the cube, we must verify that the structural classical basic systems **BCS** and **CB**$'$, defined, of course, by the axioms and rules of **BS** and **SB**$'$, plus the double negation axioms $A^{\star\star} \vdash A$, give a calculus for orthologic. Actually, **BCS** contains an extra connective, namely the connective $\rightarrow$, which is not reducible to the connectives corresponding to operations in an ortholattice. So let us call $\mathbf{SCB}^-$ and $\mathbf{SCB}'^-$ the systems obtained from **BCS** and **BCS**$'$, respectively, by suppressing the connective $\rightarrow$ and its rules. First, note that $\star R^{\emptyset}$ holds in **BCS** and hence **BCS**$'$, by propositions 5.3 and 2.4. Hence $\mathbf{SCB}^-$ and $\mathbf{SCB}'^-$ are the same and we will deal only with $\mathbf{SCB}^-$.

We prove that $\mathbf{SCB}^-$ is equivalent to system $\mathbf{GO^1}$, i.e. Nishimura system **GO** (cf. [9]), restricted to the normal form sequents, that is, sequents with at most one formula at the right. This is enough because, as proved in [9], the logic of ortholattices (as formulated for instance in [8]), is given by the normal sequents of calculus **GO**, and every normal sequent has a normal proof, i.e. one containing only normal sequents.

We write below the set of rules for system $\mathbf{GO^1}$; the sequents are of the form $\Gamma \Rightarrow \Delta$, where Γ is a finite set of formulas and Δ is a singleton $\{A\}$ or the emptyset. By a little abuse of notation, we write $\Gamma \Rightarrow A$ instead of $\Gamma \Rightarrow \{A\}$.

Axioms

$$A \Rightarrow A$$

Rules of inference

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Delta}\ L\ extension \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}\ R\ extension$$

$$\frac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi}\ cut$$

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \quad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ \wedge \Rightarrow \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \Rightarrow \wedge$$

$$\frac{\Gamma \Rightarrow A}{\Gamma, \neg A \Rightarrow}\ \neg \Rightarrow$$

$$\frac{A \Rightarrow \Delta}{\neg\Delta \Rightarrow \neg A}\ \Rightarrow \neg$$

$$\frac{A, \Gamma \Rightarrow \Delta}{\neg\neg A, \Gamma \Rightarrow \Delta}\ \neg\neg \Rightarrow \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \neg\neg A}\ \Rightarrow \neg\neg$$

Proposition 5.6 *The systems* $\mathbf{GO^1}$ *and* $\mathbf{SCB}^-$ *are equivalent.*

Proof. Apart from differences in notation, the differences between $\mathbf{GO^1}$ and $\mathbf{SCB}^-$ are essentially two. One one side, $\mathbf{GO^1}$ allows sequents with empty succedent $\Gamma \Rightarrow$ and we interpret them as sequents $\Gamma \vdash \perp$ in $\mathbf{SCB}^-$. On the other side, $\mathbf{GO^1}$ lacks disjunction $\vee$ any any constants, and we define them by putting $A \vee B = (A^\star \wedge B^\star)^\star$, $\bot \equiv A \wedge \neg A$ and $\top \equiv \neg(A \wedge \neg A)$.

Then it is not difficult to show that, under such interpretation and with such definitions, all rules of $\mathbf{GO^1}$ are derivable in $\mathbf{SCB}^-$, and conversely. $\Box$

A subtle argument now is needed to show that the full system **BCS** is strictly weaker than classical logic, i.e. **BCDS**. In fact, what is well known is that $\mathbf{GO^1}$, and hence $\mathbf{SCB}^-$, is strictly weaker than classical logic. To be able to use this fact, we need:

Proposition 5.7 **BCS** *is conservative over* $\mathbf{SCB}^-$*, in the strong sense that if* $\Gamma \vdash C$ *is derivable from the assumptions* $\Gamma_1 \vdash C_1, \ldots, \Gamma_n \vdash C_n$ *in* **BCS** *and all of* $\Gamma \vdash C$*,* $\Gamma_i \vdash C_i$ *do not contain occurrences of* $\rightarrow$*, then* $\Gamma \vdash C$ *is derivable from* $\Gamma_1 \vdash C_1, \ldots, \Gamma_n \vdash C_n$ *also in* $\mathbf{SCB}^-$*.*

Similarly, $\mathbf{BCS}'$ *is conservative over* $\mathbf{SCB}'^- = \mathbf{SCB}^-$*.*

Proof. Given a derivation Π of $\Gamma \vdash C$ from $\Gamma_1 \vdash C_1, \ldots, \Gamma_n \vdash C_n$, it is enough to substitute each occurrence of $\rightarrow$ in Π with classical implication $\supset$. Thus, since the rule $\supset U^\emptyset$ is derivable in B and $\rightarrow$ does not appear in $\Gamma \vdash C$, $\Gamma_i \vdash C_i$, one obtains a derivation Π' in $\mathbf{SCB}^-$ with equal assumptions and conclusion.

Similarly, classical implication $\supset$ satisfies the rules $\supset L^\emptyset$ and $\supset R^\emptyset$. In fact, by proposition 2.2, $\supset L^\emptyset$ is equivalent to $1 \supset C \vdash C$, which is derivable from $C^{\star\star} \vdash C$ because $1 \& C^\star = C^\star$; again by 2.2 $\supset R^\emptyset$ is equivalent to $\vdash C \supset C$, which is derivable in $\mathbf{SCB}^-$ since $C \& C^\star \vdash 1^\star$, which holds by 2.5 and 5.3, gives $1 \vdash (C \& C^\star)^\star$, by $\star PI^\emptyset$. So $\mathbf{BCS}'$ is conservative over $\mathbf{SCB}^-$ $\Box$

We now can conclude the argument as follows. Assume that $\mathbf{BCS} \cong \mathbf{BCDS}$, that is, assume that property D were derivable for $\rightarrow$ in **BCS**. Then, as already seen in the proof of proposition 4.12, also $\star R$ would be derivable in **BCS**, hence, by the preceding proposition, $\star R$ would be derivable also in $\mathbf{SCB}^-$, and so, by proposition 4.10, $\mathbf{SCB}^-$ would be equivalent to classical logic. But then, by proposition 5.6, also $\mathbf{GO^1}$ would be equivalent to classical logic, which is not the case. A fortiori, $\mathbf{BCS} \ncong \mathbf{BCDS}$.

Finally, note that $\mathbf{SCB}^-$ with $\supset$ is an example of a system in which $\supset R^\emptyset$ and $\supset L^\emptyset$ hold but $\supset R$ and $\supset L$ do fail. This proves as promised that $\mathbf{B}'$ is weaker than **BD** and $\mathbf{CB}'$ is weaker than **BCD**.

References

[1] G. Battilotti and G. Sambin, *Pretopologies and a uniform presentation of sup-lattices, quantales and frames*, 1997. to appear.

[2] J. L. Bell, *Orthologic, Forcing, and the manifestation of attributes*, North-Holland, 1983, pp. 13–36.

[3] G. Birkhoff, *Lattice Theory*, American Mathematical Society, 1967. 3rd edition.

[4] G. Cattaneo, M. L. Dalla Chiara, and R. Giuntini, *Fuzzy intuitionistic quantum logics*, Studia Logica, 52 (1993), pp. 419–444.

[5] N. J. Cutland and P. F.Gibbins, *A regular sequent calculus for quantum logic in which $\wedge$ and $\vee$ are dual*, Logique et Analyse - Nouvelle Serie -, 25 (1982), pp. 221–248.

[6] M. L. Dalla Chiara and R. Giuntini, *Paraconsistent quantum logics*, Foundations of Physics, 19 (1989), pp. 891–904.

[7] J. Girard, *Linear Logic*, Theoretical Computer Science, 50 (1987), pp. 1–102.

[8] R. I. Goldblatt, *Semantical analysis of orthologic*, Journal of Philosophical Logic, 3 (1974), pp. 19–36.

[9] H. Nishimura, *Sequential method in quantum logic*, Journal of Symbolic Logic, 45 (1980), pp. 339–352.

[10] V. Pratt, *Linear logic for generalized quantum mechanics*, ???, (1995?). disponibile via ftp://boole.stanford.edu/pub/DVI/ql.dvi.Z.

[11] G. Sambin, *Intuitionistic formal spaces and their neighbourhood*, in Logic Colloquium '88, R. Ferro, C. Bonotto, S. Valentini, and A. Zanardo, eds., Studies in Logic and Foundation of Mathematics, Amsterdam - New York, 1989, North-Holland, pp. 261–285.

[12] G. Sambin, *A new elementary method to represent every complete boolean algebra*, in Logic and Algebra, A. Ursini and P. Aglianò, eds., New York, 1996, Marcel Dekker, pp. 655–665.

[13] A. S. Troelstra, *Lectures on linear logic*, no. 29 in Lecture notes, CSLI, Stanford, 1992.