To my parents

Contents

Introduction

The main interest underlying this thesis is that of the natural, psychological, birth of logic in the human mind. The need of a psychological foundation of logics has been expressed also by some mathematician. Already a century ago, F. Enriques made the proposal of studying logic as "psychological logic" in [En]. We think that, in general, such need is present in the intuitionistic foundations, for example in Brouwer. Perhaps logic arises from the need to overcome what, at a certain point, is focused as "contradiction" in our mind. The thesis focuses on a paraconsistent logic, namely a logic which doesn't obey the Aristotelian non-contradiction principle [Pr]. Its paraconsistency is due to a specific treatment of first-order variables. Some hypothesis on the development of the meaning of variables for the human mind, that could be related to the calculus here proposed, is then illustrated in the last chapter.

A secondary motivation of the thesis is computation, since the claim is that certain pre-logical or dis-logical phenomena we can observe, underlying the logical/rational attitude of our mind, have computational explanations. We assume that a primary interest of our mind is to process assertions, namely what is considered true. Of course, one should discuss also of the meaning of "truth" in this case.

We conceive logical truth as computed by logical rules. A natural treatment of logical rules has been pursued for a long time in logic, at least since Gentzen's natural deduction [Ge]. In general, in the last decade, the study of natural computational systems has flourished [CP], in particular, of quantum computation [HP], [NC]. This is particularly intriguing for a naturalistic study of logic, since, indeed, following quantum theories of mind (see, e.g., [At]), our mind should witness the effects of the laws of quantum physics. People with a naturalistic *credo* should be inclined to quantum theories of mind. In fact, why shouldn't our mind exploit the enormous quantum computational advantage? Then quantum computation should have also important logical effects, if we assume that a primary interest of our mind is to process assertions.

The thesis touches three different logical, or, in our view, related to logical, topics. The first is a general problem of logic, namely the problem of the meaning of the logical constants, here referred to the meaning of logical constants in that introduced by rules, for which we rely on a specific approach [SBF]. The second comes from physics, namely the problem of logical models in quantum computation (see e. g. [DCGL1] and [BSm06], [BSm08]). The third is the problem of consciousness, related in particular to Hameroff-Penrose quantum theory of mind [HP], [PH]. They correspond, roughly, to the three chapters of the thesis.

The first chapter discusses some points on logical constants in the framework of *basic logic*, a substructural logic proposed as a common platform for propositional extensional logics, including some kinds of quantum logics [DCGsurvey]. The first version of basic logic [BS99] was proposed as a calculus enucleating some common semantical features of logical connectives, features obtained, in such a case, from the algebraic structures underlying several substructural logics.

Later, we developed a second version of basic logic [SBF], namely a cut-free sequent calculus, whose rules can be justified in terms of the "reflection principle", for which connectives and their rules follow from metalinguistic links between assertions. The principle allows the natural predicative extension of basic logic, given in [MS]. The principle is also exploited in the thesis to develop our paraconsistent and predicative calculus. We also furtherly discuss the problem of the meaning of symmetry and of visibility, the two features of the calculus of basic logic. In the first chapter, we analyze how coupling logical connectives could produce compound logical objects. In particular, we focus upon the problem of contexts and parallel strategies of proofs, in order to understand which compound objects are possible. We find that, in the predicative case, parallel strategies which do not increase the complexity of the objects so achieved determine an inconsistent framework. It is the framework for quantum parallelism, that we develop in the second chapter.

As is well known, the parallelism of quantum computation (see [NC], [Hi]) is due to quantum superposition associated to quantum entanglement. Quantum superposition is the presence of several different states at the same time, e.g. both the spins of a particle. This allows parallel processes of computation. The entanglement link is created when, in a system of two or more superposed particles, the states are not separable, namely they cannot be described as a product of the states of each particle. For this reason, when the superposed state of the system collapses, the resulting states of the single particles, component of the system, are not independent, in terms of statistical independence. The most important example is represented by Bell's states, couples of particles behaving as "twins", namely they collapse into the non superposed state with identical results. They enforce the effect of parallelism induced by quantum superposition, since they bound the computational complexity and allows the so called "quantum computational speed-up", peculiar of quantum computation with respect to classical computation. For, the effects of quantum superposition and entanglement are not reproducible out of a quantum environment.

The idea to obtain a calculus for quantum computation from basic logic is natural, since it was born to include quantum logics ([FS], [BF]). After a first proposal of a paraconsistent calculus within propositional logic [Ba05], we realized that our idea requires the quantifiers.

In quantum computational logics [DCGL1] propositions correspond to the qubits and the quregisters, namely to the states of the quantum computer itself, rather than to the closed subsets of a Hilbert space, as in traditional quantum logic. We also adopt such an approach. Our representation does not require the algebraic setting of Hilbert spaces, and represents quantum superposition and entanglement by means of sequents, in order to describe quantum parallelism in terms of logical proofs. Our representation allows to see the computational advantage of quantum parallelism with respect to classical computation, that consists in knocking down the exponential complexity, that was the original motivation in the proposal of quantum computation by Feynman [Fe]. The idea is that the random variable given by a measurement on a certain physical system produces the domain of a first order variable, which describes the superposed state of the system. We see that the gap existing between the description of a superposed state, and the probability distribution given by the measurement of the state, is translated into the logical gap between a predicative representation for the superposition and a propositional representation for the corresponding probability distribution. In such setting, the expressive power of logical variables seems necessary. This is confirmed by our new predicative connective for the entanglement, which exploits a variable in common to obtain a new quantifier. The variable seems to capture the holistic feature of quantum information [DCGL2]. For, a variable can glue items of information in a non-compositional way, as we discuss in the third chapter. While the algebraic definition of entanglement is negative, since it speaks of non factorizable states, our approach can represent entangled particles in a positive way. This is considered a decisive advantage in any computational and constructive setting (see e.g. [MS] for the problem of a minimal and constructive mathematical foundation).

In the second part of the chapter we begin the development of a paraconsistent and predicative sequent calculus. We also define a dual copy of the calculus, given a suitable definition of dual domain. This allows to characterize Bell's states. The closer study of the proof theoretical aspects of such calculus is under development, since our first concern was to find a correct semantical representation, from the point of view of quantum physics. The problems related to proof theory in a paraconsistent setting are quite unexplored, and could lead to very intriguing results in the topic of computability, as first pointed out by Alan Turing in [Tu], where he says:

"... if a machine is expected to be infallible, it cannot be also intelligent. There are several theorems which say almost exactly that. But these theorems say nothing about how much

intelligence may be displayed if a machine makes no pretence at infallibility."

In the thesis we hypothize some possible development, also in connection with the original motivations of quantum computation [Fe]. Another possible important development of the calculus outlined here, is in the direction of logical calculi including probabilities. For, the representation by sequents we obtain for quantum computation includes random variables in first-order domains and can deal with dependent random variables, even if in a very simplified way, up to now. This is an important challenge in logics for artificial intelligence, see [So].

The final section of the second chapter is devoted to see how the interpretation of quantum superposition and entanglement by means of sequents we have proposed, can be considered, in the framework of the interpretations of quantum mechanics (see [Ja]). We find this a very intriguing work on which one can go on, too. For, one can already see clearly, in our opinion, a connection with the typical problems of the interpretations. More specifically, we see a connection with counterfactuality and contextuality of quantum mechanics, on one side, with the hidden variables interpretation, on another side, and finally with the stochastic interpretations of quantum mechanics and the problem of causality. Such connection is due to the features of quantum parallelism enlighted by the interpretation by sequents. Then, one can see the advantage of considering quantum systems from the point of view of processes, as quantum computation permits us to do, even for the purpose of the interpretations.

In the third chapter we make a comparison between some features of the process of assertions proper of quantum computation, that we have outlined in the previous chapter, and some features of the human thinking. We support such comparison by Hameroff-Penrose quantum theory of mind, briefly introduced in the chapter. Moreover, as suggested by Stuart Hameroff, we consider "bi-logic", the logic of the unconscious, outlined by the psychoanalist I. Matte Blanco after thirty years of clinical experience. It meets the features of our calculus in a surprising way. In particular, the infinitary aspect of the human unconscious thinking, diagnosed by Matte Blanco, corresponds, in our view, to the infinitary and holistic aspect given by first-order variables to our calculus.

We conclude the chapter with some personal observations, which would deserve a much higher competence in the field of psychology, in order to be properly treated. Anyway, they are so relevant to the topic, in our opinion, that we prefer to write down them, even if in a simple way.

In the future, we would like to develop our research on the psychological foundations of human logical thinking also independently from quantum theories. We have already developed some ideas in [Ba07].

Acknowledgements

I would like to thank Maria Luisa Dalla Chiara for the support she has always given to my research, since the first "cube of logics". I wish to thank also Giovanni Sambin. I am indebted with Stuart Hameroff for his encouradgement, and in particular for his valuable suggestion of considering Matte Blanco's bi-logic. Any misunderstanding in the brief treatment of Hameroff-Penrose theory I give here, of course, is mine. Moreover, I would like to thank Roberto Giuntini, Sandro Sozzo, Francesco Valfré and Silvano Zipoli Caiani, for useful comments and help on different topics treated in the thesis. Finally I wish to thank my husband Marco, my children Marianna and Elisa, and all my family.

Chapter 1

Parallel strategies in sequent calculus

Summary: we first recall the main features of basic logic and the cube of its extensions, focusing on the problem of contexts and parallel strategies of proofs. Then we analyze the possible parallel strategies of proofs in sequent calculus, in the propositional case, first, and then in the predicative case. We find that parallel strategies which do not increase the complexity of the objects so achieved determine an inconsistent framework.

1.1 Sequents and sequent calculi

1.1.1 Sequents and contexts

Sequent calculi were introduced by Gentzen in [Ge]. A sequent is a formal writing of the following type:

$$\Gamma \vdash \Delta$$

where $\Gamma = C_1, \ldots, C_n$ and $\Delta = D_1, \ldots, D_m$ are finite lists of formulae, separated by commas. In it the sign $\vdash$ indicates the logical consequence. Then Γ and Δ are called premises and conclusions of the sequent, respectively. A sequent calculus is a set of rules transforming sequents into other sequents. A unary or binary rule is written as follows:

$$\frac{\Gamma_1 \vdash \Delta_1}{\Gamma \vdash \Delta} \qquad\qquad \frac{\Gamma_1 \vdash \Delta_1 \quad \Gamma_2 \vdash \Delta_2}{\Gamma \vdash \Delta}$$

The axioms of a sequent calculus are sequents of the following form: $A \vdash A$. Deriving a sequent $\Gamma \vdash \Delta$, in a certain sequent calculus, is to obtain it as the conclusion of a certain derivation, that consists of a suitable application of the rules of the calculus, starting from axioms.

In any sequent calculus, one can distinguish two kinds of rules:

1. Rules on the structure of sequents (*Structural Rules*)

2. Rules introducing logical connectives.

Perhaps the most important revolution in sequent calculus, after Gentzen, is due to Girard [Gi], who introduced linear logic. In his linear sequent calculi, Gentzen's structural rules of *weakening, contraction* and *exchange* are dropped (possibly some of them). This allows to distinguish the connectives of disjunction and conjunction and their rules into two forms: the additive and the multiplicative one. In the next section we will illustrate such distinction, which is at the basis of our interpretation. Now we recall that the distinction between the additive and multiplicative formulation of the rules is due to a different treatment of *contexts*. The contexts, in a sequent, are the list of formulae appearing besides the active formulae, namely the formulae which are modified by the rule we are considering. For example, in the rule [1]

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta} \to f$$

the formulae $A, B, A \to B$ are active formulae, the lists Γ and Δ are contexts. They are *separated* from the active formulae by a comma.

[1]The label $\to f$ indicates that this is the *formation* rule of the connective $\to$, in the classification of rules of basic logic, explained below.

A critical problem is the following: the use of the comma is somewhat ambiguous. For, it is also used to *join* formulae, since the interpretation of the commas in

$$A_1, \ldots, A_n \vdash B_1, \ldots, B_m$$

is "$A_1 \& \ldots \& A_n \vdash B_1 \vee \ldots \vee B_m$", where & and $\vee$ are the conjunction and the disjunction, in Gentzen's notation. In linear logic they are the multiplicative conjunction and disjunction respectively. Then, the interpretation of $A_1, \ldots, A_n \vdash B_1 \ldots, B_m$ is the following [2]:

$$A_1 \otimes \ldots \otimes A_n \vdash B_1 * \ldots * B_m$$

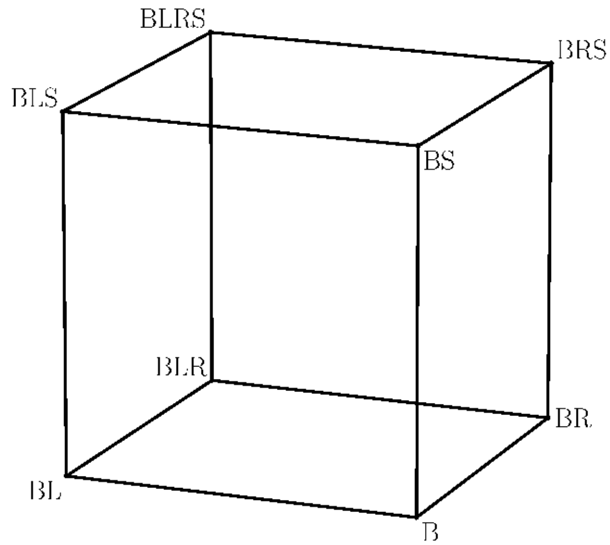
1.1.2 Basic logic and the cube of logics

A solution to the critical problem has been proposed by basic logic [SBF]. It has been a radical one, a sequent calculus **B**, given in table 1.1, where no context at all is present besides the active formulae. This is "visibility" of the formulae, in basic logic.

In this thesis, we furtherly investigate on the treatment of contexts. This is developed in a predicative extension of basic logic [MS]. For, the rules for quantifiers are the only ones in which the context matters, since restrictive conditions on the contexts are necessary in order to obtain the definability of the quantifier itself. As we shall see in details, this fact induces us to conceive an "inherently parallel" rule for the quantifier, which aims to represent quantum parallelism. As for the other connectives, parallel rules can be defined, that, however, can be simulated by a sequential application of simpler rules, thanks to the admissibility of contexts.

Our working platform will be the "cube of logics", a set of sequent calculi arising from a common kernel, the sequent calculus **B** of basic logic. In **B**, the connectives only have some minimal properties: no structural rule, except exchange, is valid. Richer logical calculi are then reached by the addition of structural rules. In this way, one can obtain

[2] We adopt here the multiplicative notation $*$ for the multiplicative disjunction, rather than the usual notation $\parr$ of linear logic, adopted in basic logic too.



cut-free sequent calculi for all the well-known extensional logics, including some kind of quantum logics.

This construction is organized in the cube represented above. In it, every vertex is a sequent calculus **C** which obeys the equation:

$$\mathbf{C} = \mathbf{B} + \text{suitable structural rules}$$

Looking at the figure, sequent calculi whose label contains **S** have the structural rules of weakening and contraction. This fact causes the identification of the multiplicative connectives with the additive ones. At least at a first stage, the identification is not convenient for our purposes, so we will work on the lower face of the cube (the linear face). Anyway, we remind that logics with **S** are important in quantum logic, since the calculus **BS** coincides with paraconsistent quantum logic [DCG]. A cut-free sequent calculus for orthologic and a formulation of classical logic as a subsystem of paraconsistent quantum logic are obtainable from it [BF].

Sequent calculi whose label contains **L** admit a context at the left, beside the active formulae of their rules. **BLS** is a calculus for intuitionistic logic, **BL** an intuitionistic and linear calculus. Sequent calculi whose label contains **R** admit a context at the right, beside the active formulae of their rules. **BRS** represents a dual version of intuitionistic logic and **BR** is a dual linear and intuitionistic calculus. We will describe the duality in some details in the following. Finally, **BLRS** is a sequent calculus for classical logic and **BLR** for linear logic (without exponentials).

Conditions **L** and **R** concerns the structure of sequents but are not expressed by means of real structural rules. This was left as an open problem in the formulation of basic logic. The problem appeared immediately related to the formulation of quantum logics, that require to drop at least some contexts. Then the choice of visibility (no context at all) was the right choice, not only from a syntactical point of view: indeed, it allows to obtain a cut-elimination proof, that extends also to the quantum logics formulated from basic logic. Besides this, visibility, dropping all contexts, allows to focus on the semantical problem of considering the comma as a link between formulae rather than as a separation from a context. A particular, interpretation of visibility allows to obtain an embedding of classical logic into paraconsistent quantum logic [B98].

In this thesis, we reach a better insight on the conditions **L** and **R**, since we show that the presence of a context at the left (resp. right), that allows to define the implication, is in contrast with the definability of the entanglement link, proper of quantum mechanics, in logic. At the end of the next chapter we shall see how the notion of “contextuality”, in the interpretations of quantum mechanics, can be approached by the treatment of contexts in sequent calculus.

1.2 Parallel strategies in sequent calculus - The propositional case

1.2.1 Propositional connectives and their natural environments

In basic logic, logical connectives and their rules are introduced following the *reflection principle*. This principle states that a logical connective is the result of importing a pre-existing meta-linguistic link between assertions into the object language. Its rules are

Table 1.1: The calculus of basic logic **B**

Axioms

$$A \vdash A$$

Structural rules

$$\frac{\Gamma, \Sigma, \Pi, \Gamma' \vdash \Delta}{\Gamma, \Pi, \Sigma, \Gamma' \vdash \Delta}\ exch\, left \qquad \frac{\Gamma \vdash \Delta, \Pi, \Sigma, \Delta'}{\Gamma \vdash \Delta, \Sigma, \Pi, \Delta'}\ exch\, right$$

Operational Rules

$$\frac{B, A \vdash \Delta}{B \otimes A \vdash \Delta}\ \otimes f \qquad \frac{\Gamma \vdash A, B}{\Gamma \vdash A * B}\ * f$$

$$\frac{A \vdash \Delta_1 \quad B \vdash \Delta_2}{A * B \vdash \Delta_1, \Delta_2}\ * r \qquad \frac{\Gamma_2 \vdash B \quad \Gamma_1 \vdash A}{\Gamma_2, \Gamma_1 \vdash B \otimes A}\ \otimes r$$

$$\frac{\vdash \Delta}{1 \vdash \Delta}\ 1 f \qquad \frac{\Gamma \vdash}{\Gamma \vdash \bot}\ \bot f$$

$$\bot \vdash \ \ \bot r \qquad \vdash 1 \ \ 1 r$$

$$\frac{B \vdash \Delta \quad A \vdash \Delta}{B \oplus A \vdash \Delta}\ \oplus f \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}\ \& f$$

$$\frac{A \vdash \Delta}{A \& B \vdash \Delta} \quad \frac{B \vdash \Delta}{A \& B \vdash \Delta}\ \& r \qquad \frac{\Gamma \vdash B}{\Gamma \vdash B \oplus A} \quad \frac{\Gamma \vdash A}{\Gamma \vdash B \oplus A}\ \oplus r$$

$$0 \vdash \Delta \ \ 0 f \qquad \Gamma \vdash \top \ \ \top f$$

$$\frac{B \vdash A}{B \leftarrow A \vdash}\ \leftarrow f \qquad \frac{A \vdash B}{\vdash A \rightarrow B}\ \rightarrow f$$

$$\frac{\vdash A \quad B \vdash \Delta}{A \rightarrow B \vdash \Delta}\ \rightarrow r \qquad \frac{\Gamma \vdash B \quad A \vdash}{\Gamma \vdash B \leftarrow A}\ \leftarrow r$$

$$\frac{A \vdash B \quad C \vdash D}{B \rightarrow C \vdash A \rightarrow D}\ \rightarrow u \qquad \frac{C \vdash D \quad A \vdash B}{C \leftarrow B \vdash D \leftarrow A}\ \leftarrow u$$

Cut rules

$$\frac{\Gamma_1 \vdash A \quad A, \Gamma_2 \vdash \Delta}{\Gamma_1, \Gamma_2 \vdash \Delta}\ cutL \qquad \frac{\Gamma \vdash \Delta_1, A \quad A \vdash \Delta_2}{\Gamma \vdash \Delta_1, \Delta_2}\ cutR$$

then a consequence of such correspondence. In the propositional case, we consider two links: *and* and *yields*. The metalinguistic link *and* correlates two logical judgements at the same level: in a *parallel* way. The metalinguistic link *yields* is the consequence relation between two logical judgements: it puts two assertions together in a *sequential* way.

Assertions can be represented by sequents, possibly with the addition of a context. The link *and* between A and B is represented in two ways, additive and multiplicative, respectively:

$$\Gamma \vdash A \quad \Gamma \vdash B$$

$$\Gamma \vdash A, B$$

The link *yields* between A and B is represented by the sequent:

$$\Gamma, A \vdash B$$

where Γ represents a context.

The links so represented are converted into three propositional linear connectives: $\&, *, \rightarrow$ (the additive conjunction, the multiplicative disjunction and the implication). This is obtained by assuming the following definitory equations:

$$\Gamma \vdash A \rightarrow B \qquad \equiv \qquad \Gamma, A \vdash B$$

$$\Gamma \vdash A\&B \qquad \equiv \qquad \Gamma \vdash A \quad \Gamma \vdash B$$

$$\Gamma \vdash A * B \qquad \equiv \qquad \Gamma \vdash A, B$$

Solving the equations one obtains the rules of the calculus. Each connective is "formed" by the rule which converts the link into the corresponding connective. Our connectives $\&, *, \rightarrow$ are formed at the right side of the sequent, hence they are called "right connectives" and their "formation rules" are the rules at the right. Moreover, each connective "reflects" its corresponding link to the other side of the sequent, by the "reflection rule", in other words, the rule at the left.

The way to solve the equation of the generic connective $\circ$, corresponding to the generic link $\cdot$, can be represented by the following equation schema:

$$\Gamma \vdash A \circ B \qquad \equiv \qquad \Gamma \vdash_{seq} A \cdot B$$

where the notation $\vdash_{seq}$ is an abbreviation for the three cases:

$$\Gamma \vdash A \quad \Gamma \vdash B$$

$$\Gamma \vdash A, B$$

$$\Gamma, A \vdash B$$

The schema of the formation rule is an immediate translation of one direction of the equation:

$$\frac{\Gamma \vdash_{seq} A \cdot B}{\Gamma \vdash A \circ B} \circ f$$

The other direction is translated into a rule termed "implicit reflection", that hasn't a valid form for sequent calculus:

$$\frac{\Gamma \vdash A \circ B}{\Gamma \vdash_{seq} A \cdot B} \circ ir$$

To reverse the rule, in order to obtain that the connective $\circ$ is introduced below, one first derives the "reflection axiom", trivializing the premise Γ in the assumption of $\circ ir$, so that the rule $\circ ir$ converts the axiom of sequent calculus $A \circ B \vdash A \circ B$ into the axiom schema

$$A \circ B \vdash_{seq} A \cdot B$$

The reflection axiom is equivalent to the implicit reflection rule, since we derive the $\circ ir$ rule by cut, following the schema: Then, a rule introducing $\circ$ at the left is derived by *cut*:

$$\frac{\Gamma \vdash A \circ B \quad A \circ B \vdash A\dot{B}}{\Gamma \vdash_{seq} A \cdot B} cut_{seq}$$

Again this is simply a general schema of the derivation, where cut_{seq} is adapted to three different cases of application of the *cut*-rule. The resulting "reflection rules", in the three cases, are the following:

$$\frac{A \vdash \Delta}{A \& B \vdash \Delta} \quad \frac{A \vdash \Delta}{A \& B \vdash \Delta} \circ r$$

$$\frac{A, B \vdash \Delta}{A * B \vdash \Delta} * r$$

$$\frac{\Gamma \vdash A \quad B \vdash \Delta}{\Gamma, A \to B \vdash \Delta} \to r$$

Such rules are equivalent to the implicit reflection rules. For, they are equivalent to the reflection axioms. On one side, one derives the conclusion of each reflection rule from its premises by cutting the corresponding reflection axiom. On the other, for every $\circ$, $\circ r$ derives the reflection axiom, from the couple of axioms $A \vdash A$ and $B \vdash B$, as we see in the following schema of derivation:

$$\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A \circ B \vdash_{seq} A, B} \circ r}{A \circ B \vdash A \circ B} \circ f$$

We term *natural environment* of a connective $\circ$ the minimum sequent calculus which allows to derive the axiom $A \circ B \vdash A \circ B$. To discover the natural environment of the three propositional right connectives, we loose the schema of derivation just shown above in the three possible cases:

$$\dfrac{\dfrac{A \vdash A}{A\&B \vdash A} \qquad \dfrac{B \vdash B}{A\&B \vdash B}}{A\&B \vdash A\&B} \begin{array}{l}\&r\\ \&f\end{array}$$

$$\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A * B \vdash A, B} *r}{A * B \vdash A * B} *f$$

$$\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A \to B, A \vdash B} \to r}{A \to B \vdash A \to B} \to f$$

In the above derivations, the sequents obtained at the intermediate stage

$A\&B \vdash A \quad A\&B \vdash B$

$A * B \vdash A, B$

$A, A \to B \vdash B$

are the reflection axioms of the corresponding connectives. The presence of the left context is required for $\to$. The natural environments of the connectives $\&, *, \to$ are then

B, **B** and **BL** respectively. Observe moreover that, in our interpretation, the two possible forms of the left rule for the additives are not simply "a matter of commutativity", rather they have to be considered the two parts of a unique reflection rule, since they are both necessary to obtain the natural environment.

1.2.2 Parallel strategies and compound objects

Since the link *and* can be represented in two ways in a sequent calculus, one obtains two different ways to implement parallel processes in logical derivations: the additive and the multiplicative. The multiplicative connectives represent the register link in a computer and can be exploited to represent the parallel processes on different registers. This is also exploited in quantum computational logics [DCGL1]. We shall see in the following sections why additivity has to be considered responsible of quantum computational parallelism.

In order to understand parallelism in logical derivations, we now analyze the relation between the two kinds of *and* in sequent calculus. We will analyze also the relation between *and* and *yields*, in order to see to what extent each parallelism is, by itself, well behaved w.r.t. the sequential relation.

Then, we assume the two kinds of *and* together, considering a compound assertion of the following form:

$$\Gamma \vdash A, C \quad \Gamma \vdash B, C$$

(where both A and B are together with C) or the additive *and* combined with *yields*, considering a compound assertion of the following form

$$\Gamma, A \vdash B \quad \Gamma, A \vdash C$$

(where both B and C are yielded by A) or, finally, the multiplicative *and* combined with *yields*

$$\Gamma, A \vdash B, C$$

(where B and C together are yielded by A). In the last case, one can consider that the number of occurrences of A in the sequent is relevant or not. In the following we adopt the second interpretation.

Is there a logical connective $t(A, B, C)$ for each of the compound links above? We have $t(A, B, C)$ when the link is effectively given in a unique way. A temptative interpretation of such effectiveness is the following: it occurs when the syntactical order is irrelevant, namely when the point is the whole thing so created, not the particular nesting of the two simple links forming the compound one. In such a case, a new compound logical object is created.

Each $t(A, B, C)$ is determined by the couple of connectives corresponding to the compound link, namely

$(\&; *)$

$(\&; \rightarrow)$

$(*; \rightarrow)$

The reflection axioms of each $t(A, B, C)$, as in the simple case, can be derived from the axioms of sequent calculus by means of the reflection rules. For each couple, we have two different syntactical possibilities, since, in deriving the axioms, one can switch the rules of the two connectives of the couple. Here are the reflection axioms so derived:

Couple $(\&; *)$:
deriving the axiom by $\&r$ first: $A * (B\&C) \vdash A, B \quad A * (B\&C) \vdash A, C$
deriving the axiom by $*r$ first: $(A * B)\&(A * C) \vdash A, B \quad (A * B)\&(A * C) \vdash A, C$

Couple $(\&; \rightarrow)$:
deriving the axiom by $\&r$ first: $A \rightarrow (B\&C), A \vdash B \quad A \rightarrow (B\&C), A \vdash C$
deriving the axiom by $\rightarrow r$ first: $(A \rightarrow B)\&(A \rightarrow C), A \vdash B \quad (A \rightarrow B)\&(A \rightarrow C), A \vdash C$

Couple $(*; \to)$:

deriving the axiom by $*r$ first: $A \to (B * C), A \vdash B, C$

deriving the axiom by $\to r$ first: $(A \to B) * (A \to C), A \vdash B, C$

We now see that, for each couple, there is a natural environment in which the compound connective is definable. To obtain this, the definability of each form of the reflection axiom is not enough: we must include the fact that the syntactical order is not relevant, too. This amounts to prove a "distributive law" between the two connectives of the couple. In the following, the apex ^{i} means that the number of occurrences of a formula is not relevant, namely weakening of the same formula and contraction are admitted.

Lemma 1.2.1 *Couple* $(\&; *)$*:*

The sequent $A * (B\&C) \vdash (A * B)\&(A * C)$ *holds in* **B**

The sequent $(A * B)\&(A * C) \vdash A * (B\&C)$ *holds in* **BR**

Couple $(\&; \to)$*:*

The sequent $A \to (B\&C) \vdash (A \to B)\&(A \to C)$ *holds in* **BL**

The sequent $(A \to B)\&(A \to C) \vdash A \to (B\&C)$ *holds in* **BL**

Couple $(*; \to)$*:*

The sequent $A \to (B * C) \vdash (A \to B) * (A \to C)$ *holds in* $\mathbf{BLR^i}$

The sequent $(A \to B) * (A \to C) \vdash A \to (B * C)$ *holds in* $\mathbf{BL^i}$

The proof consists of the following derivations:

$$\dfrac{\dfrac{\dfrac{A \vdash A \quad \dfrac{B \vdash B}{B\&C \vdash B}}{A * (B\&C) \vdash A, B}}{A * (B\&C) \vdash A * B} \qquad \dfrac{\dfrac{A \vdash A \quad \dfrac{C \vdash C}{B\&C \vdash C}\;\&r}{A * (B\&C) \vdash A, C}\;*r}{A * (B\&C) \vdash A * C}\;*f}{A * (B\&C) \vdash (A * B)\&(A * C)}\;\&f$$

that holds in **B**,

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A * B \vdash A, B}}{A * B)\&(A * C) \vdash A, B} \qquad \dfrac{\dfrac{A \vdash A \quad C \vdash C}{A * C \vdash A, C}\;*r}{A * B)\&(A * C) \vdash A, C}\;\&r}{(A * B)\&(A * C) \vdash A, B\&C}\;\&f}{(A * B)\&(A * C) \vdash A * (B\&C)}\;*f$$

that holds in **BR**,

$$\dfrac{\dfrac{A \vdash A \quad \dfrac{B \vdash B}{B\&C \vdash B}}{A \to (B\&C), A \vdash B}}{A \to (B\&C) \vdash A \to B} \quad \dfrac{\dfrac{A \vdash A \quad \dfrac{C \vdash C}{B\&C \vdash C}\,\&r}{A \to (B\&C) \vdash A, C}\to r}{A \to (B\&C) \vdash A \to C}\to f \over A \to (B\&C) \vdash (A \to B)\&(A \to C)\;\&f$$

that holds in **BL**. A derivation in **B** is also possible, substituting the two successive applications of $\to f$, $\to r$ by the basic rule $\to uni$, which allows here to derive monotonicity of $\to$ w.r.t. the subsequent.

$$\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{(A \to B, A \vdash B}}{A \to B)\&(A \to C), A \vdash B} \quad \dfrac{\dfrac{A \vdash A \quad C \vdash C}{A \to C, A \vdash C}\to r}{(A \to B)\&(A \to C) \vdash A, C}\,\&r}{\dfrac{(A \to B)\&(A \to C), A \vdash B\&C}{(A \to B)\&(A \to C) \vdash A \to (B\&C)}\to f}\,\&f$$

that holds in **BL**,

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad B \vdash B}{A \to B, A \vdash B} \quad \dfrac{A \vdash A \quad C \vdash C}{A \to C, A \vdash C}\to r}{(A \to B) * (A \to C), A \vdash B, C}*r}{(A \to B) * (A \to C), A \vdash B * C}*f}{(A \to B) * (A \to C) \vdash A \to (B * C)}\to f$$

that holds in **BL**$^{\mathbf{i}}$.

$$\dfrac{\dfrac{\dfrac{A \vdash A \quad \dfrac{B \vdash B \quad C \vdash C}{B * C \vdash B, C}*r}{A \to B * C, A \vdash B, C}\to r}{A \to B * C \vdash (A \to B), (A \to C)}\to f\|}{A \to B * C \vdash (A \to B) * (A \to C)}*f$$

that holds in **BLR**$^{\mathbf{i}}$.

Summing up, propositional compound connectives are easily definable in the extensions of basic logic. To have all of them, one needs classical linear logic without weakening. The natural environment for the couple $(\to, \&)$ is **BL**; the natural environment for the couple $(*, \&)$, namely for the distributive law of the multiplicative w.r.t. the additive connective, is **BR**; the natural environment for the couple $(\to, *)$ is **BLR**. When $*$ is present, **R** is required.

Notice, moreover, that the derivations in the above proof have a parallel character in a natural way, when they contain the additive connective, since they consist of two equal branches for the alternatives B and C. In the last case, concerning the multiplicative connective, we have parallelized the application of the rule $\rightarrow f$ on the formulae B and C appearing on the right side of the sequent. We have denoted such application by $\rightarrow \|$. Two sequential applications of two occurrences of the $\rightarrow f$ rule, on the formulae B and then C, are possible, and would produce the same result. But considering the $\rightarrow f\|$ rule permits us a parallel strategy in proofs, and hence in proof search, even for the case of multiplicative connectives. Parallel strategies are the natural way to obtain our compound objects.

1.2.3 The symmetric case

The metalinguistic link *and* can be represented at the left of the sequent sign too. In such case, it is represented by the following assertions:

$$A \vdash \Delta \quad B \vdash \Delta$$

in the additive case, and

$$A, B \vdash \Delta$$

in the multiplicative case. Moreover, we rewrite the metalinguistic link *yield* with a context at the right rather than at the left:

$$A \vdash B, \Delta$$

Then we put the definitory equations:

$$A \oplus B \vdash \Delta \qquad \equiv \qquad A \vdash \Delta \quad B \vdash \Delta$$

and

$$A \otimes B \vdash \Delta \qquad \equiv \qquad A, B \vdash \Delta$$

which define the additive disjunction $\oplus$ and the multiplicative conjunction $\otimes$ respectively. Moreover, from the third assertion, one defines exclusion, $\leftarrow$:

$$A \leftarrow B \vdash \Delta \qquad \equiv \qquad A \vdash B, \Delta$$

Solving such equations one obtains the rules of the three "left connectives" $\oplus, \otimes, \leftarrow$, which are formed at the left and reflected at the right. Their natural environments are **B**, **B** and **BR** respectively.

Then, one can also consider the three couples formed by the left connectives:

$(\oplus; \otimes)$

$(\oplus; \leftarrow)$

$(\otimes; \leftarrow)$

They gives rise to compound objects in **BL**, **BR** and $\mathbf{BLR^i}$ respectively. This result is obtained "by symmetry".

Intuitively, the left connectives are the symmetric copy of the right ones. We put in formal terms the notions of symmetric formula and of dual formula, and then we enunciate a theorem for symmetry (see [SBF]):

Definition 1.2.2 *For any formula A, we define the symmetric formula A^s of A, by the following inductive clauses:*

i) $p^s \equiv p$ *for every propositional variable p.*

ii) $(A \circ B)^s \equiv B^s \circ^s A^s$ *for every connective $\circ$, where $\circ^s$ is the left (resp. right) connective corresponding to the same link of the right (resp. left) connective $\circ$.*

For any formula A, we define the dual formula $A^\perp$ of A, by the following inductive clauses:

i) $p^\perp$ *is a literal different from p, interpretable as "a primitive negation of p", for every propositional variable p. $\perp$ is a non trivial involution on the set of literals.*

ii) $(A \circ B)^\perp \equiv B^\perp \circ^\perp A^\perp$ *for every connective $\circ$, where $\circ^\perp$ is the left (resp. right) connective corresponding to the same link of the right (resp. left) connective $\circ$.*

Then one can define the symmetric and the dual proof of a proof Π, by induction on the depth of the derivation. So one can prove the following theorem:

Theorem 1.2.3 *The sequent* $\Gamma \vdash \Delta$ *is provable in* **B** *if and only if* $\Delta^s \vdash \Gamma^s$ *and* $\Delta^\perp \vdash \Gamma^\perp$ *are provable in* **B***.*

The sequent $\Gamma \vdash \Delta$ *is provable in* **BL** *(resp.* **BR***) if and only if* $\Delta^s \vdash \Gamma^s$ *and* $\Delta^\perp \vdash \Gamma^\perp$ *are provable in* **BR** *(resp.* **BL***).*

1.3 Parallel strategies in sequent calculus - The predicative case

1.3.1 Predicative extension of basic logic

The predicative extension of basic logic has been introduced in [MS], putting the definitory equation which converts the metalinguistic link *forall* into the quantifier $\forall$. In such case, the metalinguistic assertion "*forall d* in the domain D, $A(d)$ is true" is converted into the sequent $\Gamma, z \in D \vdash A(z)$. Hence the corresponding equation is:

$$\Gamma \vdash (\forall x \in D)A(x) \quad \equiv \quad \Gamma, z \in D \vdash A(z)$$

Such equation holds under the condition "z not free in Γ". Such condition has a clear semantical motivation: for, when something depending on a generic z has to be derived, the assumptions cannot depend on the same z, otherwise we are deriving something depending on a specific z.

The assertion

$$\Gamma, z \in D \vdash A(z)$$

gathers all assertions $A(z)$ depending on the free variable z ranging on the domain D. We can picture it as a fan spreading from the common kernel given by Γ, whose branches are indexed by the z's and end with the $A(z)$'s. In this sense, the assertion could be better represented by an indexed sequent of the form

$$\Gamma \vdash_{z \in D} A(z)$$

Anyway, since $z \in D$ has also the role of an assumption, added to Γ, we prefer, now, to adopt the more traditional notation $\Gamma, z \in D \vdash A(z)$. It does not force us to exit the traditional sequent calculus, where sequents are not indexed. Such notation, moreover, allows to stress the role of $x \in D$ as an assumption, that allows to interpret $(\forall x \in D)A(x)$ as "$\forall x(x \in D \to A(x))$".

We see the solution of the definitory equation, which follows the usual pattern given in the propositional case. The rules obtained are the following [MS]:

$$\frac{\Gamma, z \in D \vdash A(z)}{\Gamma \vdash (\forall x \in D)A(x)}\ \forall f^{\dagger}$$

namely the formation rule, where † is the condition "z not free in Γ". The converse of the formation rule represents the implicit reflection rule:

$$\frac{\Gamma \vdash (\forall x \in D)A(x)}{\Gamma, z \in D \vdash A(z)}\ \forall ir$$

The explicit reflection rule is:

$$\frac{\Gamma' \vdash z \in D \quad \Gamma, A(z) \vdash \Delta}{\Gamma, (\forall x \in D)A(x), \Gamma' \vdash \Delta}\ \forall r$$

Note that, in this form, it resembles the $\to r$ rule of **BL**. As usual, it is obtained via the reflection axiom, that in turn is obtained putting $\Gamma = (\forall x \in D)A(x)$ in the implicit reflection, giving:

$$(\forall x \in D)A(x), z \in D \vdash A(z)$$

Then the following derivation, that exploits the cut rule, derives the reflection rule:

$$\frac{\dfrac{(\forall x \in D)A(x), z \in D \vdash A(z) \quad \Gamma, A(z) \vdash \Delta}{\Gamma, (\forall x \in D)A(x), z \in D \vdash \Delta}\ cutL \quad \Gamma' \vdash z \in D}{\Gamma, (\forall x \in D)A(x), \Gamma' \vdash \Delta}\ cutL$$

The above rule, applied to the axioms $A(z) \vdash A(z)$ and $z \in D \vdash z \in D$, allows to derive the reflection axiom, that in turn gives back the implicit reflection rule, by cut.

The discussion of the existential quantifier, that represents the symmetric of the universal quantifier, is postponed to the next chapter.

1.3.2 Substitution of variables by terms

Our usual way to conceive the meaning of first order variables implies that they can be substituted by any term t. This means that we adopt the following structural rule in sequent calculus:

$$\frac{\Gamma \vdash \Delta}{\Gamma[x/t] \vdash \Delta[x/t]}\,[subst]$$

(under the condition that the free variables of t are not captured by quantifiers appearing in Γ or Δ).

Then, from the reflection axiom $(\forall x \in D)A(x), z \in D \vdash A(z)$, we can derive the following axioms:

$$(\forall x \in D)A(x), t \in D \vdash A(t)$$

for every term t of the first-order language we are considering. In particular, since $t \in D$ is true if t is a closed term denoting an element of the domain, one derives the sequent

$$(\forall x \in D)A(x) \vdash A(t)$$

for every closed term t. Then, if the domain D has n elements denoted by the terms $t_1, \ldots, t_n$, one derives

$$\forall x \in D)A(x) \vdash A(t_1)\& \ldots \&A(t_n)$$

Anyway, the converse sequent

$$A(t_1)\& \ldots \&A(t_n) \vdash (\forall x \in D)A(x)$$

is underivable, without specific assumptions [3].

[3] In the infinitary case, this problem resembles that of the "ω-rule" that gives the ω completeness of arithmetic, and hence undetermines the computational content of the theory of arithmetic.

Then, even if the condition † in the $\forall f$ rule gives the quantifier an additive character, $\forall$ is much more that a "big &". We shall see a quantum interpretation of this in the following.

One can furtherly observe that, in presence of the substitution rule, one can give the following formulation to the reflection rule:

$$\frac{\Gamma' \vdash t \in D \quad \Gamma, A(t) \vdash \Delta}{\Gamma, (\forall x \in D)A(x), \Gamma' \vdash \Delta}\ \forall r - t$$

Indeed, such form has the same derivation we have just seen for the $\forall r$ rule formulated with a variable, assuming the reflection axiom in its formulation with a term t. In turn, $\forall r - t$ allows to derive the reflection axioms with terms.

The formulation $\forall r - t$ is usually considered the most general form for a $\forall$ rule in sequent calculi, since it can be "particularized" to the case in which the term t is a variable. This requires the further assumption "a variable is a term", which is commonly assumed. Anyway, this point of view is not convenient for our purposes, as we shall see in the following. Moreover the term-formulation of the rule depends on the term-formulation of the reflection axiom, which in turn is due to the substitution rule, not only to the definitory equation of the quantifier.

Notice finally that the axiom $(\forall x \in D)A(x) \vdash (\forall x \in D)A(x)$ is derivable from an axiom of the form $A(z) \vdash A(z)$, where z is a free variable, while it is not derivable from $A(t) \vdash A(t)$ where t is any term. Then the "natural environment", that we have defined above for the propositional connectives of the cube of logics, is created only by the variable-formulation of the rules, in the case of the quantifiers.

Then, in our view, which derives the rules from the definitory equations and looks for the natural environment, we require the variable-formulation.

1.3.3 Classical distributivity and the problem of complexity

We now wish to study the combination of the multiplicative connective $*$ with the quantifier $\forall$. This means, as for the propositional case, to find the minimum requirements which allow to define a combined connective. Hence we focus, first, on the problem of distributivity.

In the predicative case, distributivity is provable in the following form:

$$(\forall x \in D_1)A(x) * (\forall x \in D_2)B(x) = (\forall x \in D_1)(\forall y \in D_2)(A(x) * B(y))$$

here termed classical distributivity. Here is the proof of the two directions:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{z \in D_1 \vdash z \in D_1 \quad A(z) \vdash A(z)}{(\forall x \in D_1)A(x), z \in D_1 \vdash A(z)} \quad \dfrac{w \in D_2 \vdash w \in D_2 \quad B(w) \vdash B(w)}{(\forall x \in D_2)B(x), w \in D_2 \vdash B(w)}\ \forall r}{(\forall x \in D_1)A(x) * (\forall x \in D_2)B(x), z \in D_1, w \in D_2 \vdash A(z), B(w)}\ *r}{(\forall x \in D_1)A(x) * (\forall x \in D_2)B(x), z \in D_1, w \in D_2 \vdash A(z) * B(w)}\ *f}{(\forall x \in D_1)A(x) * ((\forall x \in D_2)B(x), w \in D_2 \vdash (\forall x \in D_1)(A(x * B(w))}\ \forall f^{\dagger}}{(\forall x \in D_1)A(x) * ((\forall x \in D_2)B(x) \vdash (\forall x \in D_1)(\forall y \in D_2)(A(x * B(y))}\ \forall f^{\dagger}$$

and

$$\dfrac{\dfrac{\dfrac{\dfrac{z \in D \vdash z \in D \quad \dfrac{w \in D_2 \vdash w \in D_2 \quad \dfrac{A(z) \vdash A(z) \quad B(w) \vdash B(w)}{A(z) * B(w) \vdash A(z), B(w)}\ *r}{(\forall y \in D_2)(A(z) * B(y)) \vdash A(z) * B(y)}\ \forall r}{(\forall x \in D_1)(\forall y \in D_2)(A(x) * B(y)), z \in D_1, w \in D_2 \vdash A(z), B(w)}\ \forall r}{(\forall x \in D_1)(\forall y \in D_2)(A(x) * B(y)), w \in D_2 \vdash (\forall x \in D_1)A(x), B(w)}\ \forall f^{\dagger}}{(\forall x \in D_1)(\forall y \in D_2)(A(x) * B(y)) \vdash (\forall x \in D_1)A(x), (\forall x \in D_2)B(x)}\ \forall f^{\dagger}}{(\forall x \in D_1)(\forall y \in D_2)(A(x) * B(y)) \vdash (\forall x \in D_1)A(x) * (\forall x \in D_2)B(x)}\ *f$$

where the two conditions † are "z not free in Γ" and "w not free in Γ". Moreover, the sequential application of the two $\forall f$ rules is always possible, when the presence of a right context is allowed (logics with "**R**" in the cube). This implies in particular that it must be $z \neq w$. For, $A(z)$ or $B(z)$, could be carried, negated, at the left, in logics with **R**. So condition † refers to them too, in such case.

The two sequential applications $\forall f^{\dagger}$ are parallelizable, as described in the following derivation:

$$\dfrac{\dfrac{\dfrac{z \in D \vdash z \in D \quad \dfrac{w \in D_2 \vdash w \in D_2 \quad \dfrac{A(z) \vdash A(z) \quad B(w) \vdash B(w)}{A(z) * B(w) \vdash A(z), B(w)}\ *r}{(\forall y \in D_2)(A(z) * B(y)) \vdash A(z) * B(y)}\ \forall r}{(\forall x \in D_1)(\forall y \in D_2)(A(x) * B(y)), z \in D_1, w \in D_2 \vdash A(z), B(w)}\ \forall r}{(\forall x \in D_1)(\forall y \in D_2)(A(x) * B(y)) \vdash (\forall x \in D_1)A(x), (\forall x \in D_2)B(x)}\ \forall f^{\dagger}\|}{(\forall x \in D_1)(\forall y \in D_2)(A(x) * B(y)) \vdash (\forall x \in D_1)A(x) * (\forall x \in D_2)B(x)}\ *f$$

which completely resembles the last derivation, containing the rule $\rightarrow \|$, of lemma 1.2.1. In both cases, we can adopt a parallel strategy which is equivalent to a sequential strategy, the last possible in logics with **R**.

When distributivity holds, one can conceive a unique semantical object given by the combination of the two connectives, since distributivity guarantees that the definition is syntax-independent. Then one can define a unique multiplicative-additive quantifier $*\forall$, putting the definitory equation:

$$\Gamma \vdash (*\forall x \in D_1, y \in D_2)(A(x); B(y)) \quad \equiv \quad \Gamma, z \in D_1, w \in D_2 \vdash A(z), B(w)$$

where the free variables z and w are not free in Γ and $z \neq w$. The object so defined coincides with $(\forall x \in D_1)A(x) * (\forall x \in D_2)B(x)$ or with $(\forall x \in D_1)(\forall y \in D_2)(A(x) * B(y))$.

The necessary requirement $z \neq w$ has a heavy computational drawback. For, it implies independent choices for $z \in D_1$ and $y \in D_2$. This yields the exponential increasing of complexity, in the number of variables, of the object combining the two parallelisms given by $*$ and $\forall$.

1.3.4 Manicheist distributivity and the problem of consistency

In order to overcome the problem of complexity, it would be crucial to have distributivity with respect to *one* variable:

$$(\forall x \in D)A(x) * (\forall x \in D)B(x) = (\forall x \in D)(A(x) * B(x))$$

here termed manicheist distributivity.

The object that could be given by such equality does not exist in logic, since the interpretation of $*$ as a disjunction, which is forced in the extensions of **B**, makes the above distributive law false. Very easy counterexample: “since every integer number is odd or even, then either every integer is odd either every integer is even”. Then logic is not used to deal with such kinds of objects. The natural environment of our good computational object is inconsistency.

One could make the objection that it is not clear what the semantical interpretation of a "multiplicative disjunction" consists of, and then try to fit the manicheist distributivity in **B** or its linear extensions. We now show how this, technically, would be possible. Let us consider the following parallel, simultaneous, application of the $\forall f$-rule to a couple of formulae both depending on the same free variable z:

$$\frac{\Gamma, z \in D \vdash A(z), B(z)}{\Gamma \vdash (\forall x \in D)A(x), (\forall x \in D)B(x)} \; \forall f^{\dagger}\|$$

$\forall f^{\dagger}\|$ allows to prove $(\forall x \in D)(A(x) * B(x)) \vdash (\forall x \in D)A(x) * (\forall x \in D)B(x)$, as follows:

$$\dfrac{\dfrac{\dfrac{z \in D \vdash z \in D \quad \dfrac{A(z) \vdash B(z) \quad B(z) \vdash B(z)}{A(z) * B(z) \vdash A(z), B(z)} *r}{(\forall x \in D)(A(x) * B(x)), z \in D \vdash A(z), B(z)} \forall r}{(\forall x \in D)(A(x) * B(x)) \vdash (\forall x \in D)A(x), (\forall x \in D)B(x)} \forall f^{\dagger}\|}{(\forall x \in D)(A(x) * B(x)) \vdash (\forall x \in D)A(x) * (\forall x \in D)B(x)} *f$$

The converse sequent is derivable as seen in the case of classical distributivity. Then $\forall f^{\dagger}\|$ would prove our new distributivity.

$\forall f^{\dagger}\|$ is an admissible rule in basic logic. For, consider what follows: In **B** (and its extensions without **R**), right contexts are not admitted. In particular, formulae cannot be carried from the right to the left of $\vdash$ in presence of another formula at the right. Then it makes a sense to consider the condition † referred to the left side of the sequent only. So, in particular, the additive character of the quantifier, referred to the invariance of the left context Γ, is preserved. Moreover, in **B** and its extensions without **R**, a two-steps sequential application of the $\forall f$ rule to $A(z)$ and then to $B(z)$ is not possible. Then, our $\forall f^{\dagger}\|$ rule is an "inherently parallel rule", where $A(z)$ is not to be considered a context for $B(z)$ and conversely. Note finally that our parallel rule satisfies Gentzen's original formulation of the † condition in the $\forall$ rule. It is a syntactical condition on the rule rather than a semantical condition on the premise of the rule itself. It says "the variable bounded by the application of $\forall$ must not occur free in the conclusion of the rule".

The problem is that, if we added the above $\forall f\|$-rule as such to linear sequent calculi, we would render their non linear extensions inconsistent, since every calculus in the cube is a conservative extension of **B**. Then one has better to hypothize the existence of a new link between $A(z)$ and $B(z)$, different from the comma, since the last is interpretable in terms of context. Such new link should allow the $\forall f^{\dagger}\|$ rule and “collapse” when usual sequent calculi are reached.

We have obtained this as an interpretation of specific features of quantum physics, and we now illustrate such an interpretation.

Chapter 2

Interpreting quantum parallelism by sequents

We propose an interpretation of quantum superposition by means of quantifiers on first-order domains equipped with probabilities. From this an interpretation of the entangled states follows. We show the necessary role of first-order variables and the meaning of substitution. Then we develop a paraconsistent sequent calculus and its dual copy, given a suitable definition of dual domain. Finally we make some proposals concerning the role of the resulting interpretation in the framework of the interpretations of quantum mechanics.

2.1 Interpreting quantum superposition

Definition 2.1.1 *Let us consider a discrete random variable Z, with set of possible outcomes B, and with associated probability measure p_Z, measuring $p\{Z = z\}$ for every $z \in B$. This determines a set $D = D(Z, p_Z)$, given by*

$$D(Z, p_Z) = \{z = (z', p\{Z = z'\}) : z' \in B\}$$

We term such set "random first order domain" associated to Z.

Let $\mathcal{A}$ be a quantum system. A quantum measurement on it gives a discrete[1] random variable Z, given by the observable. The set of the possible outcomes determines a subset of the orthonormal basis of the Hilbert space representing $\mathcal{A}$. Finally the probability p_Z is determined by the probability amplitude. Then a quantum first order domain D defined as above is associated to any measurement of $\mathcal{A}$.

A measurement on $\mathcal{A}$, under certain hypothesis, is described by an assertion, as follows:

"*forall* possible outcomes z in D, under certain hypothesis Γ, the possible result of the measurement of $\mathcal{A}$ is z."

where *forall* is the metalinguistic link introduced in 1.3.1. We rewrite the assertion, formally, as a sequent:

$$\Gamma, z \in D \vdash A(z)$$

where the first order variable z appears free in A and does not appear free in the hypothesis Γ. For, the hypothesis of a correct experiment cannot depend on its outcome. So we put the equivalence defining the quantifier *forall*, seen in 1.3.1:

$$\Gamma \vdash (\forall x \in D)A(x) \;\equiv\; \Gamma, z \in D \vdash A(z)$$

Such definition allows to gather the possible results $A(z)$, associated to the observable, into a unique object, represented by the proposition $(\forall x \in D)A(x)$[2]. Then the quantifier $\forall$ interpretes quantum superposition. The "logical glue" for quantum superposition is the variable associated with the random variable of the measurement experiment. When the superposed state is considered, the variable is bounded, and ranges over the domain given by the measurement.

By the $\forall r$ rule, from the axioms of sequent calculus, one derives the sequent

$$(\forall x \in D)A(x), z \in D \vdash A(z)$$

[1]We confine our attention to the finite spaces of quantum computation.

[2]Note that, since the measurement, which determines the domain, is performed on a state, the domain is linked to the state. Then, writing $(\forall x \in D)A(x)$, D is linked to A. This does not seem a trouble for us, since in usual first order logic such feature is present too. For example, there are propositions which make a sense on the domain of real numbers and do not on the domain of complex numbers.

namely the reflection axiom of the quantifier $\forall$. In our case, it asserts that the particle described by the proposition $(\forall x \in D)A(x)$ can be found in a state associated with any of the z's of D.

Substituting the free variable z by a *closed* term t in it, one has the sequent $(\forall x \in D)A(x), t \in D \vdash A(t)$ from which, since $t \in D$ is true, one derives the sequent

$$(\forall x \in D)A(x) \vdash A(t)$$

It asserts that the superposition $(\forall x \in D)A(x)$ is converted into $A(t)$, where t denotes a fixed element of the orthonormal basis, with its probability. The other possibilities are lost. This describes a collapse: the substitution operation destroys the superposition.

A description of the original superposition can be recovered a posteriori, by the propositional connectives, as we illustrate in the example below.

Example 2.1.2 *Let us consider a particle $\mathcal{A}$ and the random first order domain D given by the outcomes of the measurement of the spin of $\mathcal{A}$ along the z axis. D has two elements: $(|\uparrow\rangle, p\{Z = |\uparrow\rangle\})$ and $(|\downarrow\rangle, p\{Z = |\downarrow\rangle\})$, denoted by the terms $t_\uparrow$ and $t_\downarrow$ respectively. The proposition $(\forall x \in D)A(x)$ represents the superposed state of the two directions of the spin along the z-axis. The sequent $(\forall x \in D)A(x) \vdash A(t_\uparrow)$ asserts that $\mathcal{A}$ is found in the "up" direction along the z axis with the probability given by the measurement experiment. Analogously, the sequent $(\forall x \in D)A(x) \vdash A(t_\downarrow)$ says that $\mathcal{A}$ is found in the "down" direction along the z axis with the associated probability.*

From the two sequents one can derive, by the $\& f$ rule of **B***, the sequent*

$$(\forall x \in D)A(x) \vdash A(t_\uparrow)\&A(t_\downarrow).$$

The propositional formula (closed terms, no variable!) appearing on the right side of it:

$$A(t_\uparrow)\&A((t_\downarrow)$$

describes the probability distribution associated to the superposed state $(\forall x \in D)A(x)$. The sequent $(\forall x \in D)A(x) \vdash A(t_\uparrow)\&A(t_\downarrow)$, which is derivable (when a substitution rule *is allowed), states that the probability distribution follows from the superposition. The converse sequent is not derivable unless one assumes specific axioms.*

Then our logical representation can distinguish between superposition and probability distribution. The distinction is due to the presence of the variable, since it can supply the logical glue that is lost having the closed terms only. In algebraic terms, one has that real numbers are enough to describe the probability distribution, while complex numbers are required to describe the superposed state. It seems that variables are the logical way to reach what is expressed by complex numbers in algebraic terms.

2.2 A new quantifier for the entanglement

In order to import the manicheist distributivity, described in section 1.3.4, in the realm of logic, we need to distinguish the case of dependent variables from the case of independent variables. We will interpret them by different connectives, and keep both cases only in the paraconsistent setting of basic logic. Then inconsistency will be avoided in its extensions.

Let us consider a random variable Z and its associated random first order domain D. Then, a new link between two propositions A and B is definable, in terms of a common first-order variable ranging on the domain, as follows. Let us consider the sequent

$$\Gamma, z \in D \vdash A(z), B(z),$$

where z is a first order variable on D, free in A and B. Let us assume that the comma says also "there is a variable in common". This enriches the link between A and B, that would be simply put side by side otherwise.

Then let us write such new link "$,_Z$", where Z is the random variable which gives the domain of the first-order variable z. We term the new link "variable-link" and rewrite the sequent as follows:

$$\Gamma, z \in D \vdash A(z),_Z B(z),$$

Note that the link $,_Z$ may be considered even if the first order variable z becomes bounded. In fact, in that case, it is included in the random first-order domain $D(Z, p_Z)$ associated to Z.

We now put the following version of $\forall f^{\dagger}\|$-rule:

$$\frac{\Gamma, z \in D \vdash A(z),_Z B(z)}{\Gamma \vdash (\forall x \in D)A(x),_Z (\forall x \in D)B(x)} \ \forall f^{\dagger}\|$$

In it, the link $,_Z$ is still present in the conclusion, even if the first-order variable z is not free any more. This is correct for a parallel rule, since it concerns only the *forall* link, and does not act on the comma between the two formulae A and B. Hence such comma must be kept unaltered.

The variable link $,_Z$ has the character of a "semi-predicative" link. For any random variable Z, with its associated domain, we put the definitory equation of the corresponding semi-predicative multiplicative connective $\bowtie_Z$:

$$\Gamma \vdash A \bowtie_Z B \quad \equiv \quad \Gamma \vdash A,_Z B$$

that we term Bell's disjunction (with respect to Z). The formation rule of Bell's disjunction is

$$\frac{\Gamma \vdash A,_Z B}{\Gamma \vdash A \bowtie_Z B} \ \bowtie_Z f$$

Its reflection axiom is

$$A \bowtie_Z B \vdash A,_Z B$$

The "minimum" reflection rule that allows to derive the reflection axiom is the following:

$$\frac{A \vdash \Delta \quad B \vdash \Delta'}{A \bowtie_Z B \vdash \Delta,_Z \Delta'} \ \bowtie_Z r$$

Then we must adopt it as a reflection rule. In it, the notation $\Delta,_Z \Delta'$ means what follows: whenever a proposition $A \in \Delta$ and a proposition $B \in \Delta'$ depend on the random variable Z, they are linked through it, an hence the Z link must be considered linking them. Then the substitution (cut) rule that must be considered is the following:

$$\frac{\Gamma \vdash A,_Z \Delta \quad A \vdash \Delta'}{\Gamma \vdash \Delta',_Z \Delta} \ cutR_Z$$

since it allows to derive our reflection rule, cutting the reflection axiom, as we show:

$$\dfrac{\dfrac{A \bowtie B \vdash A, B \quad A \vdash \Delta}{A \bowtie_Z B \vdash \Delta}\ cut \qquad B \vdash \Delta'}{A \bowtie_Z B \vdash \Delta,_Z \Delta'}\ cut$$

By the $\forall f\|$ rule one can prove the new distributive law, written with respect to $\bowtie_Z$:

$$(\forall x \in D)A(x) \bowtie_Z (\forall x \in D)B(x) = (\forall x \in D)(A(x) \bowtie_Z B(x))$$

We term such equality Bell's distributivity.

After the definition of the semi-predicative connectives $\bowtie_Z$, a new quantifier, $\bowtie$, combining multiplicative parallelism and superposition, is definable in **B**, putting the equation:

$$\Gamma \vdash \bowtie_{x \in D} (A(x); B(x)) \quad \equiv \quad \Gamma, z \in D \vdash A(z),_Z B(z)$$

where z is not free in Γ. The following rules are derivable from such equation:

$$\dfrac{\Gamma, z \in D \vdash A(z),_Z B(z)}{\Gamma \vdash \bowtie_{z \in D} (A(x); B(x))}\ \bowtie f^{\dagger}$$

$$\dfrac{\Gamma' \vdash z \in D \quad \Gamma_1, A(z) \vdash \Delta_1 \quad \Gamma_2, B(z) \vdash \Delta_2}{\Gamma_1, \Gamma_2, \bowtie_{x \in D} (A(x); B(x)), \Gamma' \vdash \Delta_1,_Z \Delta_2}\ \bowtie r$$

The new quantifier $\bowtie$ gives an object equal to $(\forall x \in D)A(x) \bowtie_Z (\forall x \in D)B(x)$ or to $(\forall x \in D)(\forall x \in D)A(x) \bowtie_Z B(x)$. It allows to represent systems of entangled particles, as we now see.

Example 2.2.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two entangled particles, for example two electrons with opposite spin. The possible result of a measurement of the spin along the z axis, performed on $\mathcal{A}$ or on $\mathcal{B}$, is equally described by an assertion of the form*

$$\Gamma, z \in D \vdash A(z),_Z B(z)$$

where $D = \{(|\uparrow\rangle, p\{Z = \uparrow\}, (|\downarrow\rangle, p\{Z = \downarrow\})\}$, and where $A(z)$ means "$\mathcal{A}$ is found in the z direction", and $B(z)$ is "$\mathcal{B}$ is found in the direction opposite to z"[3]. *Moreover, we have the usual condition that z is not free in Γ.*

[3] $B(z)$ indicates that the state is a function of z, the free variable being z

So we put now the definitory equation:

$$\Gamma \vdash \bowtie_{x \in D} (A(x); B(x)) \quad \equiv \quad \Gamma, z \in D \vdash A(z),_Z B(z)$$

The state of the two entangled particles is then described by the proposition $\bowtie_{x \in D} (A(x); B(x))$. *The first-order variable, on the random first order domain given by the random variable describing the measurement experiment, is the glue which allows to describe the superposed state* together with *the entanglement between the two particles at the same time.*

What makes the entanglement disappear? In physics, the collapse of the wave function. In our logical terms, a substitution of the variable z by a closed term t destroys the superposition and also the entanglement, since no variable is present any more. The assertion $\Gamma \vdash A(z),_Z B(z)$, after a substitution, becomes $\Gamma \vdash A(t), B(t)$ where the comma is the usual comma of sequent calculus, since the variable has disappeared. Then, no entanglement is described at the propositional level.

2.2.1 A comparison with the classical case

Let us consider two independent measurement experiments, with respect to a couple of observables, producing two independent random variables, Z and Y, and two possibly different random first order domains D_Z and D_Y. It may even happen that the two domains coincide, anyway this fact does not affect the independence of the variables. The assertion describing the couple of measurements has the following form:

$$\Gamma, z \in D_Z, y \in D_Y \vdash A(z), B(y)$$

where z and y are free in A and B, $z \neq y$ and Γ does not contain z and y free. It corresponds to the object $*\forall$, given in section 1.2.1, defined by classical distributivity, which implies exponential growth of complexity.

We can conceive the two measurements applied to two different physical systems, for example two particles, $\mathcal{A}$ and $\mathcal{B}$. We can also conceive two independent measurements on the same physical system, say $\mathcal{A}$. In the first case the propositions $A(z)$ and $B(y)$ represent the possible value of the measurements obtained applying the observable corresponding to Z to $\mathcal{A}$ and that corresponding to Y to $\mathcal{B}$, respectively; in the second case they represent the two possible values of the two measurements performed on $\mathcal{A}$. The second case is possible only if the observables for the two experiments are compatible. Then the existence of incompatible observables in quantum mechanics should be interpreted as a way to avoid computational complexity, since the assertions containing independent variables, that are originated by couples of incompatible observables, are avoided.

We notice the following example, originated by the EPR paradox. It is a border-line event, since it can make compatible observables that would be incompatible otherwise.

Example 2.2.2 *Classical distributivity is restored considering simultaneously two incompatible observables for two entangled particles. For example, measurements of the spin along different axis, z and y, which are incompatible on the same particle, can be applied as simultaneous independent measurements on two entangled particles $\mathcal{A}$ and $\mathcal{B}$. In such case we have an assertion of the form*

$$\Gamma, z \in D_Z, y \in D_Y \vdash A(z), B(y)$$

The simpler assertion with variable link $,_Z$, namely

$$\Gamma, z \in D_Z \vdash A(z) ,_Z B(z)$$

or, as an alternative, the assertion with variable link $,_Y$,

$$\Gamma, y \in D_Y \vdash A(y) ,_Y B(y)$$

together with the corresponding $\bowtie$ logical object, are not possible when the two simultaneous independent measurements are applied. On the contrary, when one measurement (spin along z or spin along y) is applied, the other is not possible any more, for the effect of the entanglement.

Then it seems that the computational effect of the entanglement is alternative to the computational effect of incompatible observables, which become compatible in the particular case of the EPR paradox.

2.3 A paraconsistent sequent calculus for quantum computation

As we have just seen, the predicative extension of **BLR** can perform the calculus of assertions deriving from classical physics. The case of quantum mechanics requires the introduction of some new specific considerations in treating the assertions, as we have already noticed. This is given by a particular treatment of variables. Here we gather our ideas in a more formal framework. We shall obtain an enrichment of the linear intuitionistic calculus **BL**, which admits context at the left and so creates the natural environment for the implication, by a paraconsistent right side, that allows the variable-link and hence the entanglement. We indicate such calculus by $\mathbf{BL} + \mathbf{R}^{\bowtie}$.

In addition to those already considered in basic logic, let us consider a multiplicative link "$,_Z$" for every random variable Z, as we have described in 2.2. Let us put a subscript 0 to the comma denoting the usual multiplicative link, that involves no variable: then we write "$,_0$" instead of ",". Let α be a subscript of a comma. We label by α also sets: D_α is the random first order domain $D(Z, p_Z)$ when $\alpha = Z$, D_0 is simply a first order domain.

Then we define inductively non ordered α-lists of formulae as follows:

- Any formula is a α-list;
- In $\Delta_1,_\alpha \ldots {}_\alpha\Delta_n$, A in Δ_i and B in Δ_j are linked by $,_\alpha$.

The link $,_\alpha$ may be not actual, namely, if A or B do not depend on a common first-order variable on the random first-order domain D_Z, writing $A,_Z B$ is like writing $A,_0 B$. The link is actual even when the common first-order variable becomes bounded by the quantifier.

The calculus we are looking for will deal with sequents of the form

$$\Gamma \vdash \Delta_1 ,_\alpha \ldots ,_\alpha \Delta_n$$

where the variable-link is admitted at the right only. Then we shall write "," at the left and "$,_0$" at the right. We obtain several kinds of sequents:

i Sequents containing only one type of variable-link;

ii Sequents containing only one type of variable-link different from $,_0$;

iii Sequents containing different types of variable-links.

It is clear that the link is not associative, in the second and third case.

Then, the structural rule of exchange is as follows:

$$\frac{\Gamma \vdash \Delta_1 ,_\alpha \ldots ,_\alpha \Delta_n}{\Gamma \vdash \Delta_{\pi(1)} ,_\alpha \Delta_{\pi(n)}}\ exch$$

where π is a permutation of $1 \ldots n$.

The cut rules we need are:

$$\frac{\Gamma \vdash A ,_0 \Delta \quad \Gamma', A \vdash \Delta'}{\Gamma', \Gamma \vdash \Delta' ,_0 \Delta}\ cut\,f$$

that is the cut of **BL**, and

$$\frac{\Gamma \vdash A ,_\alpha \Delta \quad A \vdash \Delta'}{\Gamma \vdash \Delta' ,_\alpha \Delta}\ cutR_\alpha$$

that represents the additional cut rule we need.

The general form of the definitory equation for the multiplicatives is:

$$\Gamma \vdash A_1 \bowtie_\alpha \ldots \bowtie_\alpha A_n \qquad \equiv \qquad \Gamma \vdash A_1 ,_\alpha \ldots ,_\alpha A_n$$

We give it in an n-ary form, rather than in the binary one, since we prefer to obtain a generalized n-ary connective for the entanglement, rather than a binary one. Then the n-ary form is required, since the variable-link cannot be used as a context, and then associativity of $\bowtie_\alpha$ is not provable (see basic logic). We have already shown how to obtain the reflection rule of $\bowtie_\alpha$, by means of $cutR_\alpha$, in section 2.2. The two rules for $\bowtie_\alpha$, in the n-ary case are:

$$\frac{\Gamma \vdash A_1,_\alpha \ldots,_\alpha A_n}{\Gamma \vdash A_1 \bowtie_\alpha \ldots \bowtie_\alpha A_n} \bowtie_\alpha f \qquad \frac{A_1 \vdash \Delta_1 \ldots A_n \vdash \Delta_2}{A_1 \bowtie_\alpha \ldots \bowtie_\alpha A_n \vdash \Delta_1,_\alpha \ldots,_\alpha \Delta_n} \bowtie_\alpha r$$

If $\alpha \neq 0$, $\bowtie_\alpha$ is Bell's disjunction, in n-ary formulation. If $\alpha = 0$, the connective $\bowtie_0$ coincides with $*$. Its rules are those of $*$ in **BL**, in the n-ary form:

$$\frac{\Gamma \vdash A_1,_0 \ldots,_0 A_n}{\Gamma \vdash A_1 * \ldots * A_n} * f \qquad \frac{\Gamma_1, A \vdash \Delta_1 \quad \ldots \quad \Gamma_n, A_n \vdash \Delta_n}{\Gamma_1, \ldots A_1 * \ldots * A_n \vdash \Delta_1,_0 \ldots,_0 \Delta_n} * r$$

Indeed, the cut rule of **BL**, *cut f*, is applied to obtain the reflection rule. Then contexts at the left are present.

Then, since our calculus must be "inherently parallel", we put the following definitory equation for the quantifier:

$$\Gamma \vdash (\forall x \in D_\alpha)A_1(x),_\alpha \ldots,_\alpha (\forall x \in D_\alpha)A_n(x) \quad \equiv$$
$$\Gamma, z \in D_\alpha \vdash A_1(z),_\alpha \ldots,_\alpha A_n(z)$$

It is a generalized form of the equation for the quantifier, where $n = 1$ or $\alpha \neq 0$, and where z is not free in Γ.

If $\alpha = 0$, we need the requirement $n = 1$, otherwise we obtain the inconsistency, on one side, and the parallel behaviour of the $\forall$ is not justified by a common variable, on the other. The restriction on the variable z is applied to the premises Γ.

The formation rule obtained from the above equation is the following:

$$\frac{\Gamma, z \in D_\alpha \vdash A_1(z),_\alpha \ldots,_\alpha A_n(z)}{\Gamma \vdash (\forall x \in D_\alpha)A_1(x),_\alpha \ldots,_\alpha (\forall x \in D_\alpha)A_n(x)} \forall f^\dagger \|$$

The implicit reflection rule is its converse:

$$\frac{\Gamma \vdash (\forall x \in D_\alpha)A_1(x),_\alpha \ldots,_\alpha (\forall x \in D_\alpha)A_n(x)}{\Gamma, z \in D_\alpha \vdash A_1(z),_\alpha \ldots,_\alpha A_n(z)} \forall ir^\dagger \|$$

In order to derive the explicit reflection rule, two strategies are possible: the first consists in considering the above implicit reflection in the case $n = 1$. Then, we obtain the reflection axiom and explicit reflection rule of the universal quantifier seen in the previous chapter. This is enough to create the natural environment for the entanglement in $\mathbf{BL} + \mathbf{R}^{\bowtie}$, as we see in the derivation below:

$$\cfrac{\cfrac{\cfrac{z \in D_\alpha \vdash z \in D_\alpha \quad A_i(z) \vdash A_i(z)}{(\forall x \in D_\alpha)A_i(x), z \in D_\alpha \vdash A_i(z)}\ \forall r\ i = 1 \ldots n}{(\forall x \in D_\alpha)A_1(x) \bowtie_\alpha \ldots \bowtie_\alpha (\forall x \in D_\alpha)A_n(x) \vdash A_1(z),_\alpha \ldots,_\alpha A_n(z)}\ \bowtie_\alpha r}{\cfrac{(\forall x \in D_\alpha)A_1(x) \bowtie_\alpha \ldots \bowtie_\alpha (\forall x \in D_\alpha)A_n(x) \vdash (\forall x \in D_\alpha)A_1(x),_\alpha \ldots,_\alpha (\forall x \in D_\alpha)A_n(x)}{(\forall x \in D_\alpha)A_1(x) \bowtie_\alpha \ldots \bowtie_\alpha (\forall x \in D_\alpha)A_n(x) \vdash (\forall x \in D_\alpha)A_1(x) \bowtie_\alpha \ldots \bowtie_\alpha (\forall x \in D_\alpha)A_n(x)}\ \bowtie f}\ \forall f^\dagger\|$$

Switching the application of the reflection rules, or of the formation rules, one derives the two directions of Bell's distributivity. We leave the details.

The second strategy consists of admitting the structural rules of weakening and contraction for $,_\alpha$, when $\alpha \neq 0$. They are:

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta,_\alpha B}\ W_{\alpha\neq 0} \qquad\qquad \frac{\Gamma \vdash A,_\alpha A,_\alpha \Delta}{\Gamma \vdash A,_\alpha \Delta}\ C_{\alpha\neq 0}$$

Such rules seem quite natural, due to the meaning we attribute to the variable link. The reflection axiom is obtained trivializing the premise Γ of the definitory equation in the n possible ways, as follows:

$$\cfrac{\cfrac{(\forall x \in D_\alpha)A_i(x) \vdash (\forall x \in D_\alpha)A_i(x)}{(\forall x \in D_\alpha)A_i(x) \vdash (\forall x \in D_\alpha)A_1(x),_\alpha \ldots,_\alpha (\forall x \in D_\alpha)A_n(x)}\ W_{\alpha\neq 0}}{(\forall x \in D_\alpha)A_i(x), z \in D_\alpha \vdash A_1(z),_\alpha \ldots,_\alpha A_n(z)}\ \forall ir^\dagger\|$$

So we have obtained the following n axioms:

$$(\forall x \in D_\alpha)A_i(x), z \in D_\alpha \vdash A_i(z) \quad i = 1 \ldots n$$

Then the explicit reflection, derived cutting the axiom, looks as follows:

$$\frac{\Gamma' \vdash z \in D \quad A_i(z) \vdash \Delta_i}{\Gamma, \Gamma', (\forall z \in D)A_i(z) \vdash A_1(z),_\alpha \ldots,_\alpha \Delta_i,_\alpha \ldots,_\alpha A_n(z)}\ \forall r\| \quad i = 1 \ldots n$$

In this second perspective, deriving the natural environment for the entanglement requires contraction too. We leave the details.

As seen in section 2.2, one can summarize $\forall$ and $\bowtie_\alpha$ defining the n-ary quantifier $\bowtie$ by

$$\Gamma \vdash \bowtie_{x \in D_\alpha} (A_1(x); \ldots; A_n(x)) \quad \equiv \quad \Gamma, z \in D_\alpha \vdash A_1(z),_\alpha \ldots,_\alpha A_n(z)$$

where z is not free in Γ. It is clear that the n-ary quantifier $\bowtie$ is $\forall$ for $n = 1$.

Besides the above definitory equations, we consider the definitory equations of the other connectives and constants of propositional basic logic, added with a context at the left, in **BL**. We avoid exclusion, $\leftarrow$, which has its natural environment in logics with **R** only. Note that assuming a definitory equation for $\leftarrow$ with respect to the variable link:

$$A \leftarrow B \vdash \Delta \qquad \equiv \qquad A \vdash B,_\alpha \Delta$$

is incompatible with the parallel definitory equation of $\forall$. For, in such case, the condition $\dagger$ on variables is not well-posed, if $n \neq 1$ in the equation of $\forall$, since formulae can be carried to the left. Then $\forall||$ and $\leftarrow$ are incompatible, and then we have to avoid the exclusion in order to import the entanglement in sequent calculus. Actually, the above position is a nonsense, for $\alpha \neq 0$, since the variable-link doesn't create a context.

A second possible perpective consists of admitting the exclusion only for the 0-link, for which the entanglement doesn't exist, and that is interpretable as a separation from a context:

$$A \leftarrow B \vdash \Delta \qquad \equiv \qquad A \vdash B,_0 \Delta$$

In such case, one derives the reflection axiom $A \vdash A \leftarrow B,_0 B$ and hence the reflection rule

$$\frac{\Gamma \vdash A \quad \Gamma', B \vdash \Delta}{\Gamma, \Gamma' \vdash A \leftarrow B,_0 \Delta} \leftarrow r$$

We shall make some comment on the dichotomy entanglement/exclusion in the final section.

The existential quantifier is also avoided, with similar motivations, and inserted into the symmetric interpretation, illustrated in the next section.

We conclude with the structural rule which governs the collapse from our sequents with variable-link to normal sequents, namely substitution:

$$\frac{\Gamma \vdash \Delta_1,_\alpha \Delta_2}{\Gamma[z/t] \vdash \Delta_1[z/t],_0 \Delta_2[z/t]} \; subst$$

when t is a closed term. If t contains a variable on D_α, the variable-link remains unaltered after the substitution.

Finally, we need to stress that, in the cube of logics, one has several choices for the contexts. Then one can also conceive different calculi with entanglement, possibly without a good implication, like basic logic. This problem is intrigued with that of cut elimination, which is open, hence it has better to be discussed together. Cut elimination represents a problem with peculiar aspects, in our case, as we observe in the following. The rules derived up to now allow a good interpretation of the problem of parallelism, in our opinion. In the last section of the chapter we show that our rules are in accordance with what suggested by some well-known interpretations of quantum mechanics.

Below we summarize in a table the rules following from the definitory equations we have discussed.

2.3.1 Symmetric and dual interpretation

We have reminded in 1.2.3 how to find a symmetric and a dual copy of any logic in the cube. This fact can be extended to the calculus with entanglement at the the right $\mathbf{BL} + \mathbf{R}^{\bowtie}$, obtaining a calculus with entanglement at the left $\mathbf{BR} + \mathbf{L}^{\bowtie}$. Then, one has to deal with sequents with α-link at the left, which have the form

$$\Gamma_1 {,_\alpha} \Gamma_2 \vdash \Delta$$

and then to define the symmetric of the connectives $\bowtie_\alpha$ and $\forall$. Moreover, for the duality, we have to extend the definition of dual literal to the predicative case.

In the symmetric calculus we reintroduce the connective of exclusion $\leftarrow$, that has its natural environment in **BR**, and we drop its symmetric, the implication $\rightarrow$, since it has no natural environment there. As before, an introduction of the implication with 0-link is possible.

The symmetric of the universal quantifier $\forall$ is clearly the existential quantifier $\exists$. In our framework, it is defined by the symmetric of the definitory equation of $\forall$:

$$(\exists x \in D)A(x) \vdash \Delta \quad \equiv \quad A(z) \vdash \Delta, z \in D$$

Table 2.1: Rules for an intuitionistic calculus with entanglement

Axioms

$$A \vdash A$$

Structural rules

$$\frac{\Gamma, \Sigma, \Pi, \Gamma' \vdash \Delta}{\Gamma, \Pi, \Sigma, \Gamma' \vdash \Delta}\ exch\,left$$

$$\frac{\Gamma \vdash \Delta,_\alpha \Delta_1}{\Gamma \vdash \Delta'_1,_\alpha \Delta}\ exch\,right$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta,_\alpha B}\ W_{\alpha \neq 0}$$

$$\frac{\Gamma \vdash A,_\alpha A}{\Gamma \vdash A}\ C_{\alpha \neq 0}$$

$$\frac{\Gamma \vdash \Delta_1,_\alpha \Delta_2}{\Gamma[x/t] \vdash \Delta_1[x/t],_0 \Delta_2[x/t]}\ subst\ (t\ closed)$$

Operational Rules

$$\frac{\Gamma, B, A \vdash \Delta}{\Gamma, B \otimes A \vdash \Delta}\ \otimes f$$

$$\frac{\Gamma \vdash A_1,_0 \ldots,_0 A_n}{\Gamma \vdash A_1 * \ldots * A_n}\ * f$$

$$\frac{\Gamma_1, A_1 \vdash \Delta_1 \ \ldots \ \Gamma_n, A_n \vdash \Delta_n}{\Gamma_1, \ldots, A_1 * \ldots * A_n \vdash \Delta_1,_0 \ldots,_0 \Delta_n}\ * r$$

$$\frac{\Gamma_2 \vdash B \quad \Gamma_1 \vdash A}{\Gamma_2, \Gamma_1 \vdash B \otimes A}\ \otimes r$$

$$\frac{\Gamma \vdash A_1,_\alpha \ldots,_\alpha A_n}{\Gamma \vdash A_1 \bowtie_\alpha \ldots \bowtie_\alpha A_n}\ \bowtie_\alpha f$$

$$\frac{A_1 \vdash \Delta_1 \ldots A_n \vdash \Delta_n}{A \bowtie_\alpha \ldots \bowtie_\alpha A_n \vdash \Delta_1,_\alpha \ldots,_\alpha \Delta_n}\ \bowtie_\alpha r$$

$$\frac{\Gamma, z \in D_\alpha \vdash A_1(x),_\alpha \ldots,_\alpha A_n}{\Gamma \vdash (\forall x \in D_\alpha) A_1(x),_\alpha \ldots,_\alpha (\forall x \in D_\alpha) A_n(x)}\ \forall^\dagger||$$

$$\frac{\Gamma \vdash z \in D_\alpha \quad A_i \vdash \Delta_i}{\Gamma, (\forall x \in D_\alpha) A_i(x) \vdash \Delta_i}\ \forall||r$$

$$\frac{B \vdash A,_0 \Delta}{B \leftarrow A \vdash \Delta}\ \leftarrow f$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ \to f$$

$$\frac{\Gamma \vdash A \quad \Gamma', B \vdash \Delta}{\Gamma, \Gamma', A \to B \vdash \Delta}\ \to r$$

$$\frac{\Gamma \vdash B \quad \Gamma', A \vdash \Delta}{\Gamma', \Gamma \vdash B \leftarrow A,_0 \Delta}\ \leftarrow r$$

Cut rules

$$\frac{\Gamma_1 \vdash A, \Delta_1 \quad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_2, \Delta_1}\ cut$$

$$\frac{\Gamma \vdash \Delta_1,_\alpha A \quad A \vdash \Delta_2}{\Gamma \vdash \Delta_1,_\alpha \Delta_2}\ cutR_\alpha$$

where z is not free in Δ.

In **BR**, one can conceive a quantifier by means of the exclusion, so that $(\exists x \in D)A(x) \equiv \exists x(A(x) \leftarrow x \in D)$ which is, formally, symmetric with respect to the usual way to conceive the universal quantifier. Note that the meaning of $(\exists x \in D)A(x)$ should be $\exists x((x \in D)\&A(x))$, then we should rather have $(\exists x \in D)A(x) \equiv \exists x(A(x) \leftarrow \neg(x \in D))$. This suggests that we should rather conceive a new domain to say $\neg(x \in D)$, as we do in the following dual interpretation.

Deserving the interpretation of the domain to the next considerations, we put the parallel definitory equation of the existential quantifier, to obtain the symmetric calculus:

$$(\exists x \in D_\alpha)A_1(x),_\alpha \ldots,_\alpha (\exists x \in D_\alpha)A_n(x) \vdash \Delta \quad \equiv$$

$$A_1(z),_\alpha \ldots,_\alpha A_n(z) \vdash \Delta, z \in D_\alpha$$

The symmetric of the entanglement is then obtained by means of the symmetric of Bell's discjunction $\bowtie$, namely "Bell's conjunction", denoted by $\bowtie^s$ and defined formally by the equation:

$$A_1 \bowtie^s_\alpha \ldots \bowtie^s_\alpha A_n \vdash \Delta \qquad \equiv \qquad A,_\alpha \ldots,_\alpha A_n \vdash \Delta$$

Then every rule of $\mathbf{BL} + \mathbf{R}^{\bowtie}$ has its symmetric, obtaining $\mathbf{BR} + \mathbf{L}^{\bowtie}$. In $\mathbf{BR} + \mathbf{L}^{\bowtie}$ Bell's distributivity becomes the following:

$$(\exists x \in D_\alpha)A(x) \bowtie^s_\alpha (\exists x \in D_\alpha)B(x) = (\exists x \in D_\alpha)(A(x) \bowtie^s_\alpha B(x))$$

for $\alpha \neq 0$. If $\alpha = 0$ it becomes the false distributive law:

$$(\exists x \in D_\alpha)A(x) \otimes (\exists x \in D_\alpha)B(x) = (\exists x \in D_\alpha)(A(x) \otimes B(x))$$

that we term the perfectionist distributivity.

Even if the intuitive interpretation of quantum superposition and entanglement is obtained in $\mathbf{BL} + \mathbf{R}^{\bowtie}$, its symmetric has the advantage that it allows to see the conflict between entanglement and implication in a more intuitive way. Indeed, in $\mathbf{BR} + \mathbf{L}^{\bowtie}$ the entanglement obtained by Bell's conjunction $\bowtie^s$ and $\exists$ has a natural environment, while the usual implication $\rightarrow$ is not definable. In a second perspective, it is definable by assuming the following equation, with respect to the 0-link only:

$$\Gamma \vdash A \rightarrow B \quad \equiv \quad \Gamma,_0 A \vdash B$$

To have a dual interpretation, we have to define the dual of every primitive literal, since duality coincides with symmetry on connectives. Then we need only to define $A(z)^{\perp}$, $A[z/t]^{\perp}$ and $(z \in D)^{\perp}$.

Let us consider $D = D(Z, p_Z)$, where Z is the random variable of a measurement experiment on a quantum system represented in a finite Hilbert space C^{2^n}, which is the Hilbert space representing the quantum registers of lenghth n in a quantum computer. Then $D = D(Z, p_Z) = \{(z', p\{Z = z'\}) : z' \in B\}$ where B is a subset of an orthonormal basis of the Hilbert space. An intriguing definition of $(z \in D)^{\perp}$ is the following:

$$(z \in D)^{\perp} \equiv z \in D^{\perp}$$

where

$$D^{\perp} \equiv \{z = (NOT(z'), p\{Z = z'\})\} : z' \in B\}$$

where NOT is the unitary transformation that supplies the NOT gate in the quantum computer. Then also

$$D^{\perp} = \{z = (z', p\{Z = NOT(z)')\}) : z' \in NOT(B)\}$$

where $NOT(B)$ is the image of the subset B under the map NOT.

Let us consider a language with a term for every element of B. If t is a closed term denoting an element $(b, p\{Z = b\})$, we define $t^{\perp}$ as the term denoting $\equiv (NOT(b), p\{Z = b\})$. Then, for example, in the notations of the example 2.1.2, $t_{\downarrow}^{\perp} \equiv t_{\uparrow}^{\perp}$ and conversely (any sharp state is mapped into its "opposite").

We keep that, if a certain literal A depends on t, its dual is the same A depending on $t^{\perp}$, so we define:

$$[A(t)]^{\perp} \equiv A(t^{\perp})$$

One could furtherly say that $A(z)^{\perp}$ is $A(z^{\perp})$, where $z^{\perp}$ is something that has to be substituted with $t^{\perp}$ rather than t, but this is ininfluent when we consider a *forall* link. Then one has simply to substitute the domain D with $D^{\perp}$.

Then the dual of the assertion

$$\Gamma, z \in D \vdash A(z)$$

is

$$A(z) \vdash \Gamma^{\perp}, z \in D^{\perp}$$

and so the dual of $\Gamma \vdash (\forall x \in D)A(x)$ is $(\exists x \in D^{\perp})A(x) \vdash \Gamma^{\perp}$. The symmetric of the dual is then

$$(\Gamma^{\perp})^{s} \vdash (\forall x \in D^{\perp})A(x)$$

Then a quantum state originates a couple of the different representations in $\mathbf{BL} + \mathbf{R}^{\bowtie}$, one is the symmetric of the dual of the other. The symmetric of the dual actually represents the NOT of the quantum state we are considering.

The states that are eigenstates of the NOT gate are characterizable as those representable in a unique way. For, $D = D^{\perp}$ if and only is the measurement experiment is performed on an eigenstate. As is well known, the "most significant" states for quantum computation are eigenstates of NOT. We refer to the so-called "cat state" representable in C^2, that is $1/\sqrt{2}|0\rangle + 1/\sqrt{2}|1\rangle$, and the Bell's states representable in C^4, that is the states $1/\sqrt{2}|00\rangle \pm 1/\sqrt{2}|11\rangle$ and $1/\sqrt{2}|10\rangle \pm 1/\sqrt{2}|01\rangle$.

2.3.2 Some short considerations on cut elimination

The problem of normalization of proofs is completely under development. In general, we find it a difficult problem, for several reasons.

Technically, we can distinguish two kinds of derivations:

i Derivations with one type of variable-link only;

ii Derivations mixing different types of variable-links.

In the first case, for $\alpha = 0$, we have the derivations of **BL** enriched with the universal quantifier. In such case, one need to extend the cut-elimination procedure of basic logic to the predicative case. For a fixed $\alpha \neq 0$, we have derivations of **BL** enriched with the parallel rules for the universal quantifier, and with the connective $\bowtie_\alpha$. This is also a situation similar to that of basic logic, since α is fixed.

In the second, the mixing of two different types of subscrips is allowed by the application of the $cutR_\alpha$-rule to sequents with possibly different variable-links. In particular, we have the applications of the $\bowtie_\alpha$ r-rule (that is due to $cutR_\alpha$). This situation is complicated by the possible presence of different types of cuts.

Moreover, the problem of the meaning of the normalization of proofs in a paraconsistent setting, like ours, is very delicate, the idea cannot be simply "good=cut free". For example, if one can prove a result, in the usual logical language, by means of a paraconsistent derivation in an enriched language, the derivation doesn't satisfy the subformula property, and hence a cut must be used. It is an advantage to reach the proof, even if in a paraconsistent framework. In general, such a possibility is left open, even in the theory of arithmetic. Does the ineliminability of cut, in such case, lead to the unprovability of certain results in a consistent computational framework?

Our system is motivated by the search of alternative computational strategies, since this is the aim of quantum computation. Then normalization should be discussed as a local property, concerning the completely paraconsistent fragments, namely fragments containing only α-links, for one or more $\alpha \neq 0$, or, obviously, the completely consistent fragments, containing only $\alpha = 0$. This corresponds to what is suggested by quantum computation. In the logical system we have outlined, if we prove a result in the usual language of sequents via a paraconsistent proof, we need the application of a substitution rule, representing the unique rule which can convert an α-link into a comma. This should be like the collapse of the quantum state into a classical state, from a physical point of view. Then, since the collapse is an irreversible moment, our proof should not be normalizable, since normalization corresponds to reversibility of proofs. But we could also isolate the fragments of proofs corresponding to the quantum computational processes prior to measurement. Such fragments should be normalizable.

Moreover, as R. Feynman, introducing quantum computing, stressed, quantum physics can be simulated by a classical computer, at the price of an exponential slow-down of the computational processes [Fe]. Then it should also be possible to convert our proofs into classical proofs. Then it is also open the problem of a strategy for such conversion.

2.3.3 A forgetful substitution

We describe now a particular kind of conversions of proofs, leading to inconsistent derivations, that will be useful in the considerations of the next chapter. For every random first order domain $D \equiv D(Z, p\{Z = z\})$ one can consider the first order domain D_{Z_0} of the set of outcomes without the associated probabilities. For example, if Z is associated to a dice toss, D_{Z_0} is the set of the first six integer numbers, no reference to their probabilities. The randomness of the elements of the domain is forgotten and one has simply a first order domain. We now consider the substitution of $z \in D$ with $d \in D_{Z_0}$. Such substitution forgets the probabilities and eliminates the variable-link. Then the sequent

$$\Gamma, z \in D \vdash \Delta_1 ,_Z \Delta_2$$

is converted into the sequent

$$\Gamma[z/d], d \in D_{Z_0} \vdash \Delta_1[z/d] ,_0 \Delta_2[z/d]$$

Let us suppose to apply the forgetful substitution to the rules of $\mathbf{BL} + \mathbf{R}^{\bowtie}$. The rules $W_{\alpha \neq 0}$ and $C_{\alpha \neq 0}$, are converted into the usual rules of weakening and contraction at the right, hence the three disjunctions (additive, multiplicative and Bell's disjunction) converge to a unique disjunction, the usual intuitionistic one. The $\forall ||$ rule is converted into an inconsistent rule, which proves the manicheist distributivity. In general, derivations of $\mathbf{BL} + \mathbf{R}^{\bowtie}$ are converted into inconsistent derivations by a forgetful substitution applied to a whole proof.

An exception is given by the trivial case of having a unique certain outcome, as in the case of the measurement of sharp states. In such case, the domain $D = D_N$ is the singleton given by the unique outcome, with probability one, of a "null random variable" that we denote by N. Bell's distributivity, for N, is

$$(\forall x \in D_N)A(x) \bowtie_N (\forall x \in D_N)B(x) = (\forall x \in D_N)(A(x) \bowtie_N B(x))$$

If one applies the forgetful substitution, the equality becomes $(\forall x \in D_{N_0})A(x) * (\forall x \in D_{N_0})B(x) = (\forall x \in D_{N_0})(A(x) * B(x))$, the manicheist distributivity. In such case it is true, even if $*$ is interpreted as a normal disjunction, since the domain has one element only. $\mathbf{BL} + \mathbf{R}^{\bowtie}$ can prove Bell's distributivity and, in the particular case of a singleton as a domain, the forgetful substitution doesn't produce an inconsistent derivation.

Then the advantage of considering null random variables, namely singletons as domains, is that, even with paraconsistent derivations, truth is not lost.

2.4 Comments in the framework of the interpretations of Quantum Mechanics

Counterfactuality

Counterfactuality is at the root of Bohr's interpretation of quantum mechanics. Bohr points out the importance of the transition between potentialty and actuality. We think that sequents, representing assertions, can clearly express such point. Indeed, it seems to us that the interpretation of quantum superposition, obtained by means of the definitory equation of the metalinguistic *forall* link:

$$\Gamma \vdash (\forall x \in D)A(x) \qquad \equiv \qquad \Gamma, z \in D \vdash A(z)$$

is a way to describe the counterfactual definiteness of the outcomes of quantum measurements. It is obtained thanks to the adoption of the idea of variable, that represents the way in which our mind can deal with a mathematical/abstract object without giving it actually. Then the quantum state is decribed by means of the quantifier, that allows to grasp it as a whole. This gives a sort of “objectivity” to the quantum state. Indeed, such objectivity is very thin, since the variable, inside the quantifier, is bounded, in other words “locked”, it can get no value actually! In order to unlock the variable, one has to consider the transition

$$(\forall x \in D)A(x), z \in D \vdash A(z)$$

(obtained as the reflection axiom of the definitory equation), that describes the transition from a state to *any* of the possible outcomes of its measurements, potentially. Then, as we have seen, the transition between potentiality and actuality is obtained by substituting the variable by a closed term t, obtaining:

$$(\forall x \in D)A(x) \vdash A(t)$$

Such further step is also due to a further attitude of our mind, that produces the ability of substituting. We attribute such attitude to the “objectivization” of the variable. Only when we see the variable as an object, namely only when we can actually conceive it as an object consciously, we can substitute it, since we are aware of its status of potential representative of any of the possible values in a certain range. Such further step, in conclusion, is due to our consciousness of the meaning of the variables.

After substitution, the original superposed state is lost, as the potentiality given by the variable is lost. As we have observed, logic cannot reconstruct a quantifier by means of propositional logic, as physics cannot reconstruct the superposed state after measurement. This means that our objectivation of the variable destroys its original richness, that is hidden to our consciousness.

Hidden variables

Perhaps considering variables as logical first order variables, as our approach suggests, could allow a wider discussion concerning what variables mean, in order to discover, in particular, in which sense they should be hidden in quantum mechanics. One can hypothesize a hidden treatment of variables, due to a logical reason, proper of quantum mechanics. In particular, the logical hidden treatment of variables we have hypothesized allows to consider the variable-link between assertions, in order to represent the entanglement link, by the definitory equation:

$$\Gamma \vdash \bowtie_{x \in D_Z} (A(x); B(x)) \quad \equiv \quad \Gamma, z \in D_Z \vdash A(z),_Z B(z)$$

The variable-link, interpreted in the usual logical framework, gives false propositions. This is why, ultimately, the variable must be hidden!

We remind that a quantum system with Hilbert space $\mathcal{H}$ admits hidden variables if there exists a measurable space (Λ, Σ) such that every state $\psi \in \mathcal{H}$ can be represented as a probability measure μ_ψ on Λ, and every observable A as a measurable map: $\tilde{A} : \Lambda \to R$, whose expectation value with respect to μ_ψ is consistent with quantum mechanical predictions:

$$\langle\psi|A|\psi\rangle = \int_\Lambda \tilde{A} d\mu_\psi$$

(In such a scheme, the hidden variables $\lambda \in \Lambda$ are thought as the subquantum extension of the classical phase space (p, q) of Hamiltonian mechanics).

The proposal of hidden variable theories was made with the aim to support a realistic view of quantum mechanics, but it has been observed ([Sm]) that one could interpret a hidden variable theory as a simple counterfactual definitness, rather than an actual value definitness. Admitting such particular view, even if not contemplated, perhaps, by the original proposal, it makes a sense to look for an analogy between the above equality and the position

$$\Gamma \vdash (\forall x \in D)A(x) \qquad \equiv \qquad \Gamma, z \in D \vdash A(z)$$

An integral is like an infinitary sum and its variable is bounded. The domain of the variable, in the integral above, is the sample space Λ. In our interpretation, the quantifier is also like an infinitary sum (cf. the intuitionistic interpretation in [ML84]), its variable is bounded, and its range is a probabilistic domain. We consider logical propositions on variables, on a domain given by an observable, rather than measurable maps on the sample space representing an observable. Then it seems to us that our definitory equation resembles the equality of the hidden variable theory reminded above.

Contextuality

The hidden treatment of variables in building links between assertions, in our hypothesis, is that determined by the quantum computation. In terms of sequents, it is translated into the adoption of parallel rules, rather than having sequentializable rules in presence of contexts in the sequents. Note that a calculus performed in presence of contexts is context-insensitive; context-sensitive calculi are those in which the context matters and then, definitely, it is not a context any more. In such case, there is an actual link with the context. The problem of contexts in sequent calculus, in the interpretation by sequents, can represent a direct translation of the problem of contexts in quantum mechanics.

In particular, the von Neumann-Gleason-Kochen-Specker theorem proves that a hidden variable theory compatible with the predictions of quantum mechanics must be contextual. A measurement context is defined as a set of commuting observables. Then, any measurement restricts the possible contexts to compatible observables. The same fact can be pointed out by sequents, as we have seen in section 2.2.1. Then again we find a meet between the use of logical variables in sequent calculus, as in our interpretation of quantum computational parallelism, and the hidden variable theories and their consequences. The computational framework of the interpretation by sequents points out that contextuality creates a computational advantage, then it shouldn't be felt as a penalty.

Causality

We conclude with the problem of causality, that is also inherent the interpretation by sequents, at least at a formal level, if we consider the *yield* in metalinguistic link, that originates the logical implication, as a causal link.

Logical implication is introduced by the following definitory equation:

$$\Gamma \vdash A \to B \qquad \equiv \qquad \Gamma, A \vdash B$$

then it requires that the premises Γ are a context with respect to the antecedent A of the implication.

The variable-link, in its dual formulation of $\mathbf{BR} + \mathbf{L}^{\bowtie}$, creates assertions of the form

$$A,_Z B \vdash \Delta$$

In such case, A (or B) cannot be considered as antecedent of an implication, treating B (or A) as if it were a context. For, the variable-link creates a whole that cannot be separated. Then, where the entanglement is present, the implication is impossible, and conversely. It seems that the stochastic nature of the variable link is in contrast with the causality expressed by the implication.

Chapter 3

Quantum computation and unconscious computation

We compare the semantical features of the computational model of quantum parallelism, with some aspects of human thinking. The thesis is that human thinking adopts quantum computational strategies. This is supported by quantum theory of mind, and by Matte Blanco's bi-logic.

3.1 Logical processes for the mind

The substantial failure of artificial intelligence, in achieving an imitation of the human natural intelligence, has made clear that the logical processes that have been considered up to now are insufficient, if not unsuitable, in order to represent the humand mind processes. This is witnessed also by recent results in neuroscience, connected with the discovery of mirror neurons. Indeed, such research has shown that the comprehension of the meaning of an action isn't obtained in a *procedural* way, namely by comparing the information just achieved with those already present in a supposed database in the mind, that was the way in which A.I. intended to explain and to re-create human comprehension. In some interpretations, research on mirror neurons has shown that the comprehension of the meaning of an action is obtained since the subject “lives again” the action itself.

Then logic has the challenge to propose alternative logical links, which can reproduce the process in our mind by an “imitation of nature”. We have to discover how our mind can process assertions, without assuming logic as already given. So we think that a tool as the principle of reflection, we have exploited in justifying our logical rules, is particularly suggested in order to obtain significant logical rules. Then, one must furtherly discuss what kind of truth has to be considered. As we will see below, a paraconsistent setting is needed then.

3.1.1 Quantum mind and computation: Hameroff-Penrose theory

The crucial problem is now: what are the processes of our mind, up to now unexplored from a computational point of view, that we should consider? A further question is: why are they so hidden?

Following quantum theories of mind (see [At]), the quantum processes, which take place in our brain, contribute to form our mind. Then we could speak of quantum processes in our mind. In particular, Hameroff-Penrose quantum theory of mind sets the quantum processes of the brain in the tubulins, which are proteins forming the microtubuli, which in turn are component of the neural cells. Tubulines are dimers characterized by two different states, which are present in a quantum superposed state [HP].

Following Hameroff-Penrose theory, the quantum processes in the tubulins are at the basis of a distinction between unconscious and conscious states. While the first would coincide with the states of quantum superposition, consciousness would coincide with the moment of the decoherence of the superposition. Following Penrose, such particular decoherence wouldn’t be caused by an external factor (for example a measurement), but by a spontaneous collapse, due to a quantum gravitational treshhold, whose mathematical description cannot be translated into computational terms. This should be a point inherently non computable in physics. This fact would have the same consequence in the mind. Following

Penrose, consciousness is non computable, and, for this reason, artificial intelligence can never be obtained.

However, it seems that Hameroff-Penrose theory allows a better comprehension of our mind from a computational point of view, since it allows to answer the couple of questions posed above. For, it compels to consider the features of quantum computation in order to describe the nature of the computation of our mind. Moreover, if quantum computation takes place only in the unconscious, we cannot be aware of its consequences, contrary to the effects of classical computation, of which we are aware. This would allow to explain why a fundamental part of the links between assertions performed by the mind would be hidden to us.

Kurt Goedel, referring to the conclusions of his own incompleteness theorems, used to say that an intelligent machine cannot exist, or at least it cannot be known to us. For, if any such machine existed, and we were aware of its functioning, we should conclude that it cannot compute some true facts, since our awareness of its functioning would allow us to repeat the coding of the incompleteness theorems themselves. Then, Alan Turing observed that [Tu]

"... if a machine is expected to be infallible, it cannot be also intelligent. There are several theorems which say almost exactly that. But these theorems say nothing about how much intelligence may be displayed if a machine makes no pretence at infallibility."

Infallibility in logic means non contradiction, then one can escape the conclusions of Goedel's theorems dropping their assumption of consistency. Of course, no mathematician likes dropping such hypothesis. Anyway, if we want to observe the computational aspects of our mind, it is exactly what we need to do. Not by chance, it is also what we find in the computational models of quantum parallelism. Ultimately, the judgements "non computable", given up to now, have referred to the fact that something isn't *consciously* computable. Just connecting computation and consciousness suggests, now, that we can try to reconstruct, consciously, unconscious paraconsistent strategies of computation. There is a hope they can reach an imitation of our our processes to achieve truth: no theorem goes against this fact.

3.1.2 Matte Blanco's bi-logic

The idea of considering Matte Blanco's bi-logic was suggested by Stuart Hameroff, in order to find analogies for the computational model of the unconscious he had found with his quantum theory of mind.

As is well-known, the chilean psychoanalist I. Matte Blanco proposed a description of the logic of the unconscious, syntetized after thirty years of clinical experience, in his main book *The uncounscious as infinite sets* [MB]. Bi-logic consists on the contraposition between usual "bivalent" logic, that is proper of our conscious reasoning, and the so called "symmetric mode", or "indivisible mode" proper of the unconscious reasoning. Bivalent logic is based on the two usual truth values, that are separated, and is consistent. In the indivisible and symmetric mode, the opposites are identified and unified into a whole thing, for which negation is meaningless. It is governed by a principle of symmetry, following which every relation is considered as if it were symmetric. In particular, the part and the whole are identified. Such identification is furtherly discussed in terms of "infinite sets", since, as concluded by Matte Blanco himself, when a set is identified with a subset, they must have the same cardinality and hence the set is infinite.

Matte Blanco's method outlines precisely the fundamental logical features of the unconscious with the aim of a better comprehension of the unconscious itself. Obviously, his method has no computational purposes. Neverthless, we think that the features described by Matte Blanco meet the features of the paraconsistent calculus for quantum computation we have proposed, and some features of quantum computation in general, as we now see. This can be considered an argument in favour of quantum mind, too.

Matte Blanco clearly describes a paraconsistent logical system. The absence of negation finds its corresponding in the auto-duality we find for the quantum states that are eigenvectors of the *NOT* gate (cf. section 2.3.1). Actually, every quantum state, represented by a $\forall$, summarizes pieces of information that are considered "opposite" after measurement, while they are present together before. Moreover, implication isn't possible in presence of the variable-link, and hence negation, if considered defined by means of implication. Implication gives an asymmetric relation between propositions, hence its absence fits with the symmetric mode of the unconscious. On the other side, the entanglement link gives a symmetric kind of relation. Moreover, the idea of the "infinite sets", in our view, is in accordance with the necessity of the variables in representing quantum parallelism, as we see in more detail below. In particular, where substitution is impossible, one cannot decide the equality, and hence the domain has to be considered infinite, since counting the elements implies that one can always decide if they are equal or not. In a second perspective, one can decide that two elements are always equal and hence the domain is identified with a singleton, namely "the stereotype" for that class. This is treated in details by Matte Blanco, in commenting his infinite sets. It could correspond to a computational solution, as we see below.

3.2 Holistic thinking given by the variable

In quantum computation, the entanglement creates a holistic kind of link, where the whole thing isn't equivalent to the sum of its parts [DCGL2]. This can be read in two ways: the sum of all parts is not enough to obtain the whole thing and/or the sum of all parts is not necessary to it. This is like understanding a sentence, knowing the meaning of each word (given that this is possible) isn't enough to understand the sentence, but, on the contrary, sometimes it is possible to understand the sentence without knowing some or even a lot of its words. In this last sense, the holistic link is an advantage for our mind, that can exploit it.

We might think that logic cannot deal with holistic links, since logical connectives are defined in a compositional way, but this is not the case of predicative logic. Indeed, in order to understand $(\forall x \in D)A(x)$ we have, usually, two strategies: to consider each element of the domain or to consider a generic element, as a variable. The second strategy is necessary when the domain is infinite, then it corresponds to an “infinitary idea” of the domain. In the second case, one creates a new logical entity without composition of parts. For this reason predicative logic gives room to the entanglement link too.

Note that the variable becomes a part of our object-language in the adolescence, when we reach self-consciousness. We can retain that the logical processes of which we are aware, described in classical predicate calculus, interpret such idea of variable. Beside this, it should exist a child-level for the variable, in which it is exploited in the process of computation but not objectivized. For example, quite big children can understand the meaning of a rule as “the last going out must close the door” (that, like every rule, contains a variable), even if they are not able to understand the meaning of a juridical code or of a mathematical theory containing variables. Smaller children are not even able to understand the rule. Exploiting a sort of passage to the limit, one could say that the use of variable in mental processes is even more inward, up to being completely unknown for the subject. Moreover, it is plausible that the variable is kept as hidden as much one is far from self-consciousness.

What is the advantage of all this? An enormous computational and cognitive advantage. Children are “very bad” logicians, mainly in the first childhood, anyway they have a neat advantage in cognition. For example, children learn the mother language in the first three years, before the separation from the mother; the mother language must be achieved in the first childhood, otherwise it is not possible any more, and finally a second language can reach the level of the mother language only if acquired before the end of childhood.

As we have seen in the previous chapter, the holistic link producing the entanglement is a constraint in logic, since it is alternative to logical consequence. Then our mind, linking assertions, should adopt at least two modalities: the first one is holistic, computationally advantageous, through free unconscious associations of variables. We would like to call it “pre-logical modality”. It doesn’t consider the usual truth values, but rather a different global truth that we describe at the end. The second one is analytic and methodologic, it gives up the free associations obtained by variables in favour of a conscious use of them, and so gives room to the logical consequence. This last one is a computational mode of which we are aware, founded on the separation between the two truth values, even in a constructive setting for reasoning. It avoids the contradiction between them. The two modalities are due to different modes in self-consciousness. It is clear that the two modes are intertwined in our life and hence they give rise to different mixed types of thinking. Moreover, we can have exchanges of roles with dis-logical effects, that can also be observed in our thinking. In particular, the dis-logical effects concern the truth one concludes, applying a pre-logical modality in presence of the truth values we adopt in our logical modality. We give below some examples of what we mean.

The perfectionist and the manicheist

Manichaeist thinking is present in human thinking, both in terms of social, religious and philosophical, proposals, and in terms of individual conclusions, of which the subject is often unaware. A short example: a very good but very depressed mathematician. She would never prove that, since every integer number is odd or even, then every integer number is odd or every integer number is even. But, when she is asked "What is wrong?", she answers "everything". Then clearly she is not applying the same schema of reasoning in the two cases. Of course, in the second case the schema leads her to a wrong conclusion, but her problem is that she feels it as true. We have the case of a wrong truth.

An analogous problem arises with the perfectionist distributivity, namely, in our view, the schema that we have characterized as symmetric to the manichaeist distributivity. The idea of perfection is also present in religious and philosophical human thinking, with the idea of the "perfect being" or the "perfect aesthetics". Moreover, it is an individual attitude of mind one can often encounter, and which isn't in accordance to the usual treatment of the existential statements in mathematics. For example, people who would like to get married, but they cannot, because they are looking for a man or woman with "all the best features". They cannot accept any husband or wife, since no one corresponds to the truth they have in mind, that is necessary for them.

The manichaeist thinking is also adopted in decision making, when one needs to "hurry up" and hence renounce to the analytical thinking, usually with bad results. But, in psychological research on decision making for marketing, it has also been discovered that an unconscious strategy, rather that the usual analytical one, consciously performed, is adopted with better results. In such case, the advantage of the unconscious reasoning increases if the number of variables of the given problem is high [Dijk]. This could be a confirmation of the hypothesis that the different treatment of variables gives the computational advantage to the unconscious reasoning.

Stereotyping

Another fact one can observe is the tendency to use stereotypes, namely to identify every element of a certain class with a prototype. For example, in certain circumstances, one is lead to consider every dog as the dog of her childhood, which coincides with The Dog. In a different setting, someone who needs to reach a certainty about a certain unknown person, often makes the choice of considering the person in a certain class, applying then her stereotype for that class. For example, one may come to know that the person is a teacher, and then he applies his idea of teacher.

Stereotyping means to reduce every domain to a singleton. As we have noticed in section 2.3.3, this means that we can apply a paraconsistent reasoning without renouncing to truth. In some cases it can be that we are trying to recover a truth that has been forgotten, as in the case of the dog; in some other, we are desperately trying to obtain some kind of truth, that is a very poor one. Actually knowing something about a person is as difficult as knowing the state of a quantum system, deciding that it is a sharp state is to renunce to the richness of the system allowed by quantum physics.

Irreversible thinking

The so called "irreversible thinking" was first observed by Piaget. We describe it by a typical example: there is a box containing several pieces of two different colours (red/blue), two different sizes (big/small), two different shapes (circle/square). Then children up to four years old can form two subsets distinguishing the pieces by colour, or by size, or by shape. But, once they have made a choice of one of the three different "observables", they doesn't change her mind in favour of a second different classification. Children if encouraged, can do this at five years old. Children's thinking is irreversible up to four years old, in this example.

In our view, this corresponds to the computational advantage of quantum mechanics, created by the fact that non commuting observables are incompatible (see 2.2.1). The non-analytic thinking of children doesn't allow the increasing of complexity, while probably it allows that other associations are created, so that their thinking is different and original with respect to the adult thinking, and everyone can agree that it creates a different flavour of life. Indeed, childhood is ofted considered as "lost".

Perception and truth

One can make the hypothesis that we have a different associative thinking originated by our first attitude of mind. For example, one can associate two different objects by colour, something which was as red as ..., and so on. Usually perception favours associations, for example one can remind something forgotten after hearing a certain sound, or smelling a certain smell, or tasting a certain taste. The information we process is mainly achieved by our senses, so one can make the hypothesis that a paraconsistent setting is a particular advantage in such case. This is confirmed by the fact that most of the processes concerning perception are unconscious, since the storage of all the information achieved is impossible at the conscious level.

"None so deaf as those that won't hear". Perception requires open-minded people, without prejudices, the proverb says. One has to be ready to any truth, for a correct perception. If an object can be black or white, we have to be ready to both colours, considering that both are possible with equal probability. Let $C = \{(black, 1/2), (white, 1/2)\}$. Then the truth, namely the correct judgement, before perception, has the following form [1]:

$$(\forall x \in C)A(x)$$

Looking at the object is like a measurement, and then one has only one of the two, black or white. The difference with quantum systems is that the object is always in a sharp state, black or white, not in a superposed state of black and white. Then “black” or “white” becomes a judgement, not a prejudice. This comes from our experience. Anyway the best attitude of mind for an observer is the same attitude of the observer of a quantum system. The fact that it isn’t a quantum system is discovered later, from experience. Perhaps this induces everybody to disregard the original attitude of mind, and then the original judgements processed by our mind, before the experience of the external world, that can be only experience of the macroscopic world, gets the upper hand of us. Then one conceives the Aristotelian truth values, for which different kinds of processes are suited.

[1] This is also what is made evident by the famous pictures proposed in the 20’s by the Gestalt psychology, like, for example, the picture of the young-and-old lady.

Bibliography

[At] Atmanspacher, H., *Quantum approaches to consciousness*, in Stanford Encyclopedia of Philosophy, http://plato.stanford.edu/entries/qt-consciousness.

[BSm06] Baltag, A., Smets, S., LQP: The Dynamic Process of Quantum Information, Mathematical Structures in computer Science, Special Issue on Quantum Programming Languages, **16**(3) (2006) 491-525.

[BSm08] Baltag, A., Smets, S., A Dynamic-logical perspective on Quantum Behaviour, in I. Douven and L. Horsten eds., Studia Logica, special issue on applied logic in the philosophy of science, **89** (2008) 185-209.

[Batesi] Battilotti, G., *Logica di base attraverso il principio di riflessione*, PhD Thesis, Universita' di Siena, 1997.

[Ba05] Battilotti, G. Basic logic and quantum computing: logical judgements by an insider observer, Proc. FQI04, International Journal of Quantum Information **3**(1) (2005) 105-109. arXiv: quant-ph/0407057.

[Ba07] Battilotti, G., Logical implication and attachment bond, italian, draft (2007). Available at www.math.unipd.it/ giulia. Abstract presented at "Towards a science for consciousness", Budapest, 2007.

[Ba08] Battilotti, G., Logica, computazione e teorie quantistiche della mente, HumanaMente **5** (2008) 119-132.

[B98] Battilotti, G., Embedding classical logic into basic orthologic with a primitive modality. Logic Journal of the IGPL, Special Issue on Generalized Sequent Systems, H. Wansing ed., **6** (1998) 383-402.

[BF] Battilotti, G., Faggian, C. *Quantum Logic and the cube of Logics.* Chapter "Quantum Logic" (M.L. Dalla Chiara, R. Giuntini) in Handbook of Philosophical Logic, new edition, **6** D. Gabbay and F. Guenthner eds., Kluwer (2002).

[BS99] Battilotti, G., Sambin, G., Basic Logic and the cube of its extensions. In "Logic and foundations of mathematics", Cantini, Casari, Minari eds. Kluwer (1999) 165-186.

[BoS] Bonnay, D., Simmenauer, B., Tonk strikes back, Australasian Journal of Logic **3** (2005) 33-44.

[CP] Calude, C. S., Paun, G., Computing with cells and atoms, Taylor & Francis Publishers, London (2001).

[DCG] Dalla Chiara, M. L., Giuntini, R. Paraconsistent Quantum Logic, Foundations of Physics **19**(1989) 891-904.

[DCGL2] Dalla Chiara M.L., Giuntini R., Leporini R., Compositional and Holistic Quantum Computational Semantics, Natural Computing (2006).

[DCGsurvey] Dalla Chiara M.L., Giuntini R., *Quantum Logic*, Handbook of Philosophical Logic, new edition, **6**, D. Gabbay and F. Guenthner eds., Kluwer (2002).

[DCGL1] Dalla Chiara, M. L., Giuntini, R., Leporini, R. Quantum Computational Logics. A Survey, in V. F. Hendricks, J. Malinowski eds., Trends in Logic: 50 years of studia logica, Kluwer Academic Publishers, Dordrecht (2003) 213-255.

[Dijk] Dijksterhuis, A., Bos, M. W., Nordgren, L. F., van Baaren, R. B., On making the Right Choice: The Deliberation- Without-Attention Effect, Science **311** February 2006.

[En] Enriques, E., *Problemi della scienza*, Zanichelli, Bologna (1906).

[FS] Faggian, C., Sambin, G., From basic logic to quantum logic with cut-elimination, Proc. IQSA96, International Journal of theoretical Physics **31** (1998) 31-37.

[Fe] Feynman, R. Simulating physics with computers, IJTP **21** (1982) 467.

[Ge] Gentzen G., Untersuchungen über das logische Schliessen, *Mathematische Zeitschrift* **39** (1935), 176-210; English translation in *"The collected papers of Gerhard Gentzen"*, (M.E. Szabo ed.) North-Holland (1969) 68-131.

[Gi] Girard J-Y., Linear Logic, *Theoretical Computer Science* **50** (1987) 1-102.

[Hi] Hirvensalo, M., *Quantum Computing*, Springer, Berlin-Heidelberg (2004).

[Ja] Jammer, M., *The philosophy of quantum mechanics*, J. Wiley & Sons, New York (1974).

[Li] Liotti, G., *La dimensione interpersonale della coscienza*, Carocci, Roma (2005).

[ML84] Martin-Löf P., *Intuitionistic type theory, Notes by Giovanni Sambin of a series of lectures, etc etc*, Bibliopolis, Napoli (1984).

[MB] Matte Blanco, I. *The unconscious as infinite sets*, Duckworth, London 1975.

[HP] Hameroff, S., Penrose, R. Orchestrated Reduction of Quantum Coherence in Brain Microtubules: A model for Consciousness? in Towards a Science of Consciousness, Hameroff, Kaszniak, Scott eds., Cambridge MA MIT Press (1996) 507-540.

[MS] Maietti, M.E., Sambin, G. Toward a minimalist foundation for constructive mathematics, in "From Sets and Types to Topology and Analysis: Towards Practicable Foundations for Constructive Mathematics" (L. Crosilla, P. Schuster, eds.), Oxford UP, to appear.

[NC] Nielsen, N. A., Chuang, I. L. *Quantum Computation and Quantum Information*, Cambridge University Press (2000).

[P-emp] Penrose R., *The emperors new mind*, Oxford Press (1989).

[P-sha] Penrose, R., *Shadows of the mind* (1996).

[PH] Penrose R., Hameroff S., Consciousness as orchestrated space-time selections, Journal of Consciousness Studies **3**(1) (1996) 36-53 .

[Pr] Priest, G., *Paraconsistent Logic*, Stanford Encyclopedia of Philosophy, http://plato.stanford.edu/entries/logic-paraconsistent/

[SBF] Sambin G., Battilotti G., Faggian C., Basic logic: reflection, symmetry, visibility, The Journal of Symbolic Logic **65** (2000) 979-1013.

[Sm] Smerlak, M. *Relational Quantum Mechanics*, internal report, available at http://enslyon.free.fr/rapports/physique/Matteo_Smerlak_2.pdf

[Tu] Turing, A., Lecture to the London Mathematical Society, in *Turing ACE report of 1946 and other papers*, Carpenter et al. eds., The Charles Babbage Institute, vol. 10 MIT Press, Cambridge MA.

[So] Sossai, C., From observable-based to probabilistic reasoning, In Fuzzy Sets, Probability, and Statistics - Gaps and Bridges, 28th Linz Seminar on Fuzzy Set Theory, D. Dubois, E.P. Klement, and R. Mesiar (eds), Linz, Austria (2007) 108-111.