Quantum Logic and the Cube of Logics

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1 Quantum logic and the cube of logics

1.1 Introduction

Different forms of quantum logic can be axiomatized as a sequent calculus ([6], [11], [4], Tamura 1988, Nishimura 1994). This permits to investigate such logics more and more deeply from the proof-theoretical point of view. A sequent calculus for orthologic can be obtained from a calculus for classical logic, by requiring a special restriction on *contexts* in the rules that would permit to derive the distributive laws. The critical rules are the following: the introduction of disjunction on the left, the introduction of conjunction on the right, the rules concerning implication and negation. However, such restriction determines some serious proof-theoretical difficulties, in a situation where we want to have a sufficiently strong negation that satisfies de Morgan's laws. The shortcoming becomes apparent when we try to prove the cornerstone result, that is represented by a cut-elimination theorem. As is well known, cut-elimination essentially depends on the formulation of the rules that appear in our proofs.

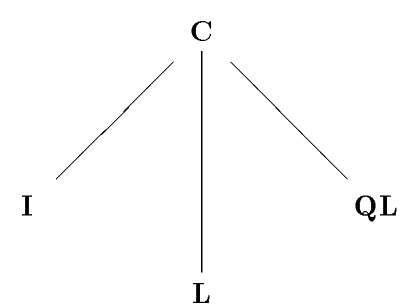
A simple and compact sequent calculus for orthologic, ([9], [7]), which admits cut-elimination by means of a neat procedure, can be obtained by a convenient strenghtening of *basic logic*. Basic logic is a new logic that has been proposed in order to find out a general structure for the space of logics (see [13], [14], [2]).

In the framework of basic logic, constraints on contexts are not considered a limitation; on the contrary, they are regarded as a positive feature, which is called *visibility*. At the same time, negation is treated by exploiting the symmetry of the calculus: the main idea is to use Girard's linear negation, which can be interpreted as an orthocomplement in a quite natural way. This approach shows that orthologic (and non-distributive logics, in general) admit a proof-theory, which turns out to be simpler than the proof-theory for classical logic. Describing quantum logic in the framework of an uniform and general setting gives some other advantages: for instance, this permits us to study various logics and their mutual relations at the same time. In particular, we obtain a whole gamma of quantum logics (including *linear orthologic*; and for each of these logics we have a proof of the cut-elimination theorem). Moreover, one obtains a new formulation for classical logic (see [8]), with respect to which orthologic and the other quantum-like logics (created by this method) turn out to be characterizable as *substructural logics*. On this basis it is easy to compare different logics, and to prove embedding results (see [1]).

1.2 Basic logic and the cube of logics

As we already know, quantum logic represents a weakening of classical logic, obtained by dropping the distributive laws. There are at least two other important ways to weaken classical logic: intuitionistic logic and linear logic ([10]). The situation can be sketched as follows:

Picture 1



It is natural to ask whether there exists a logic that represents a common denominator for **Q**, **I** and **L**, in the same way as classical logic includes all the other logics. A solution to this problem has been found in terms of a suitable sequent calculus **B**, that represents a *basic logic*.

Differently from the calculi we have considered in the previous sections, a *sequent calculus* for a given logic $\mathcal{L}$ is based on *axioms* and *rules* that govern the behaviour of *sequents*. Any sequent has the form

$$M \vdash N$$

where M,N are (possibly empty) finite multisets of formulas[1]. Axioms are particular sequents. Any rule has the form

$$\frac{M_1 \vdash N_1 \quad \ldots \quad M_n \vdash N_n}{M \vdash N}$$

where $M_1 \vdash N_1, \ldots, M_n \vdash N_n$ are the *premises*, whereas $M \vdash N$ is the *conclusion* of the rule. Rules can be *structural* or *operational*. Operational rules introduce a new connective, while structural rules deal only with the structure of the sequents (orders, repetitions, etc.).

A *derivation* is a sequence of sequents where any element is either an axiom or the conclusion of a rule whose premises are previous elements of the sequence.

Basic logic has been introduced in [3], and substantially reformulated in [14]. In its second formulation, given here in table 1.2[2], it is characterized by three strictly linked principles: *reflection*, *symmetry*, *visibility*, which we briefly illustrate now. The reflection principle represents a method that leads to the rules of the calculus, starting from metalinguistic links between assertions. Such method analyses the following equivalences, which assert a correspondence between language and metalanguage

$$M \vdash \alpha \cdot \beta \quad \text{if and only if} \quad M \vdash \alpha \circ_R \beta$$

$$\alpha \cdot \beta \vdash N \quad \text{if and only if} \quad \alpha \circ_L \beta \vdash N$$

Here the generic sign "$\cdot$", corresponding to a metalinguistic link between assertions, is translated respectively into the connective $\circ_R$, when it appears

[1] A multiset is a set of pairs such that the first element of every pair denotes any object, while the second element denotes the multiplicity in which the object appears. Two multisets are equal if and only if all their pairs are equal, that is all their objects together with their multiplicities are equal.

[2] The formulation of the rules of **B** contained in [14] is based on finite lists rather than finite multisets of formulas, and hence it contains in addition the structural rule of exchange. Here we prefer to consider multisets, in order to obtain an easier comparison with sequent calculi for quantum logics. Moreover, adopting the usual notation of quantum logic, we will denote denote formulas by $\alpha, \beta, \ldots$, rather than by $A, B, \ldots$, as it is more common in proof theory and in particular in linear logic.

Table 1: Basic sequent calculus **B**

Axioms

$$\alpha \vdash \alpha$$

Operational rules

formation	$\dfrac{\beta, \alpha \vdash N}{\beta \otimes \alpha \vdash N}\ \otimes L$	$\dfrac{M \vdash \alpha, \beta}{M \vdash \alpha \parr \beta}\ \parr R$
reflection	$\dfrac{\alpha \vdash N_1 \quad \beta \vdash N_2}{\alpha \parr \beta \vdash N_1, N_2}\ \parr L$	$\dfrac{M_2 \vdash \beta \quad M_1 \vdash \alpha}{M_2, M_1 \vdash \beta \otimes \alpha}\ \otimes R$
formation	$\dfrac{\vdash N}{1 \vdash N}\ 1L$	$\dfrac{M \vdash}{M \vdash \perp}\ \perp R$
reflection	$\perp L \quad \perp \vdash$	$\vdash 1 \quad 1R$
formation	$\dfrac{\beta \vdash N \quad \alpha \vdash N}{\beta \vee \alpha \vdash N}\ \vee L$	$\dfrac{M \vdash \alpha \quad M \vdash \beta}{M \vdash \alpha \vee \beta}\ \vee R$
reflection	$\dfrac{\alpha \vdash N}{\alpha \wedge \beta \vdash N} \quad \dfrac{\beta \vdash N}{\alpha \wedge \beta \vdash N}\ \wedge L$	$\dfrac{M \vdash \beta}{M \vdash \beta \vee \alpha} \quad \dfrac{M \vdash \alpha}{M \vdash \beta \vee \alpha}\ \vee R$
formation	$0 \vdash N \quad 0L$	$M \vdash \top \quad \top R$
formation	$\dfrac{\beta \vdash \alpha}{\beta \leftarrow \alpha \vdash}\ \leftarrow L$	$\dfrac{\alpha \vdash \beta}{\vdash \alpha \rightarrow \beta}\ \rightarrow R$
reflection	$\dfrac{\vdash \alpha \quad \beta \vdash N}{\alpha \rightarrow \beta \vdash N}\ \rightarrow L$	$\dfrac{M \vdash \beta \quad \alpha \vdash}{M \vdash \beta \leftarrow \alpha}\ \leftarrow R$
order	$\dfrac{\alpha \vdash \beta \quad \gamma \vdash \delta}{\beta \rightarrow \gamma \vdash \alpha \rightarrow \delta}\ \rightarrow U$	$\dfrac{\gamma \vdash \delta \quad \alpha \vdash \beta}{\gamma \leftarrow \beta \vdash \delta \leftarrow \alpha}\ \leftarrow U$

on the right of the sign $\vdash$, and into the connective $\circ_L$, when it appears on the left. In **B**, rules for connectives are completely determined by such equivalences. As a consequence, the meaning of a connective turns out to be uniquely determined by the correspondence with a metalinguistic link, quite independently of any link with a context. Since every metalinguistic link is translated into a connective according to two specular ways, the system of rules, obtained by this method, turns out to be strongly symmetric. In fact **B** contains, for every axiom and for every (unary or binary) rule R

$$\frac{M_i \vdash N_i}{M \vdash N}\ R$$

its symmetric rule R^s, given by

$$\frac{N_i^s \vdash M_i^s}{N^s \vdash M^s}\ R^s$$

where the map $(-)^s$ is defined by induction (on the degree of formulas), by putting $\circ_R^s \equiv \circ_L$ and $\circ_L^s \equiv \circ_R$, given a suitable correspondence between propositional variables. In accordance with the reflection principle given above, **B** satisfies the visibility property. A rule for a given connective is called *visible* when the principal formula and the corresponding secondary formulas appear in the rule without any context[3].

As an example let us refer to a rule that plays an important role in the case of quantum logic. As is well known, in classical logic, disjunction is introduced on the left according to the following rule

$$\frac{M, \alpha \vdash N \quad M, \beta \vdash N}{M, \alpha \vee \beta \vdash N}$$

In the case of **B**, instead, disjunction is introduced in the following visible form

$$\frac{\alpha \vdash N \quad \beta \vdash N}{\alpha \vee \beta \vdash N}$$

where context M has disappeared.

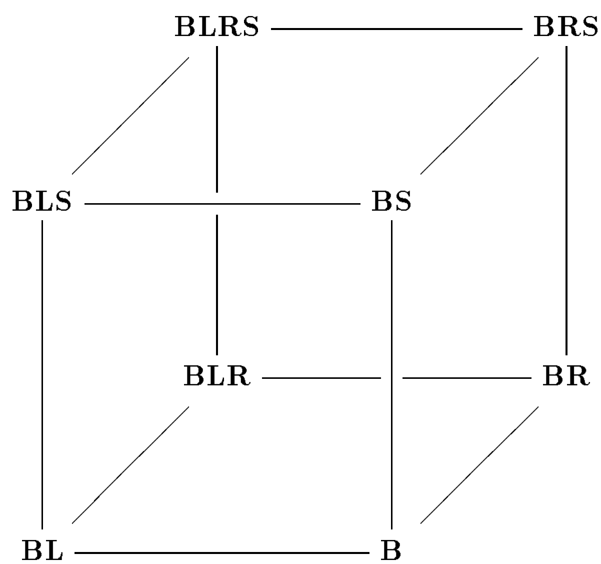
From the intuitive point of view, one can read the difference between the two cases as follows: the rule typical of classical logic attaches a meaning to the connective $\vee$ in presence of the link "," with M (such a link is to be

[3] In any operational rule, the formula in the conclusion which contains the connective introduced by the rule itself, that is the formula introduced by the rule, is called the principal formula; the formulas in the premises which are the components of the formula introduced by the rule are called the secondary formulas.

interpreted as a conjunction), whereas the visible rule is intended to explain the meaning of the connective $\vee$ by referring only to the connective itself. In particular, the visible rule does not permit us to prove the equation that links conjunction and disjunction (the distributive law of $\wedge$ with respect to $\vee$). As a consequence, any sequent calculus for a quantum logic shall adopt the visible form for the rule that concerns the introduction of disjunction on the left. As to the other rules, visibility is not strictly necessary in order to obtain an adequate sequent calculus for quantum logic. However, a more convenient strategy permits us to axiomatize quantum logic, by adding only structural rules to basic logic, without any change in the rules for the connectives. In this way, we can preserve the characteristic properties of symmetry and visibility of **B**, that turn out to be highly convenient from the proof-theoretical point of view (as we will see later).

Basic logic has been introduced in order to offer a general framework that permits us to investigate various logics, including quantum logics. Actually, no structural rule is present among its rules. Hence, as justified also by the semantics given by the principle of reflection, basic logic can be seen as "the logic of connectives", from which various stronger logics can be obtained by adding suitable structural rules, which permit us to deal with contexts. We can first distinguish three main kinds of structural rules, labelled by the letters **L**, **R** and **S**. The extensions of **B** resulting by the addition of any combination of such rules can be organized in the following cube:

Picture 2



that is conceived as an architecture whose basis is **B**. In the cube, every logic with "**S**" satisfies the structural rules of weakening and of contraction[4]; every logic with "**L**" allows left contexts in any inference rule; every logic with "**R**" allows right contexts in any inference rule. In particular, the cube solves our initial problem, sketched in picture 1. In fact, vertex **BLRS**, opposed to **B** represents classical logic, vertex **BLR** and vertex **BLS** represent respectively Girard's linear logic and to intuitionistic logic; finally, vertex **BS** corresponds to paraconsistent quantum logic (see next section). Moreover, since logics with "**R**" are simply the symmetric copy of logics with "**L**", logics containing both "**L**" and "**R**" (**BLRS**, **BLR**) or logics containing neither (**BS**, **B**), are symmetric. The study of quantum logics finds place in the diagonal of symmetric logics.

1.3 Sequent calculus for Orthologic

The logic **BS** is non-distributive. Let us consider the fragment of **BS** restricted to the connectives $\wedge$ and $\vee$. If we want to obtain, from it, a quantum logic, what is still missing is an involutive negation, satisfying de Morgan. This aim can be reached by extending the language to adopt Girard's negation. The key point is to assume as primitive symbols of the language both the propositional variables and their duals. In other words, the propositional literals are assumed to be given in pairs, consisting of a positive element (written p) and of a negative one (written $p^\perp$). On this basis, the negation of a formula is defined as follows:

$$p^{\perp\perp} \equiv p \qquad (\alpha \wedge \beta)^\perp \equiv \alpha^\perp \vee \beta^\perp \qquad (\alpha \vee \beta)^\perp \equiv \alpha^\perp \wedge \beta^\perp$$

With this choice, the calculus, denoted by $^\perp$**BS** (where the symbol $^\perp$ reminds us that the calculus is applied to a dual language), produces a logic, here called *basic orthologic*, that turns out to be equivalent to paraconsistent quantum logic (**PQL**), introduced in [5]. As we already know, **PQL** represents a weakening of orthologic, that is obtained by dropping the *non contradiction* and *excluded middle* laws. On this basis, a calculus for orthologic, denoted by $^\perp$**O**, is obtained by adding such laws. These are expressed as two new rules named *transfer*, which are structural (since they modify the structure of the sequent, without introducing any connective).

[4] In linear logic, connectives for conjunction and disjunction are distinguished into multiplicative and additive. In fact, there are two ways of formulating contexts in rules for connectives, which lead to a moltiplicative or additive form for each rule. The multiplicative and additive formulation are equivalent in presence of the structural rules of weakening and contraction. For this reason, the distinction is present in linear logic and a fortiori in basic logic, where weakening and contraction fail, and vanishes in classical logic and in orthologic.

The rules of ${}^{\perp}\mathbf{O}$ are the following, where rows (i) up to (v) constitute basic orthologic[5] whilst (vi) is *transfer*.

$$(i) \qquad \alpha \vdash \alpha$$

$$(ii) \qquad \frac{\alpha \vdash N \quad \beta \vdash N}{\alpha \vee \beta \vdash N} \vee L \qquad\qquad \frac{M \vdash \alpha \quad M \vdash \beta}{M \vdash \alpha \wedge \beta} \wedge R$$

$$(iii)$$

$$\frac{\alpha \vdash N}{\alpha \wedge \beta \vdash N} \quad \frac{\beta \vdash N}{\alpha \wedge \beta \vdash N} \wedge L \qquad\qquad \frac{M \vdash \alpha}{M \vdash \alpha \vee \beta} \quad \frac{M \vdash \beta}{M \vdash \alpha \vee \beta} \vee R$$

$$(iv) \qquad \frac{M \vdash N}{M, O \vdash P, N} \; weakening$$

$$(v) \qquad \frac{M, O, O \vdash N, N, P}{M, O \vdash N, P} \; contraction$$

$$(vi) \qquad \frac{M \vdash N}{M, N^{\perp} \vdash} \; tr1 \qquad\qquad \frac{M \vdash N}{\vdash M^{\perp}, N} \; tr2$$

The calculus ${}^{\perp}\mathbf{O}$ contains both $p, q, r\ldots$ and $p^{\perp}, q^{\perp}, r^{\perp}\ldots$. Moreover, for any rule of the calculus, the calculus shall contain also the symmetric one. As a consequence, whenever the calculus produces a derivation Π, it will also produce the dual derivation $\Pi^{\perp}$, obtained substituting every axiom $p \vdash p$ with the axiom $p^{\perp} \vdash p^{\perp}$ and every occurrence of rule with an occurrence of the symmetric rule (e.g. $\wedge R$ with $\vee L$). On this basis one has the following:

Lemma 1.1 *The following rule is derivable for* ${}^{\perp}\mathbf{O}$*:*

$$\frac{M \vdash N}{N^{\perp} \vdash M^{\perp}}$$

Proof. One can see that $M \vdash N$ is derivable by a derivation Π if and only if $N^{\perp} \vdash M^{\perp}$ is derivable by the symmetric derivation $\Pi^{\perp}$. $\square$

It is now immediate that:

[5] Note that, in ${}^{\perp}\mathbf{BS}$, weakening and contraction are redundant. In fact, one can see that such a calculus admits elimination of contraction, whilst weakening on the right and on the left can be simulated by $\wedge L$ and $\vee R$, respectively. So, PQL has a very simple formulation, given by (i), (ii), (iii).

Theorem 1.1 *$^\perp\mathbf{O}$ is a calculus for orthologic.*

We now see that the structure of the calculus $^\perp\mathbf{O}$ allows to prove the following cut-elimination result.

Theorem 1.2 *$^\perp\mathbf{O}$ admits elimination of the cuts*

$$\frac{O \vdash \mu \quad M, \mu \vdash N}{M, O \vdash N}\, cutL \qquad\qquad \frac{O \vdash \mu, P \quad \mu \vdash N}{O \vdash N, P}\, cutR$$

Sketch of the proof. Like in Gentzen, the cut-elimination procedure is obtained by induction on two parameters: degree and rank of the cut-formula[6].

The calculus $^\perp\mathbf{O}$ permits us to overcome in a simple way two questions that usually make cut elimination for orthologic so complicated: (i) constraints on contexts and (ii) negation. We give a sketch of the proof, considering the two points. The first problem is solved by visibility and the second by symmetry.

(i) As we have seen, in any calculus for quantum logic the rule that introduces $\vee$ on the left (here indicated with $\vee L$) shall have an empty context on the left. Now consider, for a generic calculus, the derivation

$$\frac{\dfrac{\alpha \vdash \gamma \wedge \delta \quad \beta \vdash \gamma \wedge \delta}{\alpha \vee \beta \vdash \gamma \wedge \delta}\, \vee L \qquad \dfrac{M, \gamma \vdash \Delta}{M, \gamma \wedge \delta \vdash \Delta}}{M, \alpha \vee \beta \vdash \Delta}\, cutL$$

In this derivation, the cut-formula is principal on the right premiss; hence the right rank is 1. In such a situation, Gentzen's procedure to lower the rank must operate on the left; this would necessarily produce the two derivations

$$\frac{\alpha \vdash \gamma \wedge \delta \quad M, \gamma \wedge \delta \vdash \Delta}{M, \alpha \vdash \Delta}\, cutL \quad \frac{\beta \vdash \gamma \wedge \delta \quad M, \gamma \wedge \delta \vdash \Delta}{M, \beta \vdash \Delta}\, cutL$$

[6] Given a derivation and a sequent containing a formula occurrence α, we can consider the paths, e.g. the successions of consecutive sequents, between that point and the point where the formula is introduced, both as an axiom, or by weakening, or as the principal formula of a rule on connectives. We define as rank the maximum among the lengths of those paths. That is, intuitively, the 'maximum length' between the formula occurrence we are examining and the point where that occurrence has been introduced.
The degree of a formula is, on the other side, its complexity, that is the number of connectives it is composed of.

Now, one would like to conclude by applying $\vee L$ in order to obtain $M, \alpha \vee \beta \vdash \Delta$. However, this step is here not allowed, unless M is empty. Such a problem does not arise for the calculus ${}^{\perp}\mathbf{O}$, because, by visibility, every principal formula has an empty context.

(ii) In ${}^{\perp}\mathbf{O}$ the only rules about negation are the structural rules of transfer. Let us consider a derivation of the form:

$$\dfrac{O \vdash \mu^{\perp} \quad \dfrac{\dfrac{\vdots\, \Pi}{M \vdash \mu}}{M, \mu^{\perp} \vdash}\, tr1}{M, O \vdash}\, cutL$$

We can reduce the rank in a quick way, by exploiting symmetry. In fact, Girard's negation has the nice property that every formula α and its dual $\alpha^{\perp}$ have exactly the same degree. The same idea can be extended to derivations, and hence to the rank of a cut. As we have seen in lemma 1.1, whenever we have a derivation Π for the sequent $M \vdash N$, we also have the dual derivation $\Pi^{\perp}$, which derives $N^{\perp} \vdash M^{\perp}$. The two derivations Π and $\Pi^{\perp}$ have exactly the same (symmetrical) structure. Hence in particular, if μ is principal, $\mu^{\perp}$ is principal. If μ has rank r, then also $\mu^{\perp}$ will have the same rank r. In such a situation, in order to raise the cut rule, we can substitute $\Pi^{\perp}$ by Π (*flipping derivation*). As a consequence, the initial derivation will be simply reduced to:

$$\dfrac{\dfrac{O \vdash \mu^{\perp} \quad \dfrac{\vdots\, \Pi^{\perp}}{\mu^{\perp} \vdash M^{\perp}}}{O \vdash M^{\perp}}\, cutL}{O, M \vdash}\, tr1$$

□

1.4 Quantum logics and classical logic

We will now consider the *symmetric diagonal* of the cube in the following diagram:

Picture 3

$$\begin{tikzcd}
\mathbf{C} \arrow[r, dash, "?"] \arrow[d, dash, "\mathbf{S}"] & {}^{\perp}\mathbf{O} \arrow[r, dash, "\mathbf{tr}"] \arrow[d, dash, "\mathbf{S}"] & {}^{\perp}\mathbf{BS} \arrow[d, dash, "\mathbf{S}"] \\
\mathbf{L} \arrow[r, dash, "?"] & {}^{\perp}\mathbf{OL} \arrow[r, dash, "\mathbf{tr}"] & {}^{\perp}\mathbf{B}
\end{tikzcd}$$

where calculus ${}^{\perp}\mathbf{O}$ appears as an intermediate point between basic orthologic and classical logic. Similarly, we have another intermediate point between basic logic and linear logic: this is given by ${}^{\perp}\mathbf{B} + \mathbf{tr}$, which represents the common denominator for orthologic and linear logic (we will call it "ortholinear logic" ${}^{\perp}\mathbf{OL}$). In the same way, ${}^{\perp}\mathbf{B}$ turns out to be the common denominator of basic orthologic and linear logic. On this basis, we obtain a whole gamma of quantum logics, which are all cut-free. The last of our logics, ${}^{\perp}\mathbf{B} + tr$, seems to be a good candidate in order to represent a *linear quantum logic* in the sense of Pratt ([12]).

So far we have only dealt with a fragment of basic logic, which has no implication connective. With this linguistic restriction, we have easily proved the equivalence between our calculi and the usual formulations of paraconsistent quantum logic and of orthologic. However, the same methods can be naturally applied to the complete versions of our calculi, preserving cut-elimination and flipping of derivations. In this way, we will have a primitive implication connective $\rightarrow$ (together with its dual $\leftarrow$) in every logic given above. An interesting question to be investigated concerns the possibility of physical interpretations of such new connectives.

In the diagram above, we have still a question mark concerning the path from orthologic to classical logic. Our question can be solved as follows:

Theorem 1.3 *A calculus for classical logic is obtained from a calculus for orthologic by adding a pair of structural rules, named separation:*

$$(vii) \qquad \frac{M, O \vdash}{M \vdash O^{\perp}}\ sep1 \qquad\qquad \frac{\vdash N, P}{N^{\perp} \vdash P}\ sep2$$

It is easy to see that, in the framework of ${}^{\perp}\mathbf{O}$, separation rules are equivalent to the following form of cut

$$\frac{M_1 \vdash \mu, N_1 \quad M_2, \mu \vdash N_2}{M_1, M_2 \vdash N_1, N_2}\ cut$$

It is well known (cf. [Dummett 1976], [Cutland e Gibbins 1982]) that adding such cut rule to orthologic yields classical logic. The theorem above gives a more effective content to this fact; for, generally, in any calculus, *cut* is well accepted only if it represents a metarule (that is eliminable).

It is natural to ask what is the meaning of *sep*. In the same way as the *tr* rules are equivalent to *tertium non datur* and *non contradiction*, the *sep* rules turn out to be equivalent to *reductio ad absurdum* [7]

$$\frac{M, \alpha^{\perp} \vdash}{M \vdash \alpha}\ RAA$$

Let us consider again our picture 3, where the question marks have been substituted by *sep*. Given the logic $\mathbf{B}$ as a basic calculus, which contains the fundametal rules for the connectives, several structural rules can be added: each rule permits us to reach a "superior" logic. The strongest element is represented by classical logic, which can be characterized as be $^{\perp}\mathbf{B} + \mathbf{S} + \mathbf{tr} + \mathbf{sep}$. With respect to such formulation for classical logic (denoted by $^{\perp}\mathbf{C}$) all the other logic in the diagram can be described as substructural logics: for, they can be obtained by dropping some structural rules. This situation holds in particular for quantum logics, which turn out to be simpler and more basic than classical logic, from the proof-theoretical point of view.

As we have seen, the examples of quantum logic (we have considered so far) are, at the same time, substructural with respect to classical logic and substructural one with respect to the other. On this basis, on can prove some embedding theorems, by convenient restriction of our structural rules to suitable kinds of formulas, by means of special modalities. In the case of linear logic, *exponentials* have been introduced in order to express weakening and contraction. In the case of quantum logics, instead, we should obtain rules of separation and of transfer in a suitable way. How to express the separation rules in orthologic, in order to obtain an embedding of classical logic into orthologic? Given $^{\perp}\mathbf{O}$, let us first assume in the language, besides the literals p and $p^{\perp}$, two new kinds of literals, $\downarrow p$ and $\downarrow p^{\perp}$. This permits us to obtain a new kind of formulas, that will be named "separable formulas", defined by the following clauses:

$$\downarrow(p) \equiv \downarrow p \qquad \downarrow(p^{\perp}) \equiv \downarrow p^{\perp} \qquad \downarrow(\downarrow p) \equiv \downarrow p \qquad \downarrow(\downarrow p^{\perp}) \equiv \downarrow p^{\perp}$$

[7] In [Gibbins 1985, pag.361], Gibbins shows that dropping the rule RAA has a direct justification in terms of *quantum mechanics*, and this is the only case of direct justification, among all the rules which must be restricted in *quantum logic*.

for basic literals

$$\downarrow(\alpha \circ \beta) \equiv \downarrow\alpha \circ \downarrow\beta$$

for every binary connective $\circ$.
Separable formulas are precisely those formulas that satisfy the separation rules, which are then defined as follows:

$$(vii') \qquad \frac{M, \downarrow O \vdash}{M \vdash \downarrow O^{\perp}} \downarrow sep1 \qquad\qquad \frac{\vdash \downarrow N, P}{\downarrow N^{\perp} \vdash P} \downarrow sep2$$

where formulas in M, N are any kind of formulas, whereas formulas in $\downarrow M$, $\downarrow N$ are separable formulae. We can now introduce the system $\downarrow{}^{\perp}\mathbf{O}$, which is defined by the rules of ${}^{\perp}\mathbf{O}$ and by the rules $\downarrow\mathbf{sep}$. In this system, the sign $\downarrow$ plays the role of a modality, that is of a unary monothonic connective, since, if $M \vdash N$, is a derivable sequent in $\downarrow{}^{\perp}\mathbf{O}$, then $\downarrow M \vdash \downarrow N$ is a derivable sequent in it too.

Let us consider now the system ${}^{\perp}\mathbf{C}$ for classical logic, and let us consider $\downarrow$ as a map from formulas of the language of ${}^{\perp}\mathbf{C}$ into formulas of the language of $\downarrow{}^{\perp}\mathbf{O}$. It is easy to show, by induction on the depth of the derivation, that the following statement holds:

Proposition 1.1 *For every M, N, $M \vdash N$ is derivable in ${}^{\perp}\mathbf{C}$ if and only if $\downarrow M \vdash \downarrow N$ is derivable in $\downarrow{}^{\perp}\mathbf{O}$.*

which proves the embedding of ${}^{\perp}\mathbf{C}$ in $\downarrow{}^{\perp}\mathbf{O}$. Then it is clear that formulas of the kind $\downarrow\alpha$ are interpretable as "the classical part of $\downarrow{}^{\perp}\mathbf{O}$". Similarly to $\perp$, the sign $\downarrow$ does not represent here a connective; therefore, there is no need of introduction rules. As a consequence, sequents like $\downarrow\alpha \vdash \alpha$ or like $\alpha \vdash \downarrow\alpha$ are not provable (differently from the exponentials in linear logic). In this way, the system $\downarrow{}^{\perp}\mathbf{O}$ is simply a way to represent the *coexistence* of classical and quantum logic: it does not assert that "classical" propositions are stronger or weaker than "quantum" propositions. All this can be proved as in [1], where an embedding of classical logic into basic orthologic is treated. All proofs needed can be adapted to the case of orthologic.

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