STRUCTURAL RULES AND IMPLICATION IN A SEQUENT CALCULUS FOR QUANTUM COMPUTATION

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Basic logic analyses how connectives are generated from metalinguistic links.
Basic logic considers the following metalinguistic links between assertions: *and*, *yield*, *forall*.

1. *yield* links two assertions at a different level, in a *sequential* way.
2. *and* links two assertions at the same level, in a *parallel* way.
3. *forall* links assertions with a variable in common. The variable is the reason of the link.

Assertions are represented by *sequents*.

Then logical connectives and their rules in sequent calculus are the result of importing the links into the object level, putting suitable *definitory equations*, and obtaining a sequent calculus.

The equations characterizing usual implication and its symmetric (exclusion) are the following

$$\Gamma \vdash A \rightarrow B\,[,\Delta] \quad \equiv \quad \Gamma, A \vdash B\,[,\Delta]$$

and

$$[\Gamma,]\,A \leftarrow B \vdash \Delta \quad \equiv \quad [\Gamma,]\,A \vdash B, \Delta$$

which aim to characterize the sequential link between A and B.

In order to solve the equations, it is necessary to consider the presence of the *context*, that is something which can be separated from the active formulae.

To what extent is implication related to the presence of the context?

In the definitory equations:

$$\Gamma \vdash A * B \quad \equiv \quad \Gamma \vdash A, B$$

and

$$A \otimes B \vdash \Delta \quad \equiv \quad A, B \vdash \Delta$$

which introduce the multiplicative disjunction and conjunction, the comma is used in order to translate the link *and*.

It is interpreted as something which joins formulae and it does not separate a context.

Sequents for quantum computation

In our model we would like to represent by a sequent

$$\Gamma \vdash A_1, \ldots, A_n$$

the information $A_1, \ldots, A_n$ one can achieve *at the same time* from a preparation of a physical system, in certain hypothesis, all this described in Γ.

If we are considering a quantum system, we need to go beyond the usual elements which can represent classical information.

The physical state inside the quantum computer is in a superposed state.
This implies that:

- Quantum information, achieved by a quantum measurement, is probabilistic.
- The inner quantum state may be an entangled state. This implies that there are items of information, inside the quantum system, that are neither independent nor functionally dependent. They are like "twins".

Then we need to consider probabilities and a new link, describing the entanglement.

The idea is to consider the quantum superposition as a physical link, as well as an information link.

We describe such link by means of the random variable determined by the measurement ("variable-link").

Quantum superposition collapses after measurement: in our representation this becomes a "collapse of the variable-link" when the variable is substituted by a closed term.

In particular, such representation yields a representation of the entanglement link. Entanglement disappears after measurement, since it is due to the superposition.

The simplest case is that in which we measure the state of one fixed particle, say $\mathcal{A}$, w.r.t. a fixed observable.

As is well known, a quantum measurement determines a random variable, given by a set of possible outcomes (the eigenstates of the observable), with associated probabilities.

If Z is a random variable so obtained, we consider the set:

$$D_Z = \{z = (z', p\{Z = z'\}) : z' \text{outcome}\}$$

We term the set D_Z the *random first order domain* associated to the random variable Z.

Superposition by a quantifier

The assertion
"*forall* possible outcomes z in D_Z, in the hypothesis Γ, the possible result of the measurement of $\mathcal{A}$ is z"
is formally written

$$\Gamma, z \in D_Z \vdash A(z)$$

Γ does not depend on z, if the measurement experiment is correct. In such hypothesis, we put the equivalence of the definitory equation of *forall*:

$$\Gamma \vdash (\forall x \in D_Z)A(x) \;\equiv\; \Gamma, z \in D_Z \vdash A(z)$$

Then the proposition $(\forall x \in D_Z)A(x)$ represents quantum superposition.

So the logical glue for quantum superposition is the variable ranging on the random first order domain of the measurement.

Substitution for collapse

The derivable sequent

$$(\forall x \in D_Z)A(x), z \in D_Z \vdash A(z)$$

says that the particle described by the proposition $(\forall x \in D_Z)A(x)$ can be found in a state associated with any of the z's of D_Z.

Substituting z by a *closed* term t, one has that the superposition $(\forall x \in D_Z)A(x)$ is converted into $A(t)$, t denoting a certain fixed outcome. The other possibilities are lost. This describes a collapse. The substitution destroys the superposition.

Then we consider the substitution rule

$$\frac{\Gamma \vdash A(z)}{\Gamma \vdash A(z/t)}$$

where t is a closed term, as a structural rule which causes the collapse of a logical system describing quantum superposition.

Separated particles

Let $\mathcal{C}$ and $\mathcal{D}$ be two particles described by a product state (separated particles). Measurements on both particles give independent outcomes. We describe them by the sequent

$$\Gamma, z \in D_Z, y \in D_Z \vdash C(z), D(y)$$

or also by

$$\Gamma, z \in D_Z, y \in D_Y \vdash C(z), D(y)$$

(possibly different domains), where z and y are different independent variables. Nothing joins $C(y)$ with $D(z)$. The comma describes simply a justaposition of the two items of information.

Representing entanglement

Let $\mathcal{A}$ and $\mathcal{B}$ be two entangled particles, for example two electrons with opposite spin.
A measurement on one of the two particles determines a simultaneous related result on the other one.
So the propositions $A(z)$ and $B(z)$ describing the two states depend on the same variable and are related, not simply justaposed.
We describe this writing

$$\Gamma, z \in D_Z \vdash A(z),_Z B(z)$$

where the indexed comma $,_Z$ says the correlation between the two measurements.

$A(z)$ does not represent a context w.r.t. $B(z)$ or conversely, since they are symultaneous and related information.

In such setting, we make the equivalence of the definitory equation of *forall* parallel:

$$\Gamma \vdash (\forall x \in D_Z)A(x),_Z (\forall x \in D_Z)B(x) \;\equiv\; \Gamma, z \in D \vdash A(z),_Z B(z)$$

Then, we interpret the indexed comma $,_Z$ by a connective $\ltimes_Z$:

$$\Gamma \vdash A \ltimes_Z B \;\equiv\; \Gamma \vdash A,_Z B$$

This creates a paraconsistent setting in which the entanglement is interpreted by a binary (n-ary) quantifier $\bowtie$, summarizing $\forall$ and $\ltimes$:

$$\bowtie_{x \in D_Z} (A(z); B(z)) = (\forall x \in D_Z)A(x) \ltimes_Z (\forall x \in D_Z)B(x) = (\forall x \in D_Z)A(x) \ltimes_Z B(x)$$

The entanglement collapse after measurement. In our terms, a substitution by a closed term t destroys the variable-link leaving a simple comma:

$$\frac{\Gamma, z \in D_Z \vdash A(z),_Z B(z)}{\Gamma, t \in D_Z \vdash A(z/t), B(z/t)}$$

since, after measurement, the information $A(z/t)$ becomes independent from $B(z/t)$.

Substitution has to be considered a *structural rule*, which makes the paraconsistent system with variable links collapse into a propositional system with contexts.

Then we think that implication, in the usual terms in which it is considered, is made possible *after* the collapse of the variable-link.

In the interpretation we have just presented, variable-links are present to the right of the sequents, then implication is possible and exclusion is not.

This corresponds to the "natural" setting of logic. It could explain why exclusion is so difficult to grasp.

One can formally obtain a dual interpretation, not intuitive, with variable links at the left, in which implication is not possible.

Implication is not really a propositional connective, since it hides first-order variables in its semantics.

The algebraic semantics of intuitionistic logic is based on the model of frames of opens in topological spaces. In frames of opens, one has the infinitary join operation and the infinitary distributive property which make possible a sound and complete interpretation of the implication connective. The infinitary properties can be grasped only through variables. In Heyting algebras, which are finitary, the equations characterizing implication have to be forced, they are not naturally true.

The BHK interpretation of implication is through the notion of *method* or *function*. Any interpretation by functions requires first order variables.

It seems that first order variables in a functional/deterministic use are in alternative to the use of variables as random variables.