NEGATION AS A PRIMITIVE DUALITY IN A MODEL FOR QUANTUM COMPUTATION

Giulia Battilotti
Dept. of Pure and Appl. Math.
University of Padova - Italy

As is well known, the measurement process of a quantum state w.r.t. an observable is a random variable, whose outcomes are associated to elements of an orthonormal basis of the Hilbert space associated to the system.

Let Z be the random variable produced by a measurement of a certain particle in a certain state. This defines a set

$$D_Z \equiv \{z = (z', p\{Z = z'\}) : z' \text{ outcome}\}$$

that depends on the state and that we will term *random first order domain*.

We represent by a sequent

$$\Gamma \vdash A_1, \ldots, A_n$$

the information $A_1, \ldots, A_n$ one can achieve *at the same time* from a preparation of a quantum system, in certain hypothesis, all this described in Γ.

This implies that we must refer to the measurement of the state.

We consider a particle $\mathscr{A}$ and an observable producing a random variable Z.

We know that

"In the measurement hypothesis Γ, the outcome is z' with probability $p\{Z = z'\}$ for all pairs $(z', p\{Z = z'\}) \in D_Z$".

More formally, we write this "*forall* $z \in D_Z$, $\Gamma \vdash A(z)$"

and finally we summarize the above assertion in the sequent

$$\Gamma, z \in D_Z \vdash A(z)$$

(Γ does not depend on z)

We put the equivalence of the definitory equation of *forall*:

$$\Gamma \vdash (\forall x \in D_Z)A(x) \;\equiv\; \Gamma, z \in D_Z \vdash A(z)$$

Then the proposition

$$(\forall x \in D_Z)A(x)$$

represents the superposed state of the particle.

If state is representable in the Hilbert space C^2, with orthonormal basis $\{|0\rangle, |1\rangle\}$ (for example: we consider Z given by the measurement of the spin of a particle w.r.t. the z axis):

The outcome is $|0\rangle$ with probability a and $|1\rangle$ with probability b $(a + b = 1)$.
The random first order domain is

$$D_Z = \{(|0\rangle, a), (|1\rangle, b)\}$$

The state is represented by

$$\alpha|0\rangle + \beta|1\rangle$$

$(\alpha, \beta \in C, |\alpha|^2 = a, |\beta|^2 = b)$ as a vector and as

$$(\forall x \in \{(|0\rangle, a), (|1\rangle, b)\})A(x)$$

as a proposition.

When $a = b = 1/2$ (uniform distribution) we have (up to phase factors) the "cat state", written

$$1/\sqrt{2}|0\rangle + 1/\sqrt{2}|1\rangle$$

as a vector of C^2.

Its domain is

$$D_U = \{(|0\rangle, 1/2), (|1\rangle, 1/2)\}$$

The state is represented by the proposition

$$(\forall x \in D_U)A(x)$$

When $a = 0$ we have the sharp state $|1\rangle$. Its domain is a singleton

$$D_1 = \{(|1\rangle, 1)\}$$

It is represented by the proposition

$$(\forall x \in D_1)A(x)$$

The case of a compound system, for example a couple of particles, $\mathcal{A}$ and $\mathcal{A}'$.

If the two particles are separated, that is, if the measurement result on the first is independent from the measurement on the second, we obtain two different independent random variables, Z and Z'.

So we define two distinct domains D_Z and D'_Z and describe the measurement of the compound system by the sequent:

$$\Gamma, z \in D_Z, z' \in D_{Z'} \vdash A(z), A'(z')$$

that is converted into $\Gamma \vdash (\forall x \in D_Z)A(x) * (\forall x \in D_{Z'})A'(x)$.

Example: the separated state
$(1/\sqrt{2}|0\rangle + 1/\sqrt{2}|1\rangle) \otimes (1/\sqrt{2}|0\rangle + 1/\sqrt{2}|1\rangle)$. The state of the sistem is represented by the *compound* proposition

$$(\forall x \in D_U)A(x) * (\forall x \in D_U)A'(x)$$

(two different occurrences of the same first order domain, not under the scope of the same quantifier).

The case of entangled particles is different. In such case one does not have independent measurements and variables.

We adopt a generalized n-ary quantifier, denoted $\bowtie^n$.

It is definable in a paraconsistent setting, in order to represent entangled states. The proposition

$$\bowtie^n_{x \in D_Z} (A_1; \ldots A_n)$$

represents the entangled state of n particles “sharing” the same random variable Z, and hence the same r.f.o.d. D_Z (in particular, $\bowtie^1$ is $\forall$).

Example: the Bell's states. They are couple of particles for which a measurement of one of the two determines the symultaeous identical (or opposite) result on the other. Then they share the same random variable.

In $C^2 \otimes C^2$ (whose orthonormal basis is $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$) their representation as vectors is the following:

$$1/\sqrt{2}|00\rangle \pm 1/\sqrt{2}|11\rangle \qquad 1/\sqrt{2}|01\rangle \pm 1/\sqrt{2}|10\rangle$$

Their representation as proposition has the form

$$\bowtie^2_{x \in D_U} (A_1(x); A_2(x))$$

where $D_U = \{(|0\rangle, 1/2), (|1\rangle, 1/2)\}$.

Note that the domain D_U is "simpler" than the state, since it is the same domain of a particle of C^2. Two particles share the same domain.

Let $D_Z = \{(z, p\{Z = z\})\}$ a domain where $z \in \{|0\rangle, |1\rangle\}$.

We put

$$D_Z^{\perp} \equiv \{(z^{\perp}, p\{Z = z\})\}$$

where the state $z^{\perp}$ is the *NOT* of z.

$D_Z^{\perp}$ is the *dual domain* of D_Z.

The proposition with the dual domain

$$(\forall x \in D_Z^{\perp})A(x)$$

denotes the *NOT* of the state denoted by $(\forall x \in D_Z)A(x)$.

1. In which terms can the definition of dual domain extend a usual duality?
2. In which terms is the proposition $(\forall x \in D_Z^{\perp})A(x)$ to be considered a logical negation?

Basic logic puts its definitory equations in *symmetric* pairs, e.g.:

$$\Gamma \vdash A\&B \qquad \equiv \qquad \Gamma \vdash A \quad \Gamma \vdash B$$

and

$$A \vee B \vdash \Delta \qquad \equiv \qquad A \vdash \Delta \quad B \vdash \Delta$$

that are solved in a symmetric way finding couples of rules "mirroring each other".

Then, one finds symmetric sequent calculi (or couples of symmetric sequent calculi) and a *symmetry theorem*:

$$\Pi \text{ proves } \ \Gamma \vdash \Delta \qquad \text{iff} \qquad \Pi^s \text{ proves } \ \Delta^s \vdash \Gamma^s$$

where $p = p^s$ on literals and Π^s has the right/left rule for $\circ^s$ where Π has the left/right rule for $\circ$.

$\circ$ and $\circ^s$ is the couple of logical constants corresponding to the same metalinguistic link: $(\&, \vee)$, $(*, \otimes)$, $(\rightarrow, \leftarrow)$.... $(\forall . \exists)$.

We see the case of $\forall$. The assertion

"*forall* $z \in D$, $\Gamma \vdash A(z)$"

considered in a symmetric way gives the assertion

"*forall* $z \in D$, $A(z) \vdash \Gamma$"

that, semantically, corresponds to the sequent $A(z) \vdash \Gamma, \neg(z \in D)$ (a negation is required), that normally is converted into $z \in D, A(z) \vdash \Gamma$, namely $(\exists z \in D)A(z) \vdash \Gamma$.

Then, formally we have a symmetric representation of the state, through the existential quantifier: $(\exists x \in D_Z)A(x)$.

In logic, the symmetry theorem becomes real when it is applied considering a duality $(-)^\perp$:

$$\Gamma \vdash \Delta \qquad \text{iff} \qquad \Delta^\perp \vdash \Gamma^\perp$$

where $p^\perp$ is the negation of p and everything else is as for symmetry. Symmetry acts as a real duality on connectives!

We can see that the duality $(-)^\perp$ extends to our case.
We put:

- $A(z)^\perp \equiv A(z)$ (where z has its values in $D^\perp$ in $A(z)^\perp$!)
- $A(z/t)^\perp \equiv A(z/t^\perp)$ (where $t^\perp$ denotes the element obtained as the *NOT* of the element denoted by t)
- $(z \in D)^\perp \equiv z \in D^\perp$

Then the dual representation of $(\forall x \in D_Z)A(x)$ is $(\exists x \in D_Z^\perp)A(x)$.

We shall see that it extends the usual propositional duality.

We need to see that $(\forall x \in D_Z^{\perp})A(x)$ is a negation for $(\forall x \in D_Z)A(x)$ (it is consistent with the usual negation).

The quantum gate *NOT*, applied to sharp states, behaves as the gate *NOT* of a classical computer. In our terms:

The proposition $A(z)$, z a variable, says that the particle is found in a generic state z' with probability $p\{Z = z'\}$ after measurement.

If we substitute the variable z by a closed term denoting a certain fixed element of D_Z, the other possibilities are lost.

Then substitution is the logical way to describe the collapse of the superposed state due to the quantum measurement.

The collapse is described by the sequent

$$(\forall x \in D_Z)A(x) \vdash A(t)$$

that is provable by a substitution rule.

In quantum mechanics there are two kinds of meaurements: non-selective and selective. Non selective measurements yield the mixed state given by all possible kinds of outcome with their probabilities. Selective measurements select one kind, one state. Then the result is a pure state with probability 1 (a sharp state).

In our setting, we represent the result of a non-selective measurement on the state $(\forall x \in D_Z)A(x)$ by the conjunction $A(t_1)\&A(t_2)$, where t_1 and t_2 are the terms denoting the elements of D_Z.

The result of the measurement is represented by a propositional and compound formula.

In order to represent a selective measurement, we consider a "forgetful substitution" which drops the real probability attributing probability 1 to the outcome $|b\rangle$.
Then we obtain the sequent $(\forall x \in D_Z)A(x) \vdash A(s)$, where s is a term denoting $(|b\rangle, 1)$.
$A(s)$ is the proposition denoting the state of the particle after measurement. $A(s)$ must be identified with a sharp quantum state. Then, only in this case, we require that the probability distribution after measurement determines the state and put the axiom $A(s) \vdash (\forall x \in \{(|b\rangle, 1)\})A(x)$.

Then it is $A(s) = (\forall x \in \{(|b\rangle, 1)\})A(x)$ (the sharp state is a propositional state).

For the same reasons, we identify $A(s)$ with $(\exists x \in \{(|b\rangle, 1)\})A(x)$ in the symmetric representation.

The propositions $A(s)$ are like propositional literals p.
Since $(\forall x \in \{(|b\rangle, 1)\})A(x)$ coincides with $A(s)$ that coincides with $(\exists x \in \{(|b\rangle, 1)\})A(x)$, it is $p^s = p$.
We obtain a consistent extension of the symmetry theorem.

The dual domain of the singleton $\{(|b\rangle, 1)\}$ is the singleton $\{(NOT|b\rangle, 1)\}$. If $s^\perp$ denotes its element, the dual of the state $A(s)$ is $A(s^\perp)$

The propositions $A(s)$ and $A(s^\perp)$ are like a couple of propositional literals: p_y and p_n, that can be interpreted as a couple of opposites.

We obtain a primitive negation.

If we put $p_y^\perp = p_n$ and conversely, we obtain a consistent extension of the duality theorem.

On the other side, a random first order domain coincides with its opposite when the domain corresponds to an eigenstate of the *NOT* gate.

In C^2, the domain $D_U = \{(|0\rangle, 1/2), (|1\rangle, 1/2)\}$ is equal to its dual.

In $C^2 \otimes C^2$ the four maximally entangled states (Bell's states) are representable by means of an entanglement quantifier which has the same domain D_U.

So, in our setting, we have formulae which coincide with their negation. We can consider them another kind of primitive literals and label them by capital letters U. We term them "uniform literals". It is $U_y \equiv U_n$.

Uniform literals aren't propositional formulae and do not coincide with their symmetric, with the existential quantifier. We must distinguish universal literals $U_\forall$ and existential literals $U_\exists$, where the symmetric of $U_\forall$ is $U_\exists$ and conversely.

On uniform literals duality coincides with symmetry: the dual of $U_\forall$ is $U_\exists$.

The role of symmetry and duality is exchanged for uniform literals!

Since $U_y \equiv U_n$, in $U \vdash U'$, U and U' can be considered as asserted and as rejected at the same time. This does not mean that the sequent $U \vdash U'$ is like the sequent $U' \vdash U$, since from $U \vdash U'$ one has $U'^s \vdash U^s$. They can be distinguished w.r.t. the turnstyle $\vdash$ by symmetry.

In any proof Π containing U, U is considered asserted as well as rejected. Only when we can substitute the variable, that is bounded in U, by a term, we fix something.

Literals U are maximal with respect to this, since any other component of a proof is obtained as a composition of elements which admit a dual different from themselves.

It is important to gather as much information as possible in literals U. This is what we represent.

The so called "massive quantum parallelism" exploits quantum superposition and Bell's states. Here we represent this when the computation is a computation of assertions, namely a logical proof.

Matte Blanco (The Unconscious as infinite sets):

there is the “bivalent mode” for the conscious thinking

there is the “indivisible/symmetric mode” for the unconscious thinking, where “...the opposites merge to sameness”.