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A predicative characterization of quantum states and Matte Blanco's bi-logic

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Abstract

We show a correspondence between a predicative characterization of quantum states, we have recently introduced, and bi-logic, a logical setting proposed by the Chilean psychoanalyst Ignacio Matte Blanco in order to describe the logic of the unconscious.

Bi-logic

Bi-logic was described in the 70's by the Chilean psychoanalyst Ignacio Matte Blanco.

Bi-logic has two modes:

- *Asymmetric mode*: proper only of the conscious reasoning.
  It can deal with non-symmetric relations
  It can separate objects
  It permits "normal" - sound - logic: two distinct truth values
- *Symmetric mode*: it is the mode of the unconscious.
  It has symmetric relations only
  It gathers, identifies objects
  It creates different links between judgements and has an unsound logical behaviour.

By symmetry: *any part is treated as the whole thing*
So: any subset and the whole set are idempotent.

So, following Matte Blanco: the "objects" of the unconscious are *infinite sets*.

Other principle linked to symmetry: *generalization*: the unconscious can make a set larger and larger.

Consequences of symmetry in logic:

- *condensation* $\longrightarrow$ in logic:
  the opposites coexist
  no mutual contradiction - no negation
- *no temporal processes*[a] $\longrightarrow$ in logic:
  no algorithmic / step-by-step processes
  no logical consequence
- *displacement* $\longrightarrow$ in logic:
  different hidden symmetric links between judgements
  correlations???

[a] MB suggests to speak about "manifestations" rather than "processes" of the unconscious.

Finite and infinite sets: Meta-level and object-level

Assume that D is any set. By the logical rules on $\exists$ and $=$, it is provable that

$$z \in D \equiv (\exists x \in D) z = x$$

However, even if we recognize that $D = \{t_1, \ldots, t_n\}$ is finite at the metalevel,

$$z \in D \equiv z = t_1 \vee \cdots \vee z = t_n$$

is not provable by the rules on the finite disjunction $\vee$[a].

If the equivalence holds in a logic, it is possible to count the elements of D in that logic: D finite in that logic.
If the equivalence does not hold, it is not possible to count the elements of D: D is infinite in that logic.

Then characterizing the set D as finite coincides with characterizing the membership predicate by means of a propositional, rather than predicative, connective.

Moreover, the equivalence

$$z \in D \equiv z = t_1 \vee \cdots \vee z = t_n$$

is provable if and only if

$$(\forall x \in D) A(x) \equiv A(t_1) \& \ldots \& A(t_n)$$

is provable for every A.

Then characterizing something by a propositional rather than predicative formula can be told in terms of finite rather than infinite sets.

[a] We consider the intuitionistic disjunction.

Virtual singletons

Following Matte Blanco, two facts:

F1 *Sets are all infinite - since the part is the whole thing*

F2 *Relations are all symmetric*

Then two questions:

Q1 *For which sets is every part equal to the whole thing?*

Q2 *For which sets is every relation symmetric?*

Only one answer: FOR SINGLETONS!

Then, to follow Matte Blanco, we need
infinite sets acting as singletons!

By extensionality: Singletons are sets V for which there is an element u of V such that, if z is any element of V, then z coincides with u. Then we write $V = \{u\}$.

Inside a logic, extensionality is translated into the following natural assumption:

$$z \in V \vdash z = u$$

(where u is a closed term of the logical language denoting the same element)

This renders singletons finite in that logic.

Finiteness of singletons seems quite unavoidable for a logician. However, singletons are not splitted by a disjunction: they are similar to infinite sets in this. They have a borderline behaviour in logic. So, is our battle hopeless?

One can characterize singletons inside a logic, even without assuming extensionality, as domains of quantifiers:

$$(\forall x \in V) A(x) \equiv (\exists x \in V) A(x)$$

for all formulae A.
Equivalently[a], one takes a duality d on propositions and puts:

$$z \in V, A(y) \vdash A(z), (y \in V)^d$$

This characterizes *virtual singletons*, possibly infinite.
Importing extensionality into our logic, $V = \{u\}$ and this becomes: $z = u, A(y) \vdash A(z), y \neq u$ (that is provable). *So, extensional singletons are the finite shadow of virtual singletons.*

Why are virtual singletons so unusual? Because any logic closed by substitution can prove that any two elements of a virtual singleton are equal.

[a] We apply the definitory equations of basic logic.

Quantum correlations and displacement

We can suitably extend the equality

$$(\forall x \in V) A(x) \vee B(x) = (\forall x \in V) A(x) \vee (\forall x \in V) B(x)$$

that is trivially sound only if V is a singleton, to virtual singletons, and obtain a generalized symmetric quantifier. It permits to represent the quantum correlations of the Bell's states.

This is obtained widening the action of virtual sigletons to the second order, considering "virtual singletons of indexes of formulae". The correlation takes place since the same variable is displace elsewhere, considering another index.

One can see that this is not compatible with the usual definition of logical implication since it implies a context-sensitive treatment of information.

Displacement is a way to link judgements that is not present in the conscious logical reasoning, however it is widely exploited by the mind. Following Matte Blanco, it takes place by symmetry, since two subclasses are identified with a larger class and then treated as identical. This is also a second-order application of symmetry.

References

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Infinite sets in a model of quantum states

Let $\mathcal{A}$ be any particle. Let us fix an observable and measure it. We have a random variable Z as an outcome:

$$D_Z = \{(s(z), p\{Z = s(z)\})\}$$

is the set of the eventual outcomes $s(z)$ with their probabilities.

We consider the proposition $A(z)$ "particle $\mathcal{A}$ is found in state $s(z)$ with probability $p\{Z = s(z)\}$" - in the variable z.
Let Γ be the hypothesis on the preparation of $\mathcal{A}$.
We write $\Gamma, z \in D_Z \vdash A(z)$ to say that $A(z)$ is a generic consequence of the preparation.
Then, by definition of $\forall$, we derive[a]

$$\Gamma \vdash (\forall x \in D_Z) A(x)$$

The proposition $(\forall x \in D_Z) A(x)$ describes the state of the particle $\mathcal{A}$. The variable acts as a glue!

Assume that the observable is discrete and that we recognize the outcomes $t_1, \ldots, t_n$ after measurement. Then $D_Z = \{t_1, \ldots, t_n\}$. The measurement says that $\Gamma \vdash A(t_i)$ for $i = 1, \ldots, n$. By definition of &, it holds

$$\Gamma \vdash A(t_1) \& \ldots \& A(t_n)$$

The proposition $A(t_1) \& \ldots \& A(t_n)$ describes the mixed state, obtained after measurement.

The consequence

$$(\forall x \in D_Z) A(x) \vdash A(t_1) \& \ldots \& A(t_n)$$

describes the collapse from the pure state to the mixed state after measurement. This is proved by substituting the free variable z by the closed terms t_i. Substitution is like "the collapse of the variable": *Substitution represents measurement.*

For any $\mathcal{A}$, and hence any A, the equivalence

$$(\forall x \in D_Z) A(x) \equiv A(t_1) \& \ldots \& A(t_n)$$

is true after measurement: by our characterization, D_Z is finite.
The equivalence is not true prior to measurement: D_Z is infinite.

Quantum superposition corresponds to the infinite character of the set D_Z.

Notice: In our model, we say that D_Z is focused/unfocused rather than finite/infinite; one can see that defining an equality predicate which renders D_Z focused, namely finite, is equivalent to dropping the phase factors in the representation of the state as a vector in the Hilbert space.

[a] We apply the definitory equations of basic logic.

Symmetry in the spin model

We measure the spin of a particle w.r.t. the z axis. The sets associated to the sharp states $\uparrow$ and $\downarrow$ are two singletons. The formulae quantified on them are then equivalent to propositional formulae, say $A_\uparrow$ and $A_\downarrow$. We put a duality $\perp$ switching $\uparrow$ and $\downarrow$. It translates the Pauli matrix σ_X (namely the *NOT* gate) into logic. We extend it to all formulae and obtain a negation (Girard's negation), such that $A \vdash B$ if and only if $B^\perp \vdash A^\perp$.

The dual states $+$ and $-$ are switched by the Pauli matrix σ_Z and are eigenvectors for σ_X. Translating all this into logic, we have a new duality $\top$ from σ_Z and extend $\perp$.
The sets associated to $+$ and $-$ contain the *two opposite pieces of information* $\uparrow$ and $\downarrow$. The corresponding predicative formulae *are fixed point for negation* $\perp$.
Such formulae satisfy $A \vdash B$ if and only if $B \vdash A$:
no direction for logical consequence.

Changing the measurement context and measuring the spin w.r.t. x would produce an objective property for $+$ and $-$, that would be represented by singletons. *However, different spin observables and so the sets associated to $+$ and $-$ are virtual singletons*, due to the duality $\top$ from σ_Z.

This is the logical framework expected by bi-logic: an asymmetric mode where negation is meaningful, a symmetric mode where negation is meaningless, due to virtual singletons.