# Embedding classical logic into basic orthologic with a primitive modality.

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**Abstract**

In the present paper we give the first proof-theoretical example of an embedding of classical logic into a quantum-like logic. This is performed in the framework of basic logic, where a proof-theoretical approach to quantum logic is convenient. We consider basic orthologic, that corresponds to a sequential formulation of paraconsistent quantum logic, and which is given by basic orthologic added with weakening and contraction, in a language with Girard's negation. In the paper we first consider a convenient cut-free calculus for classical logic, in the same language; then, in a language enriched with a new kind of literals, we introduce basic orthologic with a primitive modality, where classical logic is embeddable. Similarly to Girard's negation, our modality is not a connective but conceivable as an operator defined inductively on the set of formulae. This allows us to obtain a calculus which enjoys cut-elimination.

## Contents



## 1 Introduction

Basic logic is a new logic, introduced first in [4] and then in [19], which is weaker than linear, intuitionistic and quantum logics. It is then natural to try to compare its strenghth with that of usual logics. Since the first formulation of basic logic, we figured a possible solution. Taking example from Girard's exponentials, which allow to obtain weakening and contraction inside linear logic, we expected to solve the problem of embedding a stronger logic into a weaker one expressing what is inhibited in the weaker logic by means of modalities. To carry out this plan, anyhow, it is necessary to realize how it is possible to embed a non-quantum logic into a quantum one, by means of a modality. The only literature in quantum logic containing embedding results concerns orthomodular logic and it is developed in [8] and [9], in an algebraic setting. The present one is the first proof-theoretical solution. A preliminary exposition can be found in the author's Ph. D. Thesis, see [1].

In this paper, we consider basic orthologic, which, as seen in [12], is a sequential formulation of paraconsistent quantum logic, introduced in [11], that is a weak form of quantum logic. The calculus for basic orthologic, indicated by ${}^{\perp}\mathbf{BS}$, is obtained by the addition of weakening and contraction to basic logic, in a language without implication and with Girard's negation, indicated as usual by $\perp$. It is close to basic logic, but at the same non-linear level of classical logic, so that it allows us to concentrate on the opposition between the non-quantum features (of classical logic) and the quantum features (of basic orthologic).

The calculus for basic logic, as well as every quantum calculus (cf. [18], [10], [4], [12]) has limitations of contexts, on the right and also on the left. In particular, in basic logic and basic orthologic contexts are limited in all rules for connectives: this is called visibility, since then the principal and the secondary formulae are "visible", and it is the essential hypothesis in their cut-elimination (cf. [19]). In general, in quantum logics, contexts are necessarily limited in the introductions of disjunction on the left and conjunction on the right, to avoid the derivation of the distributive laws. Moreover, they are limited in the rules for negation and for implication (if present), in order to obtain the failure of the deduction theorem, which in turn would allow the derivation of the distributive laws. The limitation of contexts in the rules for implication and negation can be interpreted as follows: a formula cannot be separated from its contexts and moved to the other side of the sequent. So, in case of a comparison between a logic satisfying the distributive laws and the deduction theorem with a quantum-like logic, one only needs a modality by which one can express, on one side, the movement of a formula from left to right of the sequent and conversely, and, on the other side, one can obtain also the missing contexts in the rules. Solving the two problems by means of the same modality is possible, considering basic orthologic. The solution is mainly due to the choice of the language. In fact, as we realized in our first attempts, the difficulty in solving the two problems together relies in the connectives of implication and negation, which are missing in the language of basic orthologic.

Moreover, we take another important advantage from such language. In fact, as first noticed and developed in [14], [13], [12], suitable rules on $\perp$, which are to be considered structural, since $\perp$ is not a connective, can be added to basic orthologic, obtaining a calculus for orthologic and, from this last, a calculus for classical logic to which basic orthologic is substructural. Consequently, such calculi have a cut-elimination procedure, allowed by the features of visibility and symmetry, inherited from basic logic and basic orthologic. We also exploit these facts here, obtaining a convenient formulation of classical logic, denoted by ${}^{\perp}C$, which has cut-elimination, as we prove. Such formulation of classical logic consists of basic orthologic added with a pair of structural rules on $\perp$, which state the possibility to separate formulas from their context and transfer them, negated, to the other side of the sequent. They correspond to the usual introduction of negation to the left and to the right in classical logic.

Then, restricting the structural rules on $\perp$ to the case of modalized formulae, we obtain a first calculus for basic orthologic with a primitive modality, labelled $\downharpoonleft{}^{\perp}\mathbf{BS}_0$, where the sign $\downharpoonleft$ denotes the modality. It is easy to show that every rule which does not admit a context in ${}^{\perp}\mathbf{BS}$ does admit however a context of modalized formulae in $\downharpoonleft{}^{\perp}\mathbf{BS}$. So, by means of the same modality, we capture, in the case of basic orthologic, the two main characteristics of calculi for quantum logics, that are, as we said, limitations of contexts and limitation of movement from left to right and conversely.

The modality $\downharpoonleft$ we introduce is given, like $\perp$, by a unary operator, obtained by adding a new kind of primitive literals (the "classical" literals!) to the language and then defined by induction on formulae. In this sense our modality is primitive and it does not require any introduction rule, so that the only rules we need for it are the structural ones, which characterize its behaviour. A slight modification of $\downharpoonleft{}^{\perp}\mathbf{BS}_0$, labelled $\downharpoonleft{}^{\perp}\mathbf{BS}$ inherits cut-elimination from ${}^{\perp}C$, which in turn derives its cut-elimination that of basic logic. We have that basic logic is substructural to ${}^{\perp}C$, since $\perp$ is an operator and not a connective, as well as it is substructural to $\downharpoonleft{}^{\perp}\mathbf{BS}$, since $\perp$ and $\downharpoonleft$ are operators and they are not connectives. Moreover, that was the first result in this direction (cf. [12]), basic logic is substructural to orthologic too. All such extensions of basic logic have cut-elimination, due to the fact they are obtained as structural extensions. We conclude in particular that structural extensions of basic logic are precious when dealing with the proof-theoretical aspects of quantum logic (cf. [3]).

Ultimately, the logic here introduced with $\downharpoonleft{}^{\perp}\mathbf{BS}$ can be interpreted as a system which allows the coexistence of two ways of reasoning: the quantum-like one, represented by basic orthologic ${}^{\perp}\mathbf{BS}$, and the classical one, that is ${}^{\perp}C$. The first deals with any formula, the second can deal only with modalized formulas. Now, we hope that the method described above can be both exploited for other quantum calculi, investigating furtherly on their peculiarities, both applied to basic logic and its extensions, including linear extensions and obtaining a global and uniform treatment of translations into basic logic.

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## 2 A calculus for classical logic based on basic orthologic

In this section we develop some ideas which have been introduced in [12], [14], and analysed in more details in [13], [15]. For a survey on them, see also [3]. Here we adapt such ideas to the most convenient form for us.

Let us consider a language equipped with two kinds of primitive literals:

$$p_1, p_2, p_3, \ldots$$

and

$$p_1^d, p_2^d, p_3^d, \ldots$$

and with the binary connectives & and $\vee$. So, our formulae are the following: For any $i \in \omega$, $p_i$ and $p_i^d$ are formulae and, for any two formulae $A$, $B$, $A \& B$ and $A \vee B$ are formulae. Moreover, we put in the language the sign $\perp$, which denotes an operator defined on the set of formulae, by the following clauses:

$$p_i^\perp \equiv p_i^d, \qquad {p_i^d}^\perp \equiv p_i$$

on literals, and

$$(A \circ B)^\perp \equiv B^\perp \circ^\perp A^\perp$$

on formulae, where $\&^\perp \equiv \vee$, $\vee^\perp \equiv \&$.
As one can easily verify, by induction on the complexity of the formulae, any formula $A$ coincides with $A^{\perp\perp}$; so we will say that the formula $A^\perp$ is the formula "symmetric" or "dual" to the formula $A$. The degree of $A^\perp$ is equal to the degree of $A$, for any $A$, as one can see by induction, since $\perp$ is not a connective. Anyway, $\perp$ plays the role of negation, as we shall justify in more details in this section. Actually, it is Girard's negation. Such negation has been introduced in the framework of basic logic in [12] and [14] for the first time. In such papers it is also specified a suitable pair of structural rules on $\perp$, called rules of *transfer*, which allows to obtain orthologic from basic orthologic, and a second pair of structural rules, called *separation*, which allow to obtain classical logic from orthologic. Such rules, like any rule involving only $\perp$, are to be considered structural, since they modify the structure of the sequents, without introducing any connective (we remind that a structural operation for negation has been first introduced in display logic, cf. [5], [6]). In order to obtain classical logic as an extension of basic orthologic, we introduce in this paper a convenient variation of the rules for separation and transfer, which is given by the following pair of rules on $\perp$, named *st*, from *separation* and *transfer* together.

$$\frac{\Gamma \vdash \Delta, \Sigma}{\Gamma, \Sigma^\perp \vdash \Delta}\ stL \qquad\qquad \frac{\Gamma, \Sigma \vdash \Delta}{\Gamma \vdash \Delta, \Sigma^\perp}\ stR$$

Here and below, in any sequent $\Gamma \vdash \Delta$, $\Gamma$ and $\Delta$ are finite sets, hence in $\Gamma, \Gamma' \vdash \Delta$ and in $\Gamma \vdash \Delta, \Delta'$ the comma is intended as set-theoretic union, so that, in particular, the rules of contraction on the left and on the right

$$\frac{\Gamma, \Sigma, \Sigma \vdash \Delta}{\Gamma, \Sigma \vdash \Delta} \qquad \frac{\Gamma \vdash \Sigma, \Sigma, \Delta}{\Gamma \vdash \Sigma, \Delta}$$

hold by definition. As it can be seen in [19], the structural rules of weakening and contraction can be added to basic logic, obtaining a calculus where the additive and multiplicative connectives are identified, as in the passage from linear to classical logic. Basic orthologic (cf. [12] and [14]) is nothing else than such calculus in the language given above. Here, we consider a formulation of basic orthologic in which sequents contain sets of formulae, so that basic orthologic is characterized by axioms, rules of cut, rules on the connectives & and $\vee$, and the structural rules of weakening, as it can be desumed from the table below, from which basic orthologic can be obtained dropping the last line. We now introduce system ${}^{\perp}C$, characterized by the following table of rules, which is obtained by the addition of the structural rules $stL$ and $stR$ to basic orthologic.

Axioms

$$A \vdash A$$

Cuts

$$\frac{\Gamma \vdash A \quad A, \Gamma' \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta}\, cutL \qquad \frac{\Gamma \vdash \Delta, A \quad A \vdash \Delta'}{\Gamma \vdash \Delta, \Delta'}\, cutR$$

Rules on connectives

$$\frac{A \vdash \Delta}{A\&B \vdash \Delta}\, \&L \quad \frac{B \vdash \Delta}{A\&B \vdash \Delta}\, \&L \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\, \vee R \quad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\, \vee R$$

$$\frac{A \vdash \Delta \quad B \vdash \Delta}{A \vee B \vdash \Delta}\, \vee L \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A\&B}\, \&R$$

Structural rules

$$\frac{\Gamma \vdash \Delta}{\Gamma, \Sigma, \vdash \Delta}\, wL \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \Sigma}\, wR$$

$$\frac{\Gamma \vdash \Delta, \Sigma}{\Gamma, \Sigma^{\perp} \vdash \Delta}\, stL \qquad \frac{\Gamma, \Sigma \vdash \Delta}{\Gamma \vdash \Delta, \Sigma^{\perp}}\, stR$$

Intuitively, the above table of rules is symmetric, in the sense that the schemata of the rules appearing on the left column are mirrored in those appearing on the right one. This fact can be put in more formal terms. As explained in details in [19] for basic logic, any symmetry operator defined on formulae can be extended, in a natural way, to sequents and then to inference rules and to derivations themselves. In particular, considering the operator $\perp$, one can define the symmetric $(\Gamma \vdash \Delta)^\perp$ of the sequent $\Gamma \vdash \Delta$ putting

$$(\Gamma \vdash \Delta)^\perp \equiv \Delta^\perp \vdash \Gamma^\perp$$

(where, if $\Gamma = C_1, \ldots, C_n$, $\Delta = D_1, \ldots, D_m$, then $\Gamma^\perp = C_n^\perp, \ldots, C_1^\perp$ and $\Delta^\perp = D_m^\perp, \ldots, D_1^\perp$). In fact, it is $(\Gamma \vdash \Delta)^{\perp\perp} \equiv \Gamma \vdash \Delta$. Moreover, one can define the symmetric $J^\perp$ of a given rule $J$ as the rule containing as premisses the symmetric of the premisses of $J$, and as conclusion the symmetric of its conclusion. Finally, one has the notion of "symmetric proof", that is the proof obtained considering the symmetric of every axiom and rule. Note that it is $J^{\perp\perp} \equiv J$ for every rule $J$ and $\Pi^{\perp\perp} \equiv \Pi$ for every rule $\Pi$. Since the table of rules of ${}^\perp C$ is symmetric, if $\Pi$ is any proof in ${}^\perp C$, also its symmetric $\Pi^\perp$ is a proof in ${}^\perp C$. This allows to prove the following equivalence. Here and in the following, we adopt the notation $\Gamma \vdash_X \Delta$ to say that the sequent $\Gamma \vdash \Delta$ is derivable in the sequent calculus $X$.

**Lemma 2.1** *In ${}^\perp C$, for every $\Gamma$, $\Delta$,*

$$\Gamma \vdash_{{}^\perp C} \Delta \qquad \textit{if and only if} \qquad \Delta^\perp \vdash_{{}^\perp C} \Gamma^\perp$$

*by a derivation of the same length.*

**Proof.** The claim is proved by induction on the depth of the derivation. In case of the axioms it is trivially true. In the inductive case, if $\Gamma \vdash \Delta$ has been derived by means of a derivation in which $J$ is is the last rule, then $\Delta^\perp \vdash \Gamma^\perp$ is derivable by means of a derivation in which the symmetric rule $J^\perp$ is the last rule. So, $\Gamma \vdash \Delta$ is derivable from the axioms $A_1 \vdash A_1, \ldots, A_n \vdash A_n$ by means of a derivation $\Pi$ if and only if its symmetric sequent $\Delta^\perp \vdash \Gamma^\perp$ is derivable from the dual axioms $A_1^\perp \vdash A_1^\perp, \ldots, A_n^\perp \vdash A_n^\perp$, by means of the symmetric derivation $\Pi^\perp$, and conversely. □

The above theorem can be read as a metarule of the calculus which guarantees antimonotonicity of the operator $\perp$ with respect to the order given by $\vdash$. Hence, any symmetric calculus, in a language with two kinds of literals, can be equipped with an antimonotonic operator, obtaining a primitive negation in it. Moreover, the theorem is used in an essential way in proofs of cut-elimination for such calculi (see [12], [15] and [13]), in the form of the technique of "swapping", which consists of considering, instead of the derivation

$$\begin{array}{c} \vdots\, \Pi \\ \Gamma \vdash \Delta \end{array}$$

its symmetric

$$\begin{array}{c} \vdots\, \Pi^\perp \\ \Delta^\perp \vdash \Gamma^\perp \end{array}$$

Swapping does not modify the degree of the formulae, as well as their rank. A convenient definition of rank in our framework (cf. [12]) must take into account the fact that any formula $A$ appearing on the left (right) side of a sequent is the same than the formula $A^\perp$ appearing on the other side, after an *st* rule has occurred: it is the structure of the sequent which has been modified, not the formulas in it. So, to define the rank of a certain formula $A$ which appears in the sequent $\Gamma \vdash \Delta$, which has been derived by the derivation $\Pi$, consider, in $\Pi$, all the paths, i.e. the i.e. the sequences of consecutive sequents, which contain either $A$ on the same side or its dual $A^\perp$ on the other side, when an *st* rule has occurred, up to where $A$ (or $A^\perp$) is introduced. The rank is then the maximum among the leghths of all the paths.

Moreover, we have to observe that ${}^\perp C$, like other symmetric extensions of basic logic, inherits from basic logic the feature of visibility (cf. [19]). "Visibility" means that, in every rule introducing a connective, the principal and the secondary formulae are without a context, i.e., they are "visible". This is an essential hypothesis in the cut-elimination proofs given in the framework of basic logic.

In the following cut-elimination proof, the swap from $\Gamma \vdash \Delta$ to $\Delta^\perp \vdash \Gamma^\perp$ inside a derivation will be indicated by the following notation:

$$\begin{array}{c} \Gamma \vdash \Delta \\ \vdots\, swap \\ \Delta^\perp \vdash \Gamma^\perp \end{array}$$

Moreover, the notation $wL(\Sigma)$ is adopted to specify that $\Sigma$ is the set of rules introduced by weakening.

**Theorem 2.2** *Any derivation in ${}^\perp C$ which ends with an application of cutL or cutR and where no other application of cutL or cutR occurs, can be transformed into a derivation in which cutL and cutR do not appear. Hence, any derivation in ${}^\perp C$ can be transformed into a derivation in which cutL and cutR do not appear.*

**Proof.** The proof is by induction on the degree and on the right and left rank of the cut. (The degree of a cut is the degree of its cut formula, its rank is the sum of the left and right rank, that is the rank of the cut formula in the left and right premise of the cut, respectively). Let us suppose the rank is 2. If one of the premises of the cut is an axiom, it is trivially eliminable, otherwise we have the following reduction to a cut of a lower degree:

$$\dfrac{\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A\&B} \quad \dfrac{A \vdash \Delta}{A\&B \vdash \Delta}}{\Gamma \vdash \Delta} \qquad \longrightarrow \qquad \dfrac{\Gamma \vdash A \quad A \vdash \Delta}{\Gamma \vdash \Delta}$$

The case of $\vee$ is symmetric; the cases in which the cut formula is introduced by weakening will be treated among the rank reductions.

Let us assume the rank is greater than 2. We shall consider only the case of $cutL$, since the case of $cutR$ is symmetric. We have then a derivation of the following form:

$$\dfrac{\dfrac{\vdots\ \Pi_L}{\Gamma_1 \vdash A} \quad \dfrac{\vdots\ \Pi_R}{\Gamma_2, A \vdash \Delta_2}}{\Gamma_2, \Gamma_1 \vdash \Delta_2}\ cutL$$

It is not limitative to assume that the right rank is greater than 1, for, if it is 1, $\Gamma_2$ is empty and hence the same cut can be read as an instance of $cutR$. We now see how $cutL$ can be lifted along $\Pi_R$, obtaining a derivation in which either $cutL$ does not appear any more or which contains occurrences of $cutL$ of lower right rank.

The last rule of $\Pi_R$ cannot be any rule introducing a connective on the left, by visibility of the rules of ${}^{\perp}C$ and the hypothesis on the right rank. If it is any right rule $\star R$ ($\& R$, $\vee R$, $wR$, $stR$), then one can lift the applications of $cutL$ and $\star R$ (in the case of the binary rule $\& R$, the cut is duplicated).

If the last rule is $wL$, and if $A$ is not introduced by weakening, one can lift the application of $cutL$ over $wL$; if $A$ is introduced by weakening, that is $\Pi_R$ ends with

$$\dfrac{\dfrac{\vdots\ \Pi'_R}{\Gamma'_2 \vdash \Delta_2}}{A, \Gamma_2 \vdash \Delta_2}\ wL$$

then the application of $cutL$ can be avoided, obtaining:

$$\dfrac{\dfrac{\vdots\ \Pi_R}{\Gamma'_2 \vdash \Delta_2}}{\Gamma_1, \Gamma_2 \vdash \Delta_2}\ wL$$

If the last rule is $stL$, several cases are possible. The main distinction is the following:

1. $A$ does not come from the right,

2. $A$ comes from the right.

In the first case, one can again simply lift $cutL$ over $stL$. In the second case, one has to distinguish again:

$2_1$. $A$ does not appear on the left in the premiss of $stL$,

$2_2$. $A$ appears on the left in the premiss of $stL$.

In the first case, one has

$$\dfrac{\dfrac{\vdots\ \Pi_L}{\Gamma_1 \vdash A} \quad \dfrac{\dfrac{\vdots\ \Pi_R}{\Gamma'_2, \vdash A^{\perp}, \Delta'_2}}{\Gamma_2, A \vdash \Delta_2}\ stL}{\Gamma_2, \Gamma_1 \vdash \Delta_2}$$

that converts into

$$
\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\,\Pi_L\\ \Gamma_1 \vdash A\end{matrix}\qquad \begin{matrix}\vdots\,\Pi_R^{\perp}\\ A, \Delta_2'^{\perp} \vdash \Gamma_2'^{\perp}\end{matrix}}{\Gamma_1, \Delta_2'^{\perp} \vdash \Gamma_2'^{\perp}}\, cutL}{\begin{matrix}\vdots\, swap\\ \Gamma_2' \vdash, \Gamma_1^{\perp} \Delta_2'\end{matrix}}}{\Gamma_2, \Gamma_1 \vdash \Delta_2}\, stL
$$

where the rank of $A$ in $\Pi_R^{\perp}$ is the rank of $A^{\perp}$ in $\Pi_R$ and hence the right rank of the cut has decreased. In the second case, one has

$$
\dfrac{\Gamma_1 \vdash A \qquad \dfrac{\dfrac{}{\Gamma_2', A \vdash A^{\perp}, \Delta_2'}\,\star}{\Gamma_2, A \vdash \Delta_2}\, stL}{\Gamma_2, \Gamma_1 \vdash \Delta_2}
$$

where one has to consider the rule $\star$. Note that, to conclude $\Gamma_2', A \vdash A^{\perp}, \Delta_2'$ no $stL$ or $stR$ on $A$ or $A^{\perp}$ can have been applied, since our sequents contain sets of formulae. So $\star$ is in one of the two following classes of rules:

$C_1$. $stL$, $stR$, or applications of $wL$ and $wR$ which do not move or introduce $A$ and $A^{\perp}$

$C_2$. applications of $wL$ or $wR$, introducing $A$ or $A^{\perp}$, respectively, or rules for connectives which introduce $A$ on the left or $A^{\perp}$ on the right.

If $\star$ is a rule of the first class, it is possible to modify the derivation of the right premise of the cut as follows:

$$
\dfrac{\dfrac{\dfrac{\Gamma_2'', A \vdash A^{\perp}, \Delta_2''}{\Gamma_2'', A \vdash \Delta_2''}\, stL}{\Gamma_2', A \vdash \Delta_2'}\,\star}{\Gamma_2, A, \vdash \Delta_2}\, stL
$$

(where the second occurrence of $stL$ may not occurr, if it is $\Delta_2' = \Delta_2$). Now one can lift the application of the cut over $stL$ and $\star$ up to the upper application of $stL$, obtaining a cut of the form:

$$
\dfrac{\Gamma_1 \vdash A \qquad \dfrac{\dfrac{}{\Gamma_2'', A \vdash A^{\perp}, \Delta_2''}\,\star'}{\Gamma_2'', A \vdash \Delta_2''}\, stL}{\Gamma_2'', \Gamma_1 \vdash \Delta_2''}\, cutL
$$

where $\star'$ is a rule of the first or of the second class and the right rank is lower, since $\star$ now appears below.

So, let us suppose we have arrived to a rule of the second class, after $n$ steps. (If $n = 0$, that is no rule of the first class occurred, and if $\Delta_2' \neq \Delta_2$ in the original cut, one has identically to substitute the application of $stL$ with two applications, and then to lift the cut over the second of them). Now there are four possibilities:

If $\star$ is $wR$, one simply avoids to introduce $A^\perp$ by weakening, so that the application of $stL$ immediately below disappears.

If $\star$ is $wL$, that is we have:

$$\dfrac{\Gamma_1 \vdash A \qquad \dfrac{\dfrac{\Gamma_2^{n+1} \vdash A^\perp, \Delta_2^n}{\Gamma_2^n, A \vdash A^\perp, \Delta_2^n}\; wL(A,\overline{\Gamma})}{\Gamma_2^n, A \vdash \Delta_2^n}\; stL}{\Gamma_2^n, \Gamma_1 \vdash \Delta_2^n}\; cutL$$

one can avoid $wL$ and apply cut as follows:

$$\dfrac{\Gamma_1 \vdash A \qquad \begin{array}{c} \vdots\; swap \\ A, \Delta_2^{n\perp} \vdash \Gamma_2^{(n+1)\perp} \end{array}}{\Gamma_1, \Delta_2^n \vdash \Gamma_2^{(n+1)^\perp}}\; cutL$$

and then one can swap again, apply $wL$ of the remaining formulae $\overline{\Gamma}$ and then $stL$ to $\Gamma_1$.

If $\star$ is a unary or binary rule $\circ R$, introducing a connective on the right, that is one has (with abuse of notation):

$$\dfrac{\Gamma_1 \vdash A \qquad \dfrac{\dfrac{\Gamma_2^n, A \vdash B_i}{\Gamma_2^n, A \vdash A^\perp}\; \circ R}{\Gamma_2^n, A \vdash}\; stL}{\Gamma_2^n, \Gamma_1 \vdash}$$

the following reduction can be applied:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_1 \vdash A \qquad \Gamma_2^n, A \vdash B_i}{\Gamma_2^n, \Gamma_1 \vdash B_i}\; cutL}{\Gamma_2^n, \Gamma_1 \vdash A^\perp}\; \circ R \qquad \begin{array}{c} \vdots\; swap \\ A^\perp \vdash \Gamma_1^\perp \end{array}}{\Gamma_2^n, \Gamma_1 \vdash \Gamma_1^\perp}\; cutL}{\Gamma_2^n, \Gamma_1 \vdash}\; stL$$

In such derivation the above occurrence (or the above occurrences, in the binary case) of $cutL$ has lower right rank. Once such occurrence has been eliminated, it remains to reduce a cut whose right rank (that is the rank of $A^\perp$ in $A^\perp \vdash \Gamma^\perp$) is the left rank of the original cut, while the left rank is 1. By means of another swap, one has then a cut in which the original left premise $\Gamma \vdash A$, together with the original left rank, is restored, and in which the right rank is 1.

If $\star$ is a unary or binary rule $\circ L$, introducing a connective on the left, one has:

$$\dfrac{\Gamma_1 \vdash A \qquad \dfrac{\dfrac{B_i \vdash A^\perp, \Delta_2^n}{A \vdash A^\perp, \Delta_2^n}\; \circ L}{A \vdash \Delta_2^n}\; stL}{\Gamma_1 \vdash \Delta_2^n}\; cutL$$

which reduces as follows:

$$\dfrac{\dfrac{\Gamma_1 \vdash A \quad \begin{array}{c}\vdots\ swap\\ A, {\Delta_2^n}^\perp \vdash B_i^\perp\end{array}}{\dfrac{\Gamma_1, {\Delta_2^n}^\perp \vdash B_i^\perp}{\Gamma_1, {\Delta_2^n}^\perp \vdash A^\perp}\ (\circ L)^\perp}\ cutL}{}$$

$$\begin{array}{c}\vdots\ swap\end{array}$$

$$\dfrac{\dfrac{\Gamma_1 \vdash A \quad A \vdash \Gamma_1^\perp, \Delta_2^n}{\Gamma_1 \vdash \Gamma_1^\perp, \Delta_2^n}}{\Gamma_1 \vdash \Delta_2^n}\ stL$$

where $(\circ L)^\perp$ is the dual rule of $\circ L$ (that is $\& R$ if $\circ L$ is $\vee L$ and $\vee R$ if $\circ L$ is $\& L$). In such derivation we have first to reduce one or two cuts of lower right rank, and then a cut of right rank equal to 1.

Now all the cases have been considered. It is easy to realize that, by applying the procedure just described, either we eliminate the cut or we reduce to having a $cutL$ whose right rank is 1, that, by visibility of the rules of ${}^\perp C$, has the following form:

$$\dfrac{\begin{array}{c}\vdots\ \Pi_L\\ \Gamma_1 \vdash A\end{array} \quad \dfrac{}{A \vdash \Delta_2}\ \star L}{\Gamma_1 \vdash \Delta_2}$$

where $\star L$ is the rule introducing $A$ on the left. Such an application of the cut rule can be considered as an application of $cutR$, to which the symmetric procedure of that just described can be applied. The application of such procedure, in turn, either eliminates the *cut* or produces a *cut* of rank 2. □

In basic logic, as well as in the quantum-like calculi which extend it and, in general, in any sequent calculus for quantum logics (cf. [18], [10]), a form of the rules which includes contexts is underivable, at least for some of them. For, otherwise, the distributive laws would be derivable (cf. [19] and see also next proposition 3.5). We stress moreover that, in basic logic, visibility is an intrinsic proof-theoretical feature which corresponds to the interpretation of the meaning of its connectives given by the reflection principle (cf. [2], [1], [19]). Contrary to this, every rule of ${}^\perp C$ admits also a form with contexts (a full form, in the terminology of basic logic, which is then indicated by the apex $(.)^f$). A similar result is known as display theorem in display logic, cf. [5].

**Lemma 2.3** *The following rules are derivable in* ${}^\perp C$*:*

*Cut*

$$\dfrac{\Gamma \vdash \Delta, A \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ cut^f$$

*Rules on connectives*

$$\dfrac{\Gamma, A \vdash \Delta}{\Gamma, A\& B \vdash \Delta}\ \& L^f \qquad\qquad \dfrac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \vee R^f$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \vee L^f \qquad\qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \& B, \Delta} \& R^f$$

**Proof.** We see below the proof of $\vee L^f$ by means of $stL$, $stR$ and $\vee L$.

$$\cfrac{\cfrac{\cfrac{\Gamma, A \vdash \Delta}{A \vdash \Gamma^\perp, \Delta}\, stR \quad \cfrac{\Gamma, B \vdash \Delta}{B \vdash \Gamma^\perp, \Delta}\, stR}{A \vee B \vdash \Gamma^\perp, \Delta}\, \vee L}{\Gamma, A \vee B \vdash \Delta}\, stL$$

The proof of the other derived rules is completely similar. $\square$

The above lemma allows to prove that ${}^\perp C$ is equivalent to **LK**. To verify this fact, let us first define an interpretation $i_1$ of the formulae in the language of ${}^\perp C$ into the formulae of a language including a set of literals $p_1, p_2, p_3, \ldots$ and the connectives $\&$, $\vee$ and $\neg$. Such interpretation is given by the clauses:

$$i_1(p_i) \equiv p_i \qquad\qquad i_1(p_i^d) \equiv \neg p_i \qquad\qquad i_1(A \circ B) \equiv i_1(A) \circ i_1(B)$$

for any binary connective $\circ$. Then one can see that $i_1(A^\perp) = \neg(i_1(A))$, by induction on the degree of $A$. Conversely, we define an interpretation $i_2$ of the classical formulae into the formulae of ${}^\perp C$, given by the clauses:

$$i_2(p_i) \equiv p_i \qquad\qquad i_2(\neg A) \equiv (i_2(A))^\perp \qquad\qquad i_2(A \circ B) \equiv i_2(A) \circ i_2(B)$$

for any binary connective $\circ$. One can see that $i_1(i_2(A)) = A$, for every classical formula, and that $i_2(i_1(A)) = A$ for every formula of ${}^\perp C$. The equivalence of ${}^\perp C$ and **LK** can now be proved as follows:

**Proposition 2.4** *${}^\perp C$ is equivalent to* **LK**, *that is*

$$\Gamma \vdash_{{}^\perp C} \Delta \qquad \textit{if and only if} \qquad i_1(\Gamma) \vdash_{\mathbf{LK}} i_1(\Delta)$$

*and*

$$\Gamma \vdash_{\mathbf{LK}} \Delta \qquad \textit{if and only if} \qquad i_2(\Gamma) \vdash_{{}^\perp C} i_2(\Delta)$$

**Proof.** It is easy to prove that $\Gamma \vdash_{{}^\perp C} \Delta$ implies $i_1(\Gamma) \vdash_{\mathbf{LK}} i_1(\Delta)$, and that $\Gamma \vdash_{\mathbf{LK}} \Delta$ implies $i_2(\Gamma) \vdash_{{}^\perp C} i_2(\Delta)$, by induction on the depth of the derivation. To prove the first, the only fact to notice is that $stL$ and $stR$ become the introductions of $\neg$ to the left and to the right, respectively. To prove the second, note that conversely the introductions of $\neg$ are translated into the $st$ rules, and, as for the rules of **LK** introducing $\&$ and $\vee$, apply the lemma above. Then one has in particular that $i_2(\Gamma) \vdash_{{}^\perp C} i_2(\Delta)$ implies $i_1(i_2(\Gamma)) \vdash_{\mathbf{LK}} i_1(i_2(\Delta))$ and that $i_1(\Gamma) \vdash_{\mathbf{LK}} i_1(\Delta)$ implies $i_2(i_1(\Gamma)) \vdash_{{}^\perp C} i_2(i_1(\Delta))$. So, since it is $i_1(i_2(A)) = A$, for every classical formula and $i_2(i_1(A)) = A$ for every formula of ${}^\perp C$, one obtains the two equivalences. $\square$

## 3 Basic orthologic with a primitive modality

The result we have reached in the previous section could be put in the following equality (already obtained in [12], [13])

classical logic = basic orthologic + structural rules

Now, given such an equality, we can easily obtain an embedding of classical logic into basic orthologic, by forcing the structural rules $stL$ and $stR$ to act only in the case of modalized formulae. In this, we take example from the treatment of the structural rules of weakening and contraction by means of exponentials in linear logic. The novelty of our approach is that we introduce the modality $\downharpoonleft$ by means of new literals, and then we extend it to all formulae by an inductive definition. In this sense, we say that our modality is "primitive" since it is present as a distinguished set of formulae, where the distinction has been made at the origin, in the literals, before any rule on it. These facts will be better explained in this section.

To obtain basic orthologic equipped with a primitive modality, we need a language equipped with the usual binary connectives & and $\vee$ and with four kinds of primitive literals:

$$p_1, p_2, p_3, \ldots$$
$$p_1^d, p_2^d, p_3^d, \ldots$$
$$q_1, q_2, q_3, \ldots$$
$$q_1^d, q_2^d, q_3^d, \ldots$$

The intended meaning of the literals $p$ and $q$ is clarified by the definition of the operator $\downharpoonleft$ on the atomic formulae, as follows:

$$\downharpoonleft p_i \equiv q_i, \qquad \downharpoonleft p_i^d \equiv q_i^d, \qquad \downharpoonleft q_i \equiv q_i, \qquad \downharpoonleft q_i^d \equiv q_i^d,$$

Such an operator is then extended to all formulae putting:

$$\downharpoonleft(A \circ B) \equiv \downharpoonleft A \circ \downharpoonleft B.$$

for any connective $\circ$.

Note that the degree of $\downharpoonleft A$ is equal to the degree of $A$. Moreover, we extend the duality $\perp$ to the new literals, as follows:

$$q^\perp \equiv q^d \qquad\qquad (q^d)^\perp \equiv q$$

We now see that the operator $\perp$ is still a duality, that $\downharpoonleft$ is an idempotent operator and that the operators $\perp$ and $\downharpoonleft$ commute. Then it will be possible to study the structural operator $\perp$ in a modal setting too (for the treatment of structural operations in modal logic, in the framework of display logic, cf. [20]).

**Proposition 3.1** *For every formula $A$ in the above language, it is*

$$A^{\perp\perp} = A, \qquad \downharpoonleft\downharpoonleft A = \downharpoonleft A, \qquad (\downharpoonleft A)^\perp = \downharpoonleft(A^\perp)$$

*where the equality means that the formula on the right and the formula on the left are the same formula.*

**Proof.** It is easy to check this by induction on the complexity of the formula. As for the base of the induction, note that $x^{\perp\perp} = x$, $\downharpoonleft\downharpoonleft x = \downharpoonleft x$, $\downharpoonleft(x^{\perp}) = (\downharpoonleft x)^{\perp}$, for any literal $x$ of any kind; then, to obtain the thesis, suppose that $A$ is $B \circ C$ and apply the inductive definitions of $\perp$ and $\downharpoonleft$. $\square$

Since the operator $\downharpoonleft$ is idempotent, the formulae made out of literals $q$ and $q^d$, that are the formulae preceded by $\downharpoonleft$, are a proper subset of the set of all formulae. Then it makes sense to consider notions which apply to $\downharpoonleft$-formulae only. We introduce here the couple of structural rules $\downharpoonleft stL$ and $\downharpoonleft stR$, that is the rules *st* which apply only to $\downharpoonleft$-formulae:

$$\frac{\Gamma \vdash \Delta, \downharpoonleft\Sigma}{\Gamma, \downharpoonleft\Sigma^{\perp} \vdash \Delta}\ \downharpoonleft stL \qquad\qquad \frac{\Gamma, \downharpoonleft\Sigma \vdash \Delta}{\Gamma \vdash \Delta, \downharpoonleft\Sigma^{\perp}}\ \downharpoonleft stR$$

Now we can consider a new calculus, which we shall call $\downharpoonleft^{\perp}\mathbf{BS}_0$, in the new language, whose rules are those of ${}^{\perp}C$, with the exception of the *st* rules, which are substituted by the $\downharpoonleft st$ rules. Note that it is, by definition,

$$\downharpoonleft^{\perp}\mathbf{BS} = \text{basic orthologic} + \text{structural rules}$$

where however basic orthologic is intended in the new language.

Then $\downharpoonleft^{\perp}\mathbf{BS}_0$ is a symmetric system in which the operator $\downharpoonleft$ acts as a unary monotonic connective. In fact, one obtains the following structure theorem, which extends lemma 2.1:

**Lemma 3.2** *In* $\downharpoonleft^{\perp}\mathbf{BS}_0$ *the following metarules hold:*

$$\Gamma \vdash_{\downharpoonleft^{\perp}\mathbf{BS}_0} \Delta \textit{ if and only if } \Delta^{\perp} \vdash_{\downharpoonleft^{\perp}\mathbf{BS}_0} \Gamma^{\perp}$$

*by a derivation of the same length; and*

$$\textit{if } \Gamma \vdash_{\downharpoonleft^{\perp}\mathbf{BS}_0} \Delta, \textit{ then } \downharpoonleft\Gamma \vdash_{\downharpoonleft^{\perp}\mathbf{BS}_0} \downharpoonleft\Delta$$

*by a derivation of the same length.*

**Proof.** The first claim is proved as in lemma 2.1, exploiting proposition 3.1; the second holds because any rule of $\downharpoonleft^{\perp}\mathbf{BS}_0$ can be applied in particular to modalized formulae. So, if $\Gamma \vdash \Delta$ is derivable from the axioms $A_1 \vdash A_1, \ldots, A_n \vdash A_n$, the same derivation, applied to the axioms $\downharpoonleft A_1 \vdash \downharpoonleft A_1, \ldots, \downharpoonleft A_n \vdash \downharpoonleft A_n$, derives $\downharpoonleft\Gamma \vdash \downharpoonleft\Delta$. $\square$

Moreover, $\downharpoonleft^{\perp}\mathbf{BS}_0$ satisfies visibility. The following result, analogous to the result obtained in lemma 2.3, shows that contexts in rules of $\downharpoonleft^{\perp}\mathbf{BS}_0$ can be allowed, if they are made of $\downharpoonleft$-formulae only.

**Lemma 3.3** *The following rules are valid in* $\downharpoonleft^{\perp}\mathbf{BS}_0$*:*

*Cuts*

$$\frac{\Gamma \vdash \downharpoonleft \Delta, A \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \downharpoonleft \Delta, \Delta'}\ cutL^{\downharpoonleft} \qquad\qquad \frac{\Gamma \vdash \Delta, A \quad A, \downharpoonleft \Gamma' \vdash \Delta'}{\Gamma, \downharpoonleft \Gamma' \vdash \Delta, \Delta'}\ cutR^{\downharpoonleft}$$

*Rules on connectives*

$$\frac{\downharpoonleft \Gamma, A \vdash \Delta}{\downharpoonleft \Gamma, A \& B \vdash \Delta}\ \&L^{\downharpoonleft} \qquad\qquad \frac{\Gamma \vdash A, \downharpoonleft \Delta}{\Gamma \vdash A \vee B, \downharpoonleft \Delta}\ \vee R^{\downharpoonleft}$$

$$\frac{\downarrow\Gamma, A \vdash \Delta \quad \downarrow\Gamma, B \vdash \Delta}{\downarrow\Gamma, A \vee B \vdash \Delta}\ \vee L^{\downarrow} \qquad\qquad \frac{\Gamma \vdash A, \downarrow\Delta \quad \Gamma \vdash B, \downarrow\Delta}{\Gamma \vdash A \& B, \downarrow\Delta}\ \& R^{\downarrow}$$

**Proof.** Any rule $J^{\downarrow}$ can be derived by means of the corresponding rule $J$ of ${}^{\perp}\mathbf{BS}_0$, by applying $\downarrow stL$ and $\downarrow stR$. □

Comparing the tables of rules of ${}^{\perp}C$ and of $\downarrow{}^{\perp}\mathbf{BS}_0$, one can easily find an embedding of classical logic into basic orthologic with a primitive modality. This is obtained considering a translation $f$ of $\downarrow{}^{\perp}\mathbf{BS}_0$-formulae into ${}^{\perp}C$-formulae, defined putting

$$f(p_i) \equiv p_i \qquad f(p_i^d) \equiv p_i^d \qquad f(q_i) \equiv p_i \qquad f(q_i^d) \equiv p_i^d$$

and $f(A \circ B) \equiv f(A) \circ f(B)$. Such translation simply forgets the $\downarrow$ of the $\downarrow$-formulae of $\downarrow{}^{\perp}\mathbf{BS}_0$; in fact one can prove by induction on the degree of the formula that

$$f(\downarrow A) = A$$

for every formula $A$. Conversely, one defines a map $d$ of ${}^{\perp}C$-formulae into $\downarrow{}^{\perp}\mathbf{BS}_0$- formulae, putting:

$$d(p_i) \equiv q_i \qquad d(p_i^d) \equiv q_i^d$$

and $d(A \circ B) \equiv d(A) \circ d(B)$. It maps the formulae of ${}^{\perp}C$ into the $\downarrow$-formulae of $\downarrow{}^{\perp}\mathbf{BS}_0$. Note that $f(d(A)) = A$ for every formula of ${}^{\perp}C$, as it is easy to see. So, we obtain a formal proof of our embedding theorem:

**Theorem 3.4** *For any pair of finite sets of formulae $\Gamma$, $\Delta$, in the language of ${}^{\perp}C$, one has:*

$$\Gamma \vdash_{{}^{\perp}C} \Delta \qquad \textit{if and only if} \qquad d(\Gamma) \vdash_{\downarrow{}^{\perp}\mathbf{BS}_0} d(\Delta)$$

**Proof.** One can see, by induction on the derivation, that $\Gamma \vdash_{{}^{\perp}C} \Delta$ implies $d(\Gamma) \vdash_{\downarrow{}^{\perp}\mathbf{BS}_0} d(\Delta)$ and that $\Gamma \vdash_{\downarrow{}^{\perp}\mathbf{BS}_0} \Delta$ implies $f(\Gamma) \vdash_{{}^{\perp}C} f(\Delta)$. Then, if $d(\Gamma) \vdash_{\downarrow{}^{\perp}\mathbf{BS}_0} d(\Delta)$, one has $f(d(\Gamma)) \vdash_{{}^{\perp}C} f(d(\Delta))$, that is $\Gamma \vdash_{{}^{\perp}C} \Delta$. □

After this theorem, we can say that $\downarrow$-formulae of $\downarrow{}^{\perp}\mathbf{BS}_0$ are, so to say, "the classical formulae of $\downarrow{}^{\perp}\mathbf{BS}_0$". Besides classical formulas, we have also "basic" formulas, that is formulas containing only literals $p_i$ or $p_i^d$. Then, we have also "non classical" formulas, that are those containing at least one literal $p_i$ or $p_i^d$, as well as "non basic" formulas, containing at least one literal $q_i$ or $q_i^d$. Dealing with such "mixed" formulae, $\downarrow{}^{\perp}\mathbf{BS}_0$ takes the interesting feature of distributivity of classical formulae with respect to any kind of formula.

**Proposition 3.5** *In $\downarrow{}^{\perp}\mathbf{BS}_0$, the distributive laws are provable in the following form:*

$$\downarrow C \& (A \vee B) \vdash (\downarrow C \& A) \vee (\downarrow C \& B) \qquad (A \vee \downarrow D) \& (B \vee \downarrow D) \vdash (A \vee B) \& \downarrow D$$

**Proof.** The two sequents $\downarrow C, A \vdash (\downarrow C \& A) \vee (\downarrow C \& B)$ and $\downarrow C, B \vdash (\downarrow C \& A) \vee (\downarrow C \& B)$ are derivable from the axioms $\downarrow C \vdash \downarrow C$, $A \vdash A$ and $\downarrow C \vdash \downarrow C$, $B \vdash B$, respectively, by means of weakening, $\& R$ and $\vee R$, that are rules of basic orthologic. Now, one can conclude $\downarrow C, A \vee B \vdash (\downarrow C \& A) \vee (\downarrow C \& B)$ by $\vee L^{\downarrow}$, which holds by lemma 3.3. Finally, one has the conclusion $\downarrow C \& (A \vee B) \vdash (\downarrow C \& A) \vee (\downarrow C \& B)$ by two applications of the rule $\& R$ to the axioms $\downarrow C \vdash \downarrow C$ and $A \vee B \vdash A \vee B$, each of them followed by a cut. Symmetrically one can derive the other distributive law, by means of the rule $\& R^{\downarrow}$ and hence exploiting again lemma 3.3. $\square$

In basic logic, the validity of the distributive laws of the multiplicative conjunction (disjunction) w.r.t. the additive disjunction (conjunction) is equivalent to the presence of contexts in the rules $\vee L$ and $\& R$, respectively, as it is shown in details in [19]. In such case, the distributive laws admit a natural cut-free derivation. The same proof cannot be adapted to $\downarrow^{\perp}\mathbf{BS}_0$, and actually, as one can see, no cut free derivation of the above distributive laws is possible in it. So, the cut rules allows to prove, in case of "non basic" formulae, more than what basic orthologic would prove.

To obtain a cut-free system, it is necessary to limit some possibilities on "non basic" formulae. This is obtained limiting the rules of basic orthologic, so that they cannot produce non basic formulae, unless they are really classical formulae. To do this, we need a formal definition of "non classical formula": the literals $p_i$ and $p_i^d$ are non classical formulae and, if $\alpha$ is a non classical formula and $B$ is any formula, then $\alpha \circ B$ is a non classical formula.

Now we can introduce the calculus $\downarrow^{\perp}\mathbf{BS}$, which is defined by the following axioms and rules of inference. In such system, some rules on connectives are splitted into two, with respect to those of basic orthologic. ($A, B$... are formulae; $\alpha, \beta$... are non classical formulae and $\Gamma, \Delta$... are sets of formulae).

Axioms

$$A \vdash A$$

Cuts

$$\frac{\Gamma \vdash A \quad A, \Gamma' \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta}\ cutL \qquad\qquad \frac{\Gamma \vdash \Delta, A \quad A \vdash \Delta'}{\Gamma \vdash \Delta, \Delta'}\ cutR$$

Rules on connectives

$$\frac{\alpha \vdash \Delta}{\alpha \& B \vdash \Delta}\ nc\&L \quad \frac{\beta \vdash \Delta}{A \& \beta \vdash \Delta}\ nc\&L \qquad\qquad \frac{\Gamma \vdash \beta}{\Gamma \vdash A \vee \beta}\ nc \vee R \quad \frac{\Gamma \vdash \alpha}{\Gamma \vdash \alpha \vee B}\ nc \vee R$$

$$\frac{\downarrow A \vdash \Delta}{\downarrow A \& \downarrow B \vdash \Delta}\ c\&L \quad \frac{\downarrow B \vdash \Delta}{\downarrow A \& \downarrow B \vdash \Delta}\ c\&L \qquad\qquad \frac{\Gamma \vdash \downarrow B}{\Gamma \vdash \downarrow A \vee \downarrow B}\ c \vee R \quad \frac{\Gamma \vdash \downarrow A}{\Gamma \vdash \downarrow A \vee \downarrow B}\ c \vee R$$

$$\frac{A \vdash \Delta \quad B \vdash \Delta}{A \vee B \vdash \Delta} \vee L \qquad\qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B} \& R$$

Structural rules

$$\frac{\Gamma \vdash \Delta}{\Gamma, \Sigma, \vdash \Delta} wL \qquad\qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \Sigma} wR$$

$$\frac{\Gamma \vdash \Delta, \downarrow\Sigma}{\Gamma, \downarrow\Sigma^{\perp} \vdash \Delta} \downarrow stL \qquad\qquad \frac{\Gamma, \downarrow\Sigma \vdash \Delta}{\Gamma \vdash \Delta, \downarrow\Sigma^{\perp}} \downarrow stR$$

As it is immediate to realize, $\downarrow^{\perp}\mathbf{BS}$ has the same results given in 3.2 and 3.3 for $\downarrow^{\perp}\mathbf{BS}_0$. Moreover, it inherits also the result given in theorem 3.4, that is classical logic is embeddable in $\downarrow^{\perp}\mathbf{BS}$ too. So, $\downarrow^{\perp}\mathbf{BS}$ proves, on classical formulae $\downarrow A$, exactly the theorems provable in $\downarrow^{\perp}\mathbf{BS}_0$, that is the theorems of classical propositional logic. It is also easy to realize that the cut-free derivations for basic formulae which can be performed in $\downarrow^{\perp}\mathbf{BS}$ are exactly the cut free derivations which can be performed in $\downarrow^{\perp}\mathbf{BS}_0$, that is, the derivations of basic orthologic.

The two systems $\downarrow^{\perp}\mathbf{BS}$ and $\downarrow^{\perp}\mathbf{BS}_0$ differ in their behaviour on non basic (or non classical) formulae. Actually, $\downarrow^{\perp}\mathbf{BS}_0$ proves more than basic orthologic on this kind of formulae (e.g. distributivity) while $\downarrow^{\perp}\mathbf{BS}$ can prove less. Sequents of the form $q_i \& p_j \vdash q_i$, which are derivable in $\downarrow^{\perp}\mathbf{BS}_0$, by means of the rule $\& L$ inherited from basic orthologic, are underivable in $\downarrow^{\perp}\mathbf{BS}$ (incidentally, sequents of the form $q_i \& p_j \vdash p_j$ are derivable yet). An unpleasant consequence is that axioms $A \vdash A$, where $A$ is neither basic nor classical, are not derivable from the literals which form the formula $A$. Also the distributive laws derived in lemma 3.5 are underivable, since their derivation requires sequents of the form $\downarrow C \& A \vdash \downarrow C$, or their symmetric, in an essential way.

The loss of lemma 3.5 is compensated by the following lemma, which allows to obtain cut-elimination for $\downarrow^{\perp}\mathbf{BS}$ as a consequence of that for ${}^{\perp}C$.

**Lemma 3.6** *If the sequent $\Gamma \vdash \downarrow\Delta$ is derived in $\downarrow^{\perp}\mathbf{BS}_0$ by a cut-free proof, then there exists a cut-free proof $\Pi'$ and a finite set of formulae $\overline{\Gamma}$, such that the same sequent is derivable as follows*

$$\frac{\begin{array}{c}\vdots\ \Pi' \\ \downarrow\Gamma' \vdash \downarrow\Delta\end{array}}{\Gamma \vdash \downarrow\Delta} wL(\overline{\Gamma})$$

*where $\Gamma = \overline{\Gamma}, \downarrow\Gamma'$. Symmetrically for any sequent of the form $\downarrow\Gamma \vdash \Delta$.*

**Proof.** By induction on the depth of the derivation of the sequent. If $\Gamma \vdash \downarrow\Delta$ is the axiom $\downarrow A \vdash \downarrow A$, we are finished; in the inductive cases we shall see that the premises of the last rule applied are also sequents of the same form, so that the inductive hypothesis can be applied, obtaining a proof and a set of formulas (or two, in the binary cases), from which then $\Pi'$ and $\overline{\Gamma}$ can be obtained, essentially permuting the applications of the last rule and of the weakening obtained by the inductive hypothesis. Let us see the details.

- $wR$ If the last rule applied is $wR$, then it has to be $\downarrow\Delta = \downarrow\Delta', \downarrow\Delta''$, where $\downarrow\Delta''$ has been introduced by $wR$ from the premise $\Gamma \vdash \downarrow\Delta'$. So, by the inductive hypothesis, one has a proof $\Pi'_1$ followed by $wL(\overline{\Gamma}_1)$, from which one obtains a proof $\Pi'$ given by $\Pi'$ followed by $wR(\downarrow\Delta'')$ and the set $\overline{\Gamma} = \overline{\Gamma}_1$, satisfying the thesis.

- $wL$ If the last rule applied is $wL(\Gamma'')$, its premise has the form $\Gamma' \vdash \downarrow\Delta$. Then we have $\Pi'_1$ and $\overline{\Gamma}_1$ by the inductive hypothesis, and then $\Pi' = \Pi'_1$ and $\overline{\Gamma} = \overline{\Gamma}_1, \Gamma''$ satisfy the thesis.

- $\downarrow stR$ The premise of $\downarrow stR$ must be of the form $\Gamma, \downarrow\Delta'' \vdash \downarrow\Delta'$. Such sequent, by the inductive hypothesis, is derivable by applying $wL(\overline{\Gamma}_1)$ to $\Pi'_1$, but then $\Pi'_1$ followed by $wL(\downarrow\Delta'')$ and then by $\downarrow stR(\downarrow\Delta'')$ is a good $\Pi'$, while $\overline{\Gamma} = \overline{\Gamma}_1 - \downarrow\Delta''$.

- $\downarrow stL$ In this case it is enough to choose $\Pi' = \Pi'_1 + \downarrow stL$ and $\overline{\Gamma} = \overline{\Gamma}_1$.

- $\&L$ If $A\&B \vdash \Delta$ follows from $A \vdash \Delta$ by means of one of the two $\&L$ rules, then by the inductive hypothesis it must be $A = \downarrow A$ unless $A$ has been introduced by weakening. Then either we already have a derivation of the sequent $\vdash \downarrow\Delta$, to which $wL(A\&B)$ can be applied, or the rule $\&L$ must have the form $c\&L$, so that $B = \downarrow B$ and hence $A\&B = \downarrow A\&\downarrow B = \downarrow(A\&B)$, so that the original derivation satisfies the thesis.

- $\vee L$ In this case we have $A \vee B \vdash \downarrow\Delta$ which follows from the premises $A \vdash \downarrow\Delta$ and $B \vdash \downarrow\Delta$. As in the previous case, if in one of them the left formula is introduced by weakening from $\vdash \downarrow\Delta$, then one can introduce by weakening the formula $A \vee B$ itself, otherwise we have $A = \downarrow A$ and $B = \downarrow B$ by the inductive hypothesis, so that it is $A \vee B = \downarrow(A \vee B)$ and the original derivation satisfies the thesis.

- $\&R$ By hypothesis, the formula introduced by the last rule must have the form $\downarrow A\&\downarrow B$, and hence we have the derivations $\Pi_1$ of $\Gamma \vdash \downarrow A$ and $\Pi_2$ of $\Gamma \vdash \downarrow B$, which by the inductive hypothesis can be converted into a derivation $\Pi'_1$ of $\downarrow\Gamma_1 \vdash \downarrow A$ followed by $wL(\overline{\Gamma}_1)$ and into a derivation $\Pi'_2$ of $\downarrow\Gamma_2 \vdash \downarrow B$ followed by $wL(\overline{\Gamma}_2)$, respectively. Then we obtain the derivation:

$$
\dfrac{\dfrac{\dfrac{\vdots\ \Pi'_1}{\downarrow\Gamma_1 \vdash \downarrow A}}{\downarrow\Gamma_1, \downarrow\Gamma_2 \vdash \downarrow A} \quad \dfrac{\dfrac{\vdots\ \Pi'_2}{\downarrow\Gamma_2 \vdash \downarrow B}}{\downarrow\Gamma_1, \downarrow\Gamma_2 \vdash \downarrow B}}{\dfrac{\downarrow\Gamma_1, \downarrow\Gamma_2 \vdash \downarrow A\&\downarrow B}{\Gamma \vdash \downarrow A\&\downarrow B}\ wL(\overline{\Gamma}_1, \overline{\Gamma}_2)}
$$

(possibly with some redundancy in the weakenings).

$\vee R$ Here the formula introduced by $\vee R$ must be of the form $\downarrow A \vee \downarrow B$, so that in the premise we have, e.g., the sequent $\Gamma \vdash \downarrow A$ to which we apply the inductive hypothesis, obtaining $\Pi_1'$ and $\overline{\Gamma}_1$. Then we have finally $\Pi_1'$ followed by $\vee R$ and $wL(\overline{\Gamma'}_1)$.

$\square$

The lemma allows to exploit the cut-elimination procedure for ${}^{\perp}C$ described in the previous section, obtaining cut-elimination for $\downarrow^{\perp}\mathbf{BS}$.

**Theorem 3.7** *Any derivation in $\downarrow^{\perp}\mathbf{BS}$ which ends with an application of $cutL$ or $cutR$ and where no other application of $cutL$ or $cutR$ occurs, can be transformed into a derivation in which $cutL$ and $cutR$ do not appear. Hence, any derivation in $\downarrow^{\perp}\mathbf{BS}$ can be transformed into a derivation in which $cutL$ and $cutR$ do not appear.*

**Proof.** As it is easy to control, the modification of the rules $\&L$ and $\vee R$ of $C^{\perp}$ which occurs in $\downarrow^{\perp}\mathbf{BS}$ is irrelevant in the cut elimination procedure described in theorem 2.2, both in the degree reduction than in the rank reduction. The only additional difference between ${}^{\perp}C$ and $\downarrow^{\perp}\mathbf{BS}$ relies in the rules $st$, which, in case of $\downarrow^{\perp}\mathbf{BS}$, are applied only to the $\downarrow$-formulae. The reductions described in the procedure of theorem 2.2, when applied to derivations of $\downarrow^{\perp}\mathbf{BS}$, do not introduce any new application of such rules, except in the case of $\downarrow stL$, in the subcase 2, that is when the cut formula itself comes from the right. But then the cut formula is certainly an $\downarrow$-formula and the derivation is as follows:

$$\dfrac{\begin{matrix}\vdots\ \Pi_L \\ \Gamma_1 \vdash \downarrow A\end{matrix} \qquad \dfrac{\Gamma_2', \vdash \downarrow A^{\perp}, \Delta_2'}{\Gamma_2, \downarrow A \vdash \Delta_2}\ \downarrow stL}{\Gamma_2, \Gamma_1 \vdash \Delta_2}\ cutL$$

When reducing this cut, a problem arises, since the derivations one obtains contain several applications of the rule $\downarrow stL$ to the dual $\Gamma_1^{\perp}$ of the set of rules $\Gamma_1$. Such applications are not allowed in $\downarrow^{\perp}\mathbf{BS}$, unless $\Gamma_1$ consists of $\downarrow$-formulae only. By the above lemma, one can modify $\Pi_L$ obtaining

$$\dfrac{\begin{matrix}\vdots\ \Pi_L' \\ \downarrow\Gamma_1' \vdash \downarrow A\end{matrix}}{\Gamma_1 \vdash \downarrow A}\ wL(\overline{\Gamma})$$

and then one can modify the above derivation of $\Gamma_2, \Gamma_1 \vdash \Delta_2$ substituting $\Pi_L$ with $\Pi_L'$ and applying $wL$ after the cut rule, obtaining:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \Pi_L' \\ \downarrow \Gamma_1' \vdash \downarrow A\end{matrix} \qquad \dfrac{\Gamma_2', \vdash \downarrow A^{\perp}, \Delta_2'}{\Gamma_2, \downarrow A \vdash \Delta_2}\ \downarrow stL}{\Gamma_2, \downarrow \Gamma_1' \vdash \Delta_2}\ cutL}{\Gamma_2, \Gamma_1 \vdash \Delta_2}\ wL(\overline{\Gamma})$$

In such derivation the cut rule can be eliminated by the same procedure seen in 2.2. In fact such procedure reduces the right rank of $cutL$, without any assumption on the left rank, so that it is insensitive to the change of left rank which may occurr substituting $\Pi_L$ with $\Pi'_L$. Then $cutL$ is reduced to a cut whose right rank is 1, that appears as follows:

$$\dfrac{\begin{array}{c}\vdots\ \Pi'_L \\ \downarrow\Gamma_1 \vdash \downarrow A\end{array} \quad \dfrac{}{\downarrow A \vdash \Delta_2}\,\star}{\downarrow\Gamma_1 \vdash \Delta_2}$$

where $\star$ is $\vee L$ or $\& L$. Now, one has possibly to apply again the lemma (in its dual form). Such application does not modify the fact that $\downarrow A$ is a principal formula in the right premise of the cut, so that the reduction of the left rank can again be performed as in theorem 2.2. $\square$

From lemma 3.6 and from the above cut-elimination result, we can derive the following feature of the modality $\downarrow$ in $\downarrow^{\perp}$**BS**:

**Proposition 3.8** *The schemata $\downarrow A \vdash A$ and $A \vdash \downarrow A$ are not provable in $\downarrow^{\perp}$**BS**, and, equivalently, the following rules*

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \downarrow A \vdash \Delta}\ \downarrow L^f \qquad\qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash \downarrow A, \Delta}\ \downarrow R^f$$

*are not valid in $\downarrow^{\perp}$**BS**.*

**Proof.** By lemma 3.6 and 3.7, a proof of a sequent of the form $B \vdash \downarrow A$ is possible only if $\vdash \downarrow A$ is derivable or $B = \downarrow C$ for some formula $C$. Symmetrically for a sequent of the form $\downarrow A \vdash B$. Then in particular the schemata $\downarrow A \vdash A$ and $A \vdash \downarrow A$ are not true. Finally, one can see that the rules $\downarrow L^f$ and $\downarrow R^f$ are derivable from the schemata $\downarrow A \vdash A$ and $A \vdash \downarrow A$, respectively, by an application of $cutL$ and $cutR$, respectively; conversely, the schemata are derivable applying the rules to the axiom $A \vdash A$. $\square$

If we had adopted a connective for $\downarrow$ and, with it, introduction rules like the above ones, as it is actually done for exponentials in linear logic ([16]), the application of our cut-elimination procedure would have been impossible. In fact, if one of the two rules is added to $\downarrow^{\perp}$**BS**, lemma 3.6 fails. Moreover, in any symmetric calculus, the rules $\downarrow L^f$ or $\downarrow R^f$ could be present only together with their symmetric. This would force us to introduce the symmetric $\downarrow^{\perp}$ of the connective $\downarrow$, and hence a second modality, different from $\downarrow$, as it is also done in classical linear logic, with the two exponentials ! and ?. In fact, putting $\downarrow^{\perp} \equiv \downarrow$ would lead to $\downarrow A = A$, since then the symmetric of $\downarrow L^f$ would be exactly $\downarrow R^f$, so that, by 3.8, the two schemata $\downarrow A \vdash A$ and $A \vdash \downarrow A$ would hold at the same time. On the other side, putting $\downarrow^{\perp} \neq \downarrow$ would have required an interpretation for the presence of two modalities in the calculus. Ultimately, since we did not see any reason for which one of the two schemata above should hold, we made the choice to try to avoid both of them, and our primitive modality was a solution. So we have obtained a system in which two ways of reasoning, the classical one and the quantum-like one, coexist and do not interfere.

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