Interpreting quantum parallelism by sequents

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Abstract

We introduce an interpretation of quantum superposition in predicative sequent calculus, in the framework of basic logic. Then we introduce a new predicative connective for the entanglement. Our aim is to represent quantum parallelism in terms of logical proofs.

1 Introduction

Basic logic [9] was first proposed as a platform for cut-free sequent calculi of propositional extensional logics, including some kind of quantum logics [3]. The natural idea was then to obtain a calculus for quantum computation. After some attempt with propositional logic, we realized that our idea requires the quantifiers, first introduced in basic logic in [8].

In quantum computational logics [4] propositions correspond to the qubits and the quregisters, that is to the states of the quantum computer itself, rather than to the closed subsets of a Hilbert space, as in traditional quantum logic. We also adopt such an approach. Our representation does not require the algebraic setting of Hilbert spaces, and represents quantum superposition and entanglement by means of sequents, in order to describe quantum parallelism in terms of logical proofs. The complete calculus associated to our representation is under development [1]. Anyway, our representation already allows to see the computational advantage of quantum parallelism with respect to classical computation, that consists in knocking down the exponential complexity. This was the original motivation in the proposal of quantum computation by Feynman [6].

The idea is that the random variable given by an experiment on a certain physical system produces the domain of a first order variable, which describes the superposed state of the system. We see that the gap existing between the description of a superposed state, and the probability distribution given by the measurement of the state, is translated into the logical gap between a predicative representation for the superposition and a propositional representation

for the corresponding probability distribution. In such setting, the expressive power of logical variables seems necessary. This is confirmed by our new predicative connective for the entanglement, which exploits a variable in common to obtain a new quantifier. The variable seems to capture the holistic feature of quantum information [5]. In fact, a variable can glue items of information in a non-compositional way (see [2] for philosophical considerations). In particular, while the algebraic definition of entanglement is *negative*, since it speaks of *non factorizable* states, our approach can represent entangled particles in a positive way. This is considered a decisive advantage in any computational and constructive setting.

2 Basic Logic

As it is well known, one can distinguish two kinds of rules in sequent calculus: rules on the structure of sequents (*structural rules*) and rules introducing logical connectives. Basic logic **B** is a core for sequent calculus since it aims to characterize the "logic of connectives". Well known logics are then recovered by the addition of structural rules. In basic logic, connectives are characterized after metalinguistic links. We consider the following metalinguistic links between assertions: *and*, *yield*, *forall*.

1. *yield* links two assertions at a different level, in a *sequential* way.
2. *and* links two assertions at the same level, in a *parallel* way.
3. *forall* gathers assertions depending on a common variable.

When assertions are represented by sequents, logical connectives and their rules are the result of importing the links into the object level, putting and solving the following definitory equations:

1. $\Gamma \vdash A \to B \quad \equiv \quad \Gamma, A \vdash B$
   where $\Gamma, A \vdash B$ represents the sequential link between A and B, in a context Γ.
2. $\Gamma \vdash A \& B \quad \equiv \quad \Gamma \vdash A \quad \Gamma \vdash B$
   $\Gamma \vdash A * B \quad \equiv \quad \Gamma \vdash A, B$
   where the couple $\Gamma \vdash A \quad \Gamma \vdash B$ is the additive translation of *and*; $\Gamma \vdash A, B$ is the multiplicative translation of *and*[1].
3. $\Gamma \vdash (\forall x \in D)A(x) \quad \equiv \quad \Gamma, z \in D \vdash A(z)$, z not free in Γ
   where $\Gamma, z \in D \vdash A(z)$ gathers all assertions $A(z)$ depending on the free variable z ranging on the domain D.

[1] For the connective which translates the multiplicative way to conceive *and*, we write a simple multiplication symbol "$*$" rather than adopting the usual notation of the connective "par" of linear logic, also adopted for basic logic.

For the solution of the definitory equations we refer to basic logic. We remind the rules obtained as a solution of the $\forall$ equation [8]:

$$\frac{\Gamma, z \in D \vdash A(z)}{\Gamma \vdash (\forall x \in D)A(x)} \forall f^{\dagger} \qquad \frac{\Gamma' \vdash z \in D \quad \Gamma, A(z) \vdash \Delta}{\Gamma, (\forall x \in D)A(x), \Gamma' \vdash \Delta} \forall r$$

where † is the condition "z not free in Γ". Such condition has a clear semantical motivation and is necessary for consistency. Technically, it gives the quantifier an additive character. Then we can consider $\forall$ an additive connective, a sort of "big &", in particular. Anyway, it is much more, due to the presence of the variable. We discuss below a physical interpretation of this.

3 Interpretation of quantum superposition

We remind that, in probability theory, an *experiment* is defined as a random variable Z, with a set of possible outcomes B, and with an associated probability measure p_Z. Then any experiment determines a set $D = D(Z, p_Z)$, given by

$$D(Z, p_Z) = \{(z, p\{Z = z\}) : z \in B\}$$

Let $\mathcal{A}$ be a quantum system. A quantum measurement on it is an experiment, where Z is given by the observable, the set of the possible outcomes determines an orthonormal basis of the Hilbert space of $\mathcal{A}$ and p_Z comes from the probability amplitude. Then a set D defined as above is associated to any measurement of $\mathcal{A}$.

A measurement on $\mathcal{A}$, under certain hypothesis, is described by an assertion, as follows:
"*forall* possible outcomes z in D, under certain hypothesis Γ, the possible result of the measurement of $\mathcal{A}$ is z."

We rewrite it, formally, as a sequent:

$$\Gamma, z \in D \vdash A(z)$$

where the first order variable z appears free in A and does not appear free in the hypothesis Γ. In fact, the hypothesis of a correct experiment cannot depend on its outcome. So we put the equivalence given at point 3. in the previous section, defining the quantifier *forall*:

$$\Gamma \vdash (\forall x \in D)A(x) \;\equiv\; \Gamma, z \in D \vdash A(z)$$

Such definition allows to gather the possible results $A(z)$, associated to the observable, into a unique object, represented by the proposition $(\forall x \in D)A(x)$. Then the quantifier $\forall$ interpretes quantum superposition. The "logical glue" for quantum superposition is the variable associated with the random variable of the experiment. When the superposed state is considered, the variable is bounded, and ranges on the domain given by the experiment.

By the $\forall r$ rule, from the axioms of sequent calculus, one derives the sequent

$$(\forall x \in D)A(x), z \in D \vdash A(z)$$

that is the so called "reflection axiom" in basic logic. In our case, it asserts that the particle described by the proposition $(\forall x \in D)A(x)$ can be found in a state associated with any of the z's of D.

Substituting the free variable z by a *closed* term t in it, one has the sequent $(\forall x \in D)A(x), t \in D \vdash A(t)$ from which, since $t \in D$ is true, one derives the sequent

$$(\forall x \in D)A(x), t \in D \vdash A(t)$$

It asserts that the superposition $(\forall x \in D)A(x)$ is converted into $A(t)$, where t denotes a fixed element of the orthonormal basis, with its probability. The other possibilities are lost. This describes a collapse: the substitution operation destroys the superposition.

A description of the original superposition can be recovered a posteriori, by the propositional connectives, as we illustrate in the example below.

Example
Let us consider a particle $\mathcal{A}$ and the set D given by the outcomes of the measurement of the spin of $\mathcal{A}$ along the z axis. D has got two elements: $(|\uparrow\rangle, p\{Z = |\uparrow\rangle\})$ and $(|\downarrow\rangle, p\{Z = |\downarrow\rangle\})$, denoted by the terms $t_\uparrow$ and $t_\downarrow$ respectively. The proposition $(\forall x \in D)A(x)$ represents the superposed state of the two directions of the spin along the z-axis. The sequent $(\forall x \in D)A(x) \vdash A(t_\uparrow)$ asserts that $\mathcal{A}$ is found in the "up" direction along the z axis with the probability given by the measurement experiment. Analogously, the sequent $(\forall x \in D)A(x) \vdash A(t_\downarrow)$ says that $\mathcal{A}$ is found in the "down" direction along the z axis with the associated probability.
From the two sequents one can derive, by the $\&f$ rule of **B**, the sequent

$$(\forall x \in D)A(x) \vdash A(t_\uparrow)\&A(t_\downarrow).$$

The propositional formula (closed terms, no variable!) appearing on the right side of it:

$$A(t_\uparrow)\&A((t_\downarrow)$$

describes the probability distribution associated to the superposed state $(\forall x \in D)A(x)$. The sequent $(\forall x \in D)A(x) \vdash A(t_\uparrow)\&A(t_\downarrow)$, that is derivable (when a substitution rule is allowed), states that the probability distribution follows from the superposition. The converse sequent is not derivable unless one assumes specific axioms.

Then our logical representation can distinguish between superposition and probability distribution. The distinction, in the logical setting, is due to the presence of the variable.

The role of the variable in representing quantum parallelism is enforced by the considerations below.

4 Multiplicative and quantum parallelism

The multiplicative connectives represent the register link in a computer and are exploited to represent the parallel processes on different registers. We now wish to study the combination of the multiplicative connective $*$ with the quantifier $\forall$, that, as just seen, can represent quantum superposition in sequent calculus. Our aim is to obtain a view of quantum parallelism in terms of sequents.

The technique of the definitory equations allows to study easily the combinations of connectives. Here we skip the details [1] and focus on the distributive law of the multiplicative connective $*$ w.r.t. the quantifier $\forall$. Distributivity is provable in the following form:

$$(\forall x \in D_1)A(x) * (\forall x \in D_2)B(x) = (\forall x \in D_1)(\forall w \in D_2)(A(x) * B(w))$$

here termed classical distributivity. The variables $x \in D_1$ and $w \in D_2$ are different and range on possibly different domains. This means that the variables on which propositions A and B depend and which are bounded by the $\forall$ are different and independent. In fact, as one can see in proving one direction of the above equality, the $\forall f^{\dagger}$ rule cannot be applied, when the free variable to be bounded by the quantifier is the same for A and B, due to the restriction † on the free variable the rule itself contains.

When distributivity holds, one can conceive a unique semantical object given by the combination of the two connectives, since distributivity guarantees that the definition is syntax-independent. Then one can define a unique multiplicative-additive quantifier $*\forall$, putting the definitory equation:

$$\Gamma \vdash (*\forall x \in D_1, w \in D_2)(A(x); B(w)) \quad \equiv \quad \Gamma, z \in D_1, y \in D_2 \vdash A(z), B(y)$$

where the free variables z and y are not free in Γ and $z \neq y$. The object so defined coincides with $(\forall x \in D_1)A(x) * (\forall x \in D_2)B(x)$ or with $(\forall x \in D_1)(\forall w \in D_2)(A(x) * B(w))$.

The necessary requirement $z \neq y$ has a heavy computational drawback. In fact, it implies independent choices for $z \in D_1$ and $y \in D_2$. This yields the exponential increasing of complexity, in the number of variables, of the object combining the two parallelisms given by $*$ and $\forall$.

In order to overcome the problem of complexity, it would be crucial to have distributivity with respect to *one* variable:

$$(\forall x \in D)A(x) * (\forall x \in D)B(x) = (\forall x \in D)(A(x) * B(x))$$

Unfortunately, the object that could be given by such equality does not exist in logic, since the interpretation of $*$ as a disjunction, which is forced in the extensions of **B**, makes the above distributive law false, as one can easily realize.

Where could such object be set? First, notice that the following "parallel" application of the $\forall f$-rule, would allow to prove the false version of distributivity:

$$\frac{\Gamma, z \in D \vdash A(z), B(z)}{\Gamma \vdash (\forall x \in D)A(x), (\forall x \in D)B(x)} \; \forall f \|$$

Such rule consists of a simultaneous application of the $\forall f$ rule to the couple of formulae $A(z)$ and $B(z)$. It is technically admissible in **B**, since it preserves the additive character of the quantifier, that would be spoiled by a two-steps application of a $\forall f$ rule to $A(z)$ and then to $B(z)$, when z is the same free variable. [2]. But, if we added the $\forall f\|$-rule as such to sequent calculus, we would render logic inconsistent!

5 A new quantifier for the entanglement

In order to import the second case of distributivity in the realm of logic, we need to distinguish the case of dependent variables from the case of independent variables, interpreting them by different connectives. We will keep both cases only in the paraconsistent setting of basic logic. Then inconsistency can be avoided in its extensions [1].

Let us consider a random variable Z and its associated domain, as described in section 3. Then, a new link between two propositions A and B is definable, in terms of a common first-order variable ranging on the domain, as follows. Let us consider the sequent

$$\Gamma, z \in D \vdash A(z), B(z),$$

where z is free in A and B. Let us assume that the comma says also "there is a variable in common". This enriches the link between A and B, that would be simply put side by side otherwise.

Then let us write this "$,_Z$", where Z is the aleatory variable which gives the domain of the first-order variable z. We term such new link "variable-comma" and rewrite the sequent as follows:

$$\Gamma, z \in D \vdash A(z),_Z B(z),$$

Note that the link $,_Z$ may be present even if the first order variable z becomes bounded. In fact, in that case, it is included in the first-order domain D associated to the experiment, that is $D(Z, p_Z)$.

We now put the following version of $\forall f\|$-rule, valid for the $,_Z$ link only:

$$\frac{\Gamma, z \in D \vdash A(z),_Z B(z)}{\Gamma \vdash (\forall x \in D)A(x),_Z (\forall x \in D)B(x)} \forall f\|$$

In it, the link $,_Z$ is still present in the conclusion, even if the first-order variable z is not free any more. This is correct for a parallel rule, since it concerns only the *forall* link, and does not act on the comma between the two formulae A and B. Hence such comma must be kept unaltered.

The variable-comma $,_Z$ has the character of a "semi-predicative" link. We put the definitory equation of the corresponding semi-predicative multiplicative connective $\bowtie_Z$:

$$\Gamma \vdash A \bowtie_Z B \quad \equiv \quad \Gamma \vdash A,_Z B$$

[2] It also satisfies Gentzen's original formulation of the condition in the $\forall$ rule, that is "the variable bounded by the application of $\forall$ must not occurr free in the conclusion of the rule".

By the $\forall f\|$ rule one can prove the new distributive law, written with respect to $\bowtie_Z$ (where the subscript $_Z$ may be redundant):

$$(\forall x \in D)A(x) \bowtie_Z (\forall x \in D)B(x) = (\forall x \in D)(A(x) \bowtie_Z B(x))$$

We term such equality Bell's distributivity.

As a consequence, a new quantifier $\bowtie_{x\in D} (A(x); B(x))$, combining multiplicative parallelism and superposition, is definable in $\mathbf{B}$, putting the equation:

$$\Gamma \vdash \bowtie_{x\in D} (A(x); B(x)) \quad \equiv \quad \Gamma, z \in D \vdash A(z),_Z B(z)$$

where z is not free in Γ. The following rules are derivable from such equation (see [1]):

$$\frac{\Gamma, z \in D \vdash A(z),_Z B(z)}{\Gamma \vdash \bowtie_{z\in D} (A(x); B(x))} \bowtie f^\dagger$$

$$\frac{\Gamma' \vdash z \in D \quad \Gamma_1, A(z) \vdash \Delta_1 \quad \Gamma_2, B(z) \vdash \Delta_2}{\Gamma_1, \Gamma_2, \bowtie_{x\in D} (A(x); B(x)), \Gamma' \vdash \Delta_1,_Z \Delta_2} \bowtie r$$

The new quantifier $\bowtie$ is equal to $(\forall x \in D)A(x) \bowtie_Z (\forall x \in D)B(x)$ or to $(\forall x \in D)(\forall x \in D)A(x) \bowtie_Z B(x)$. It allows to represent systems of entangled particles, as we now see.

Let $\mathcal{A}$ and $\mathcal{B}$ be two entangled particles, for example two electrons with opposite spin. The possible result of a measurement of the spin along the z axis, performed on $\mathcal{A}$ or on $\mathcal{B}$, is equally described by an assertion of the form

$$\Gamma, z \in D \vdash A(z),_Z B(z)$$

where $D = \{(|\uparrow\rangle, p\{Z =\uparrow\}, (|\downarrow\rangle, p\{Z =\downarrow\})\}$, and where $A(z)$ means "$\mathcal{A}$ is found in the z direction", and $B(z)$ is "$\mathcal{B}$ is found in the direction opposite to z"[3]. Moreover, we have the usual condition that z is not free in Γ.

So we put now the definitory equation:

$$\Gamma \vdash \bowtie_{x\in D} (A(x); B(x)) \quad \equiv \quad \Gamma, z \in D \vdash A(z),_Z B(z)$$

The state of the two entangled particles is then described by the proposition $\bowtie_{x\in D} (A(x); B(x))$. The first-order variable, corresponding to the *unique* random variable describing the experiment, is the glue which allows to describe the superposed state *together with* the entanglement between the two particles at the same time.

What makes the entanglement disappear? In physics, the collapse of the wave function. In our logical terms, a substitution of the variable z by a closed term t destroys the superposition and also the entanglement, since no variable is present any more. The assertion $\Gamma \vdash A(z),_Z B(z)$, after a substitution, becomes $\Gamma \vdash A(t), B(t)$ where the comma is the usual comma of sequent calculus, since the variable has disappeared. Then, no entanglement is described at the propositional level.

[3] $B(z)$ indicates that the state is a function of z, the free variable being z

6 A comparison with the classical case

Let us consider two independent experiments, proucing two independent random variables, Z and Y, and so two possibly different domains D_Z and D_Y. It may even happen that the two domains coincide, anyway this fact does not affect the independence of the variables. The assertion describing the couple of experiments has got the following form:

$$\Gamma, z \in D_Z, y \in D_Y \vdash A(z), B(y)$$

where $z \neq y$ and Γ does not contain z and y free. It corresponds to the object $*\forall$, given in section 4, defined by classical distributivity, which implies exponential growth of complexity.

We can conceive the two experiments applied to two different physical systems, for example two particles, $\mathcal{A}$ and $\mathcal{B}$. We can also conceive two independent experiments on the same physical system, say $\mathcal{A}$. In the first case the propositions $A(z)$ and $B(y)$ represent the possible value of the measurements obtained applying the observable corresponding to Z to $\mathcal{A}$ and that corresponding to Y to $\mathcal{B}$, respectively; in the second case they represent the two possible values of the two measurements performed on $\mathcal{A}$. The second case is possible only if the observables for the two experiments are compatible. Then incompatible observables of quantum mechanics should be interpreted as a way to avoid computational complexity.

Classical distributivity is restored considering simultaneously two incompatible observables for two entangled particles. For example, measurements of the spin along different axis, z and y, which are incompatible on the same particle, can be applied as simultaneous independent measurements on two entangled particles $\mathcal{A}$ and $\mathcal{B}$. In such case we have an assertion of the form $\Gamma, z \in D_Z, y \in D_Y \vdash A(z), B(y)$. The simpler assertion with variable-comma $,_Z$, $\Gamma, z \in D_Z \vdash A(z),_Z B(z)$, or, as an alternative, the assertion with variable-comma $,_Y$, $\Gamma, y \in D_Y \vdash A(y),_Y B(y)$, together with the corresponding $\bowtie$ logical object, are not possible when the two simultaneous independent measurements are applied. On the contrary, when one measurement (spin along z or spin along y) is applied, the other is not possible any more, for the effect of the entanglement. Then the computational effect of the entanglement is alternative to the computational effect of compatible observables.

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