SEQUENT CALCULUS AND QUANTUM PARALLELISM

Giulia Battilotti
Department of Philosophy
University of Florence

I

Challenge: to find an explanation to the quantum computational speed up in terms of logical proofs.

To obtain logical proofs we consider sequent calculus. A sequent is an object of the form

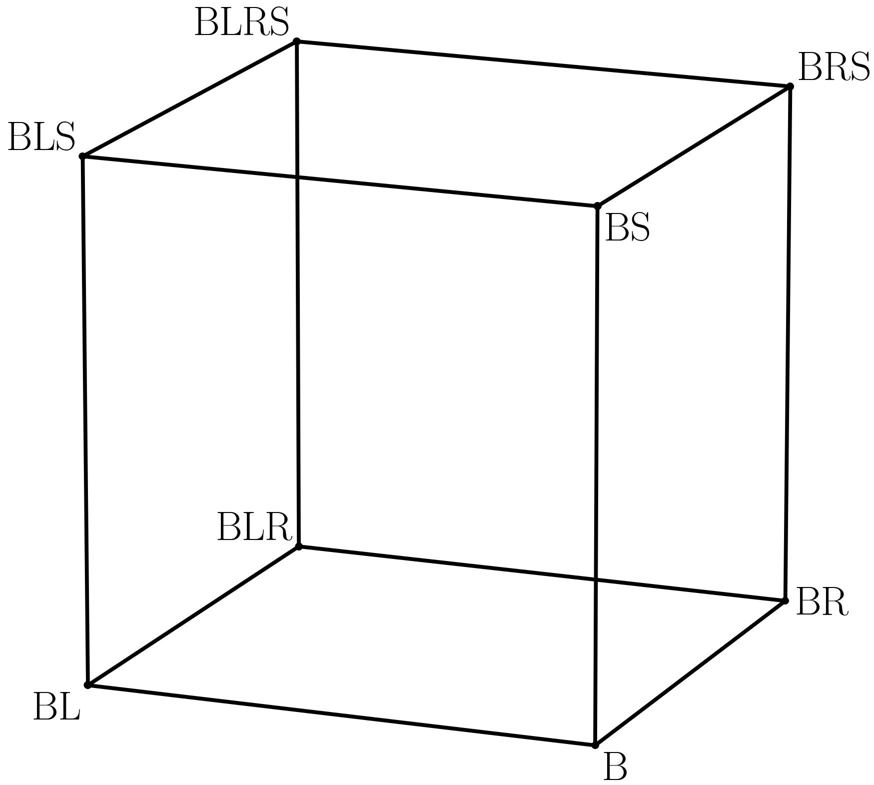
$$A_1, \ldots A_n \vdash B_1 \ldots B_m$$

(summing up $\Gamma \vdash \Delta$). The calculus is given by rules on sequents. We distinguish two kinds of rules:

1. Rules on the structure of sequents (*Structural Rules*)
2. Rules introducing logical connectives.

The cube of logics

Basic logic $\mathbb{B}$ is a core for sequent calculus.



Connectives from metalinguistic links

Basic logic considers the following metalinguistic links between assertions: *and*, *yield*, *forall*.

1. *yield* links two assertions at a different level, in a *sequential* way.
2. *and* links two assertions at the same level, in a *parallel* way.
3. *forall* links assertions with a variable in common. The variable is the reason of the link.

Assertions are represented by *sequents*.
Then logical connectives and their rules in sequent calculus are the result of importing the links into the object level obtaining a sequent calculus.

Definitory equations

1. $\Gamma \vdash A \to B \quad \equiv \quad \Gamma, A \vdash B$
   where $\Gamma, A \vdash B$ represents the sequential link between A and B, in a context Γ.
2. $\Gamma \vdash A \& B \quad \equiv \quad \Gamma \vdash A \quad \Gamma \vdash B$
   $\Gamma \vdash A \cdot B \quad \equiv \quad \Gamma \vdash A, B$
   where the couple $\Gamma \vdash A \quad \Gamma \vdash B$ is the additive translation of *and*; $\Gamma \vdash A, B$ is the multiplicative translation of *and*.
3. $\Gamma \vdash (\forall x \in D)A(x) \quad \equiv \quad \Gamma, z \in D \vdash A(z)$, z not free in Γ
   where $\Gamma, z \in D \vdash A(z)$ gathers all assertions $A(z)$ depending on a free variable on the domain D.

Rules of $\forall$:

$$\frac{\Gamma, z \in D \vdash A(z)}{\Gamma \vdash (\forall z \in D)A(x)}\,\forall f \qquad\qquad \frac{\Gamma' \vdash z \in D \quad \Gamma, A(z) \vdash \Delta}{\Gamma, (\forall x \in D)A(x), \Gamma' \vdash \Delta}\,\forall r$$

Multiplicative + additive parallelism

The distributive law $A \cdot (B\&C) = (A\&B) \cdot (A\&C)$ is provable in $\mathbf{BR}$ and its extensions.

Distributivity is extended to the predicative case as follows:

$$(\forall x \in D_1)A(x) \cdot (\forall x \in D_2)B(x) = (\forall x \in D_1)(\forall y \in D_2)(A(x) \cdot B(y))$$

(classical distributivity). One has to require that x and y are independent variables.

Computational drawback: exponential increasing of complexity in the number of independent variables.

Dependent variables

Distributivity with *dependent* variables:

$$(\forall x \in D)A(x) \cdot (\forall x \in D)B(x) = (\forall x \in D)(A(x) \cdot B(x))$$

fails. It is false!!!

Computational advantage: no exponential increasing of complexity.

It would be proved by a parallel application of the $\forall f$-rule:

$$\frac{\Gamma, z \in D \vdash A(z), B(z)}{\Gamma \vdash (\forall x \in D)A(x), (\forall x \in D)B(x)} \; \forall f \|$$

A new link

We consider a common variable as a further link.

In $\Gamma \vdash A(z), B(z)$ the comma says "there is a variable in common, or there used to be a variable in common above in the derivation".
We write $,_z$ and put the definitory equation:

$$\Gamma \vdash A \bowtie B \quad \equiv \quad \Gamma \vdash A,_z B$$

New $\forall f\|$-rule:

$$\frac{\Gamma, z \in D \vdash A(z),_z B(z)}{\Gamma \vdash (\forall x \in D)A(x),_z (\forall x \in D)B(x)}\ \forall f\|$$

Then distributivity is:

$$(\forall x \in D)A(x) \bowtie (\forall x \in D)B(x) = (\forall x \in D)(A(x) \bowtie B(x))$$

(Bell's distributivity).

A new quantifier

A new multiplicative-additive quantifier

$$\bowtie_{x \in D} (A(x); B(x))$$

equal to $(\forall x \in D)A(x) \bowtie (\forall x \in D)B(x)$ or to $(\forall x \in D)(\forall x \in D)A(x) \bowtie B(x)$ is definable to exploit dependent variables.

It is defined by the equation:

$$\Gamma \vdash \bowtie_{x \in D} (A(x); B(x)) \quad \equiv \quad \Gamma, z \in D \vdash A(z),_z B(z)$$

Rules of $\bowtie$:

$$\frac{\Gamma, z \in D \vdash A(z),_z B(z)}{\Gamma \vdash \bowtie_{z \in D} (A(x); B(x))} \bowtie f$$

$$\frac{\Gamma' \vdash z \in D \quad \Gamma_1, A(z) \vdash \Delta_1 \quad \Gamma_2, B(z) \vdash \Delta_2}{\Gamma_1, \Gamma_2, \bowtie_{x \in D} (A(x); B(x)), \Gamma' \vdash \Delta_1, \Delta_2} \bowtie r$$

What makes the system collapse?

Substitution of the variable z by a closed term t destroys the $,_z$-link.

$\Gamma \vdash A(z),_z B(z)$ becomes $\Gamma \vdash A(t), B(t)$ where the comma is the usual comma of sequent calculus.

Experiments

Let $\mathcal{A}$ be a quantum system.

A quantum measurement on $\mathcal{A}$ is an *experiment*. The set of its possible outcomes determines an orthonormal basis B of the Hilbert space of $\mathcal{A}$. The hypothesis Γ of the experiment cannot depend on its outcome.

In probability theory an *experiment* is a random variable Z with an associated probability measure. In quantum mechanics such measure is given by a probability amplitude. Let

$$D = D(Z, p_Z) = \{(z, p\{Z = z\}) : z \in B\}$$

such set.

Superposition by a quantifier

The assertion
"*forall* possible outcomes z in D, in the hyporthesis Γ, the possible result of the measurement of $\mathcal{A}$ is z"
is formally

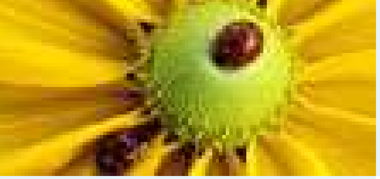
$$\Gamma, z \in D \vdash A(z)$$

Since Γ does not depend on z, we put the equivalence of the definitory equation of *forall*:

$$\Gamma \vdash (\forall x \in D)A(x) \ \equiv \ \Gamma, z \in D \vdash A(z)$$

Then the proposition $(\forall x \in D)A(x)$ represents quantum superposition.

So the logical glue for quantum superposition is the variable associated with the random variable of the experiment.



Substitution for collapse

The derivable sequent

$$(\forall x \in D)A(x), z \in D \vdash A(z)$$

says that the particle described by the proposition $(\forall x \in D)A(x)$ can be found in a state associated with any of the z's of D.

Substituting z by a *closed* term t, one has that the superposition $(\forall x \in D)A(x)$ is converted into $A(t)$, t corresponding to a fixed element of the orthonormal basis. The other possibilities are lost. This describes a collapse. Substitution destroys superposition.

Example

$\mathcal{A}$ a particle, D given by outcomes of the measurement of the spin of $\mathcal{A}$ along the z axis.

D has got two elements: $|\uparrow\rangle$ and $|\downarrow\rangle$. $(\forall x \in D)A(x)$ represents the superposed state of the two directions of the spin along the z-axis.

$(\forall x \in D)A(x) \vdash A(|\uparrow\rangle)$ says that $\mathcal{A}$ is found in the "up" direction along the z-axis.

Representing entanglement

Let $\mathcal{A}$ and $\mathcal{B}$ be two entangled particles, for example two electrons with opposite spin. The possible result of a measurement of the spin along the z axis, performed on $\mathcal{A}$ or on $\mathcal{B}$, is equally described by an assertion of the form

$$\Gamma, z \in D \vdash A(z),_z B(z)$$

Since we have put

$$\Gamma \vdash \bowtie_{x \in D} (A(x); B(x)) \quad \equiv \quad \Gamma, z \in D \vdash A(z),_z B(z)$$

the corresponding state is then described by the proposition

$$\bowtie_{x \in D} (A(x); B(x)).$$

Again the variable is our unique glue. As seen before:

- We have Bell's distributivity for entangled particles.
- Substitution by a closed term destroys the superposition, hence the entanglement.

Separated particles

Let $\mathcal{C}$ and $\mathcal{D}$ be two particles described by a product state (separated). A measurement of $\mathcal{C}$ is described simply by $\Gamma, z \in D \vdash C(z)$.

Measurements on both particles are also possible and described by

$$\Gamma, z \in D, y \in D \vdash C(z), D(y)$$

Measurements on both particles and on different axis (e.g. the z-axis for $\mathcal{C}$ and the y-axis for $\mathcal{D}$) are also possibles and described by

$$\Gamma, z \in D_Z, y \in D_Y \vdash C(z), D(y)$$

(different domains).

In both cases, one has classical distributivity with exponential growth of complexity.

Thanks

I wish to thank the organizers, the audience and Sandro Sozzo.