QUANTUM SEQUENTS

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We aim to put assertions from quantum mechanics in terms of sequents.

A sequent is an object of the form

$$A_1, \ldots A_n \vdash B_1 \ldots B_m$$

summing up $\Gamma \vdash \Delta$. It can represent the assertion of the conclusions Δ under the premises Γ. The turnstyle $\vdash$ represents a consequence relation. Read it *yield*.

A sequent calculus can derive assertions from other assertions. There are underivable assertions: $A \vdash A$, our axioms. Other assertions are derived by rules on sequents. For example:

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}\& f$$

We adopt the view of basic logic, developed as a common platform for sequent calculi of extensional logics.

One derives the rules of logical connectives putting definitory equations for connectives themselves, of the form

$$\Gamma \vdash A \circ B \quad \equiv \quad \Gamma \vdash A \approx B$$

where $\circ$ is the connective defined in terms of the metalinguistic link $\approx$. Examples:

$$\Gamma \vdash A \& B \quad \equiv \quad \Gamma \vdash A \;\; and \;\; \Gamma \vdash B$$

$$\Gamma(-z) \vdash (\forall x \in D) A(x) \quad \equiv \quad \Gamma(-z), z \in D \vdash A(z)$$

Moreover, one has a Leibnitz-style definition of the equality relation:

$$\Gamma', \Gamma(t/s), t = s \vdash \Delta(t/s), \Delta' \quad \equiv \quad \Gamma', \Gamma \vdash \Delta, \Delta'$$

We consider a preparation of a physical system. The preparation and all the measurement hypothesis are described in the set of premises Γ.
We represent by the sequent

$$\Gamma \vdash A_1, \ldots, A_n$$

the information $A_1, \ldots, A_n$ one can achieve from the preparation by a measurement.

Quantum measurements enables us to distinguish three logical levels:

- quantum states prior to measurement: predicative level
- density operators: propositional level with probabilities
- classical states/sharp states: propositional level

A discrete random variable Z yields a set

$$D_Z \equiv \{z = (s(z), p\{Z = s(z)\})\}$$

where $s(z)$ is the outcome and $p\{Z = s(z)\} > 0$ is its frequency. We term D_Z *random first order domain*.

We say that a random first order domain D_Z is *focused* w.r.t. an equality predicate $=$ if and only if it holds

$$z \in D_Z \vdash (z = t_1) \vee \cdots \vee (z = t_m)$$

where the terms $t_i = (s(t_i), p\{Z = s(t_i)\})$, $i = 1, \ldots m$, denote the outcomes of the random variable with their probabilities.

Otherwise, D_Z is *unfocused*.

The measurement process of a quantum state w.r.t. an observable is a random variable.
Its outcomes can be associated to elements of an orthonormal basis of the Hilbert space associated to the system.

Let Z be the random variable produced by a measurement of a certain particle in a certain state. This defines the random first order domain

$$D_Z \equiv \{z = (s(z), p\{Z = s(z)\}) : s(z) \text{ state of the outcome}\}$$

Propositions to represent probability distributions

We consider $D_Z = \{t_1, \ldots, t_m\}$ the r.f.o.d. of outcomes of a measurement of a particle $\mathcal{A}$.
We write $A(t_i)$ for the proposition "The particle $\mathcal{A}$ is found in state $s(t_i)$ with probability $p\{Z = s(t_i)\}$".

Let Γ represent a set of premises for the measurement. One has that Γ *yield* $A(t_i)$, for $i = 1 \ldots m$. This is written $\Gamma \vdash A(t_i)$ as sequents.

Then one has

$$\Gamma \vdash A(t_1) \& \ldots \& A(t_m)$$

where $\&$ is the additive conjunction. The proposition

$$A(t_1) \& \ldots \& A(t_m)$$

represents our knowledge of the state after measurement, namely the probability distribution of the outcomes.

Propositions to represent quantum states

To describe the quantum state prior to measurement, one drops the identification of the outcomes, namely, D_Z is unfocused. Then we consider variables rather than closed terms. We describe the outcomes of the measurement by the assertion:

"In the measurement hypothesis Γ, the state of the outcome is $s(z)$ with probability $p\{Z = s(z)\}$ for all pairs $z = (s(z), p\{Z = s(z)\}) \in D_Z$".

More formally, we write this assertion

$$\text{“}\textit{forall } z \in D_Z, \Gamma \vdash A(z)\text{”}$$

Then we import the metalinguistic assumption "*forall* $z \in D_Z$ as a further premise in the sequent, writing:

$$\Gamma, z \in D_Z \vdash A(z)$$

(where Γ does not depend on z, since the measurement hypothesis do not depend on the outcome.)

We put the definition:

$$\Gamma \vdash (\forall x \in D_Z)A(x) \equiv \Gamma, z \in D_Z \vdash A(z)$$

which summarizes the assertion by means of the quantifier $\forall$.

The first order variable z (associated to the random variable Z) is used as a logical glue for the different outcomes.
In this sense we claim that the proposition

$$(\forall x \in D_Z)A(x)$$

can attribute a superposed state to the particle $\mathscr{A}$.

Performing a quantum measurement determines a collapse.

In our terms we consider the "collapse of the variable" due to a substitution by a closed term.

We consider the provable sequent

$$(\forall x \in D_Z)A(x), z \in D_Z \vdash A(z)$$

The substitution z/t yields

$$(\forall x \in D_Z)A(x), t \in D_Z \vdash A(t)$$

from which

$$(\forall x \in D_Z)A(x) \vdash A(t)$$

If $t_1, \ldots, t_n$ denote the n elements of D_Z, one obtains (by & rule):

$$(\forall x \in D_Z)A(x) \vdash A(t_1)\& \ldots \& A(t_n)$$

The proposition $A(t_1)\& \ldots \& A(t_n)$ represents the mixed state given by the outcome of the quantum measurement.

We have represented a non selective quantum measurement.

To represent a selective measurement, yielding a pure state:

We consider a substitution which "forgets" the probability and gives probability 1 to the result:

$$(\forall x \in D_Z)A(x) \vdash A_f(s)$$

where s is a term denoting a state $|b\rangle$ with probability 1 after the measurement: $s = (|b\rangle, 1)$. The r.f.o.d is the singleton $\{(|b\rangle, 1)\}$.

For every formula $A(x)$, we have

$$A(s) \vdash (\forall x \in \{(|b\rangle, 1)\})A(x)$$

Since it is also $(\forall x \in \{(|b\rangle, 1)\})A(x) \vdash A(s)$, one has the equality

$$(\forall x \in \{(|b\rangle, 1)\})A(x) = A(s)$$

In particular, if the state is sharp, the r.f.o.d. is a singleton and $A_f(s)$ and $A(s)$ are the same.
Sharp states can be identified with propositional formulae.

But, for $n > 1$

$$(\forall x \in D_Z)A(x) \neq A(t_1)\& \ldots \& A(t_n)$$

For, the sequent $A(t_1)\& \ldots \& A(t_n) \vdash (\forall x \in D_Z)A(x)$ is not derivable.

One can prove that:
The sequent $A(t_1)\& \ldots \& A(t_n) \vdash (\forall x \in D_Z)A(x)$ holds for every A if and only if the domain D_Z is focused.

As soon as one can focus, the interference disappears!

This enables us to characterize quantum states predicatively.

Note that focusing D_Z requires that an equality predicate

$$z = t_i$$

should be definable *in a uniform way* on it. In terms of vectors in Hilbert spaces, this means to choose a unique phase factor.
True if and only if the domain is a singleton.

If we consider more than one particle, and consider an observable, we may obtain an assertion of the form $\Gamma \vdash A_1, \ldots, A_n$, $n > 1$.

For example we have a couple of particles, $\mathcal{A}$ and $\mathcal{A}'$.
If the two particles are separated, that is, if the measurement result on the first is independent from the measurement on the second, we have two independent random variables, Z and Z'.

So we define two domains D_Z and D'_Z and describe the measurement of the compound system by the sequent:

$$\Gamma, z \in D_Z, z' \in D_{Z'} \vdash A(z), A'(z')$$

that is converted into

$$\Gamma \vdash (\forall x \in D_Z)A(x) * (\forall x \in D_{Z'})A'(x)$$

(where $*$ is the multiplicative disjunction of linear logic).
One derives:

$$(\forall x \in D, w \in D')A(x) * B(w) = (\forall x \in D)A(x) * (\forall x \in D')B(x)$$

that is a distributive law of classical logic.

With entangled particles, one does not have independent measurements and hence independent variables.

Again, the variable can act as a glue.
In a paraconsistent setting one can define a generalized n-ary quantifier, in order to represent entangled states. It is denoted $\bowtie^n$ (in particular, $\bowtie^1$ is $\forall$).

The proposition

$$\bowtie^2_{x \in D_Z} (A_1; A_2)$$

represents the entangled state of 2 particles "sharing" the same random variable Z, and hence the same r.f.o.d. D_Z.

It comes from the following definition:

$$\Gamma \vdash \bowtie^2_{x \in D_Z} (A_1; A_2) \quad \equiv \quad \Gamma, z \in D_Z \vdash A_1(z),_Z A_2(z)$$

where A_1 and A_2 depend on the same variable z and the indexed comma $,_Z$ indicates the correlation between the outcomes for the two particles.

Qubits

We consider the measurement of the spin w.r.t. the z axis. In the Hilbert space C^2 we consider the orthonormal basis $\{|\downarrow\rangle, |\uparrow\rangle\}$. We write the state of q as a vector evidentiating its *relative phase* ϕ:

$$|q\rangle = \alpha|\downarrow\rangle + e^{i\phi}\beta|\uparrow\rangle$$

Different qubits yielding the same probability distribution are characterized by ϕ. So the unfocused domain $D_Z = \{(\downarrow, \alpha^2), (\uparrow, \beta^2)\}$, corresponds to the family of vectors $\alpha|\downarrow\rangle + e^{i\phi}\beta|\uparrow\rangle$, $\phi \in [0, 2\pi)$.
Two qubits in the same family can be distinguished by measurement if and only if they are orthogonal. This gives $\alpha^2 = \beta^2 = 1/2$ and $\phi' - \phi = \pi$.
We consider $\phi = 0$ and $\phi = \pi$, and characterize the couple of orthogonal vectors $|+\rangle$ and $|-\rangle$:

$$|+\rangle = 1/\sqrt{2}|\downarrow\rangle + 1/\sqrt{2}|\uparrow\rangle \qquad |-\rangle = 1/\sqrt{2}|\downarrow\rangle - 1/\sqrt{2}|\uparrow\rangle$$

Domains characterized in C^2

So the measurement basis $|\downarrow\rangle$ and $|\uparrow\rangle$ allows to characterize:

- Two singletons $D_\uparrow = \{(\uparrow, 1)\}$ and $D_\downarrow = \{(\downarrow, 1)\}$, relative to the measurement of qubits in the basis state.
- Two unfocused copies of the domain $D = \{(\downarrow, 1/2), (\uparrow, 1/2)\}$ (uniform distribution). We shall label them D^+ and D^-.

D^+ and D^- are equal as sets, from an extensional point of view. The labels $+$ and – give an "intensional" characterization, to represent qubits in states $|+\rangle$ and $|–\rangle$.

Propositions from qubits

A qubit in state $\downarrow$ is represented by the proposition $(\forall x \in D_{\downarrow})A(x)$
A qubit in state $\uparrow$ is represented by the proposition $(\forall x \in D_{\uparrow})A(x)$.

Moreover:
A qubit in state $|+\rangle$ is represented by the prop. $(\forall x \in D^{+})A(x)$.
A qubit in state $|-\rangle$ is represented by the prop. $(\forall x \in D^{-})A(x)$.

So, for different qubits, we have two different lists of pairs of propositions.

We find out how such propositions can be characterized by introducing a different representation.

Definitory equations can be put in *symmetric* pairs, as follows:

$$\Gamma \vdash A \circ B \qquad \equiv \qquad \Gamma \vdash A \approx B$$

and

$$A \circ^s B \vdash \Delta \qquad \equiv \qquad A \approx B \vdash \Delta$$

so that logical connectives come out in symmetric pairs $(\circ, \circ^s)$, each pair corresponding to the same metalinguistic link $\approx$: $(\&, \vee)$, $(*, \otimes)$, ... $(\forall, \exists)$.

Symmetric equations are solved in a symmetric way, finding couples of rules "mirroring each other". So, one finds out symmetric sequent calculi (or couples of symmetric sequent calculi) and a *symmetry theorem*:

$$\Pi \text{ proves } \ \Gamma \vdash \Delta \qquad \text{iff} \qquad \Pi^s \text{ proves } \ \Delta^s \vdash \Gamma^s$$

where $p = p^s$ on literals and Π^s has the right/left rule for $\circ^s$ where Π has the left/right rule for $\circ$.

In logic, the symmetry theorem becomes real when it is applied considering a duality $(-)^{\perp}$, that means a negation:

$$\Gamma \vdash \Delta \qquad \text{iff} \qquad \Delta^{\perp} \vdash \Gamma^{\perp}$$

where $p^{\perp}$ is the negation of p for every literal p (Girard's duality) and everything else is as for symmetry. Symmetry acts as a real duality on connectives!

The duality theorem, but not the symmetry theorem, can be extended to admit contexts, as follows:

$$\Gamma', \Gamma \vdash \Delta, \Delta' \qquad \text{iff} \qquad \Gamma', \Delta^{\perp} \vdash \Gamma^{\perp}, \Delta'$$

This gives the duality the real meaning of negation.

Symmetry and sharp states

Formally, one could consider a symmetric representation for the state of a particle, via the existential quantifier $\exists$.
We consider the symmetric representation for a particle $\mathscr{A}$ in one of the states $|\downarrow\rangle, |\uparrow\rangle, |+\rangle, |-\rangle$.
The two sharp states $|\downarrow\rangle$ and $|\uparrow\rangle$ are associated to the propositions $(\exists x \in D_{\downarrow})A(x)$ and $(\exists x \in D_{\uparrow})A(x)$.
One easily proves that

$$(\forall x \in D_{\downarrow})A(x) = (\exists x \in D_{\downarrow})A(x)$$
$$(\forall x \in D_{\uparrow})A(x) = (\exists x \in D_{\uparrow})A(x)$$

Since $\forall = \exists$ on singletons.
Then we make a unique list of pairs, shorthanded $A_{\downarrow}, A_{\uparrow}$. As observed, they are in turn equal to propositional formulae. They are like pairs of propositional literals in a logical language (Girard literals) and we apply Girard duality to them:

$$A_{\downarrow}^{\perp} \equiv A_{\uparrow} \qquad A_{\uparrow}^{\perp} \equiv A_{\downarrow}$$

In our case, Girard's duality describes the action of the Pauli matrix σ_X, that is the *NOT* gate, on the sharp states.

$(\exists x \in\{\mathrm{u}\})A(x) = (\forall x \in\{\mathrm{u}\})A(x)$ holds for every singleton $\{u\}$. It is equivalent to the following sequent:

$$A(y), z = u \vdash A(z), y \neq u$$

that is derivable by equality rules. In turn, when $\{u\} = D_\downarrow$ or $\{u\} = D_\uparrow$, we rewrite it:

$$A(y), z \in D \vdash A(z), y \in D^\perp$$

where D is $D_\downarrow$ or $D_\uparrow$ and the duality $\perp$ switches $\downarrow$ and $\uparrow$.

Extending symmetry

We extend such a kind of sequents to unfocused domains, by the following axioms, termed *phase axioms*:

$$A(y), z \in D \vdash A(z), y \in D^{\top}$$

where D is D^+ or D^- and $\top$ is an operator switching $+$ and $-$: we term it *phase duality*.

Adopting phase axioms, one can prove the equalities:

$$(\forall x \in D^+)A(x) = (\exists x \in D^+)A(x)$$
$$(\forall x \in D^-)A(x) = (\exists x \in D^-)A(x)$$

So our logic can be equipped with a second list of pairs of literals, A^+, A^-, that we term *phase literals*, switched by phase duality:

$$A^{+\top} \equiv A^- \qquad A^{-\top} \equiv A^+$$

Note that they are not equivalent to propositional formulae, since the state is not sharp.
Phase duality describes the action of the Pauli matrix σ_Z, on the

Virtual singletons

We can extend the action of duality and phase duality, consistently with the action of the *NOT* and the σ_X gate, to all literals, putting:

$$A_{\downarrow}{}^{\top} \equiv A_{\downarrow} \quad A_{\uparrow}{}^{\top} \equiv A_{\uparrow} \qquad A^{+\perp} \equiv A^{+} \qquad A^{-\perp} \equiv A^{-}$$

One can see that:

- phase axioms are inconsistent on focused domains *of cardinality greater than 1*
- phase axioms characterize singletons *when substitution is allowed*, namely when measurement is considered.

Then, on one side, singletons are comparable to the intensional domains D^{+} and D^{-}: they are not splitted by a disjunction.
On the other, D^{+} and D^{-} are like singletons.

The analogy with the behaviour of singletons can be extended to a couple of qubits $\mathscr{A}$ and $\mathscr{B}$, described by propositions A and B. For couples of separated particles, we have seen that the state of the couple is described by the proposition

$$(\forall x \in D, w \in D')A(x) * B(y) = (\forall x \in D)A(x) * (\forall x \in D')B(x)$$

If and only if $D = D' = \{u\}$ one derives

$$(\forall x \in D)A(x) * B(x) = (\forall x \in D)A(x) * (\forall x \in D)B(x)$$

One can derive it via the equality rules.

An analogous derivation could derive the same equality for any domain D satisfying the phase axiom. In a “normal” logical setting, this gives inconsistency.

Bell's states

We consider a couple of particles in a Bell's state. To write down the outcomes of their measurements, we adopt the writing:

$$\Gamma, z \in D^{+} \vdash A(z),_i B(z) \qquad \Gamma, z \in D^{-} \vdash A(z),_i B(z)$$

$$\Gamma, z \in D^{+} \vdash A(z),_o B(z) \qquad \Gamma, z \in D^{-} \vdash A(z),_o B(z)$$

where the indexed commas $,_{i/o}$ describe the identical or opposite correlation between the outcomes in the measurements of the two particles.

By phase axioms, one can derive the following "parallel" rule for the quantifier:

$$\frac{\Gamma, z \in D^{\pm} \vdash A(z),_{i/0} \pm B(z)}{\Gamma \vdash (\forall x \in D^{\pm})A(x),_{i/o} (\forall x \in D^{\pm})B(x))}$$

Bell's states as virtual singletons

Translating the indexed commas into new connectives $\bowtie_i, \bowtie_o$, one can prove the analogous of the equality just seen for singletons:

$$(\forall x \in D^{\pm})A(x) \bowtie_{i\o} B(x) = (\forall x \in D^{\pm})A(x) \bowtie_{i\o} (\forall x \in D^{\pm})B(x)$$

This allows to define a generalized quantifier to represent the four Bell's states:

$$1/\sqrt{2}|00\rangle \pm 1/\sqrt{2}|11\rangle \qquad 1/\sqrt{2}|01\rangle \pm 1/\sqrt{2}|10\rangle$$

as four propositions, on the domains D^+ and D^-:

$$(\forall x \in D^{\pm})A(x) \bowtie_i B(x) \qquad (\forall x \in D^{\pm})A(x) \bowtie_o B(x)$$

We can extend phase duality to such propositions, switching the domains D^+ and D^-. Moreover, we extend Girard's duality to them by the identity, since it is the identity for the domains D^+ and D^-. Then the representation of Bell's states gives other phase literals.

Conclusions

Phase duality $\top$ is naturally induced by duality $\perp$ itself.

It is hidden by measurement.

The information contained in phase literals is independent of the orientation of the consequence relation $\vdash$. It is asserted and refuted at the same time.

Then it allows to process information as if it were asserted as well as refuted in the same process of derivation.

Thank you for your attention!