Instability paths in the Kirchhoff–Plateau problem

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Liquid films spanning rigid boundaries

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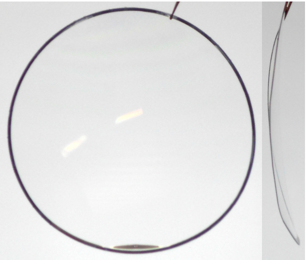
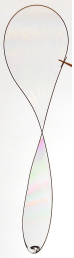
How can we control those shapes?
The only controllable entity is the shape of the rigid boundary.

Flexible frames





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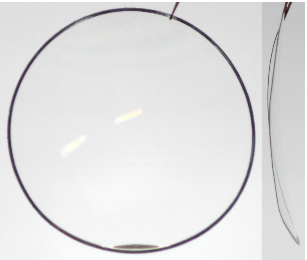

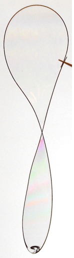

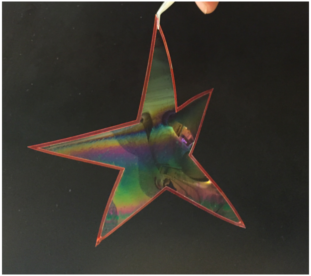

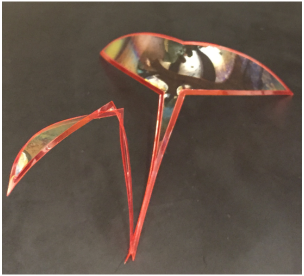

The Kirchhoff–Plateau problem

The Kirchhoff–Plateau problem concerns the **equilibrium shapes** of a system in which a flexible filament in the form of a closed loop is spanned by a soap film, with the filament being modeled as a **Kirchhoff rod** and the action of the spanning surface being solely due to **surface tension**.

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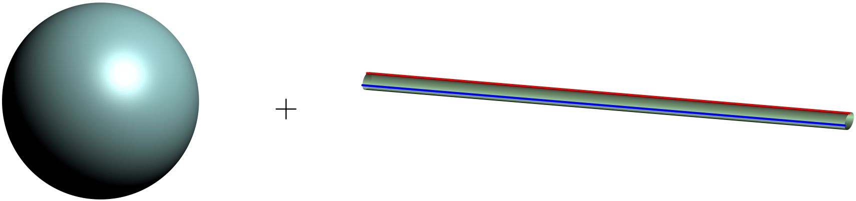
The energetic approach

We define an energy and look for configurations that minimize it.



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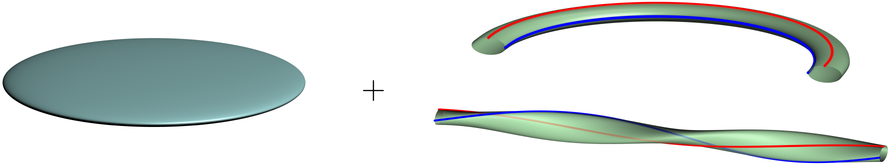


The rod midline

The configuration of a rod of (relaxed) length L is fixed once we have assigned, for any value of the parameter $s \in [0, L]$ the position $\boldsymbol{x}(s)$ of its midline and the orientation of the material cross section at s.
This can be done by assigning a Lipschitz continuous field $\boldsymbol{k} : [0, L] \to \mathbb{R}^3$ and solving the Cauchy problem

$$\begin{cases} \boldsymbol{x}(0) = \boldsymbol{x}_0\,, \\ \boldsymbol{x}'(0) = \boldsymbol{t}_0\,, \\ \boldsymbol{x}''(s) = \boldsymbol{k}(s)\,. \end{cases} \tag{1}$$

We have a total of **nine degrees of freedom**, but only **three of them describe the shape** of the midline: the components of the vector field $\boldsymbol{k}$, which determine the direction of the tangent field $\boldsymbol{x}'$ and its modulus.

The inextensibility constraint

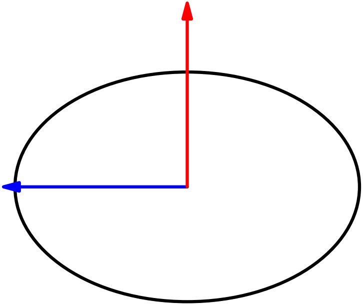
The inextensibility constraint can be enforced by the following modification of the previous equations, which ensures that $|\boldsymbol{x}'(s)| = 1$ for any s, and that the parameter s corresponds to arclength.

$$\begin{cases} \boldsymbol{x}(0) = \boldsymbol{x}_0\,, \\ \boldsymbol{x}'(0) = \boldsymbol{t}_0\,, \\ \boldsymbol{x}''(s) = \boldsymbol{k}(s) - [\boldsymbol{k}(s)\cdot\boldsymbol{x}'(s)]\boldsymbol{x}'(s) =: \boldsymbol{\kappa}(s)\,. \end{cases} \tag{2}$$

The field $\boldsymbol{\kappa}$ is the vector curvature field. Its two components in the plane orthogonal to $\boldsymbol{x}'$ are the **two degrees of freedom** that describe the shape of the midline in the inextensible case.

The material cross sections

We assume, for definiteness, that the rod has an elliptical cross section: for any $s \in [0, L]$ the ellipse is in the plane orthogonal to a director $\boldsymbol{d}_3(s)$, and its minor axis is aligned with another director $\boldsymbol{d}_1(s)$, so that the major axis is aligned with $\boldsymbol{d}_2 := \boldsymbol{d}_3 \times \boldsymbol{d}_1$. Moreover, the center of the ellipse is on the midline at $\boldsymbol{x}(s)$.



The material cross sections

Since the directors always form an orthonormal triple, we can completely describe the orientation of the material cross section at s by assigning a *Darboux vector field* $\boldsymbol{u} : [0, L] \to \mathbb{R}^3$ and solving the Cauchy problem

$$\begin{cases} \boldsymbol{d}_k(0) = \boldsymbol{e}_k\,, \\ \boldsymbol{d}_k'(s) = \boldsymbol{u}(s) \times \boldsymbol{d}_k(s)\,, \\ \text{for } k = 1, 2, 3\,. \end{cases} \tag{3}$$

We have **six degrees of freedom** to describe the material cross sections: the relative orientation at $s = 0$ with respect to the midline, and the components of the Darboux vector field.
It is usually assumed the impenetrability condition $\boldsymbol{d}_3 \cdot \boldsymbol{x}' > 0$.

The unshearability constraint

The unshearability constraint is a strong assumption requiring that the material cross section be always orthogonal to the midline:

$$\boldsymbol{d}_3 \cdot \boldsymbol{x}' = 1\,.$$

This implies that the two flexural components of the Darboux vector field $\boldsymbol{u}$ involved in the evolution of the material director $\boldsymbol{d}_3$ must coincide with the components of the vector curvature $\boldsymbol{\kappa}$ in the plane orthogonal to $\boldsymbol{x}'$.

Current shape and intrinsic shape

The current shape of an inextensible and unshearable rod is then characterized by two fields,

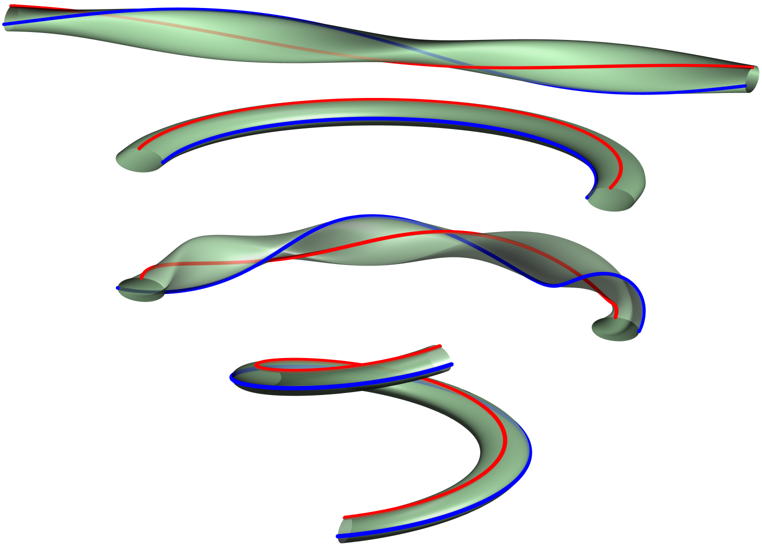
$$\boldsymbol{x}'' \cdot \boldsymbol{d}_1 =: \kappa_1 \qquad \text{and} \qquad \boldsymbol{x}'' \cdot \boldsymbol{d}_2 =: \kappa_2\,,$$

associated with the geometry of the midline, and a third one,

$$\boldsymbol{u} \cdot \boldsymbol{x}' = \boldsymbol{d}_1 \times \boldsymbol{d}_1' \cdot \boldsymbol{x}' =: \omega\,,$$

associated with the twist of the material cross sections.
An intrinsic shape of the rod can be described by assigning the three fields $\bar{\kappa}_1$, $\bar{\kappa}_2$, and $\bar{\omega}$, which define the intrinsic vector curvature $\bar{\boldsymbol{\kappa}}$ and the intrinsic Darboux field $\bar{\boldsymbol{u}}$.

Examples



A quadratic deformation energy for the rod

If we assume homogeneous and independent bending and twisting rigidities, a quadratic energy penalizing only modifications of the shape of the rod with respect to an intrinsic preferred shape takes the form

$$\int_0^L \frac{1}{2}\left[a_1(\kappa_2-\bar{\kappa}_2)^2+a_2(\kappa_1-\bar{\kappa}_1)^2+a_3(\omega-\bar{\omega})^2\right]\,ds\,,$$

where a_1 and a_2 are the flexural rigidities and a_3 is the torsional rigidity.

The energy of the entire system

The total energy of the system is given by the energy due to the surface tension of the soap film plus the elastic energy of the rod:

$$\int_{\mathcal{S}} \sigma \, dA + \int_0^L \frac{1}{2} \left[a_1(\kappa_2 - \bar{\kappa}_2)^2 + a_2(\kappa_1 - \bar{\kappa}_1)^2 + a_3(\omega - \bar{\omega})^2 \right] ds \, .$$

To express the equilibrium and stability condition it is convenient to introduce a global parametrization of the surface, whose restriction to the boundary will furnish a parametrization of the midline of the rod, to be completed with a parametrization of the director field $\boldsymbol{d}_1$ of the rod.

The simplest equilibria: disks with untwisted boundary

In our experiments we considered rods with the following properties:

- the total length of the midline is $2\pi R > 0$;
- there is no intrinsic torsional strain, i.e. $\bar{\omega} = 0$;
- the intrinsic curvature vector has constant modulus and it is parallel to the director $\boldsymbol{d}_2$, i.e. $\bar{\kappa}_1 = 0$ and $\bar{\kappa}_2 = 1/R_0$ for some $R_0 \geq R$;
- the flexural rigidities are anisotropic, i.e. $a_1 > a_2$.

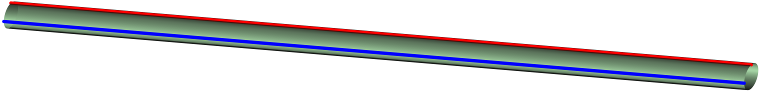
Moreover we look for solutions which are planar disks, whose bounding rod is characterized by the following properties:

$$\kappa_1^2 + \kappa_2^2 = \frac{1}{R^2}\,, \text{ with } \kappa_1 \text{ and } \kappa_2 \text{ constant, and } \omega = 0\,.$$

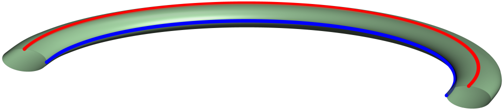
Intrinsic shapes with different curvatures

We define the curvature mismatch as $\zeta := 1 - \dfrac{R}{R_0}$

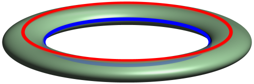
- $\zeta = 1$



- $\zeta = 0.5$

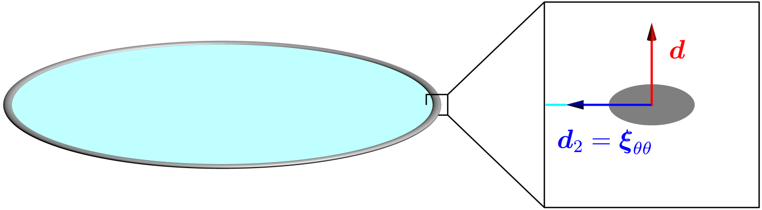


- $\zeta = 0$

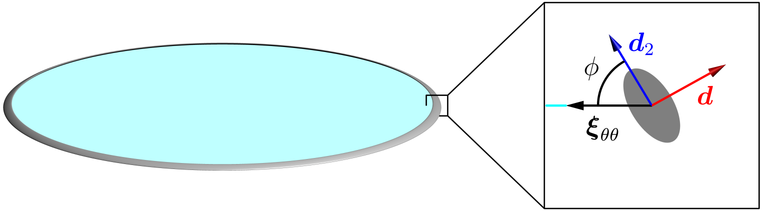


Base solution: flat circular disk

Regime of small curvature mismatch: $\zeta < \dfrac{a_2}{a_1}$.

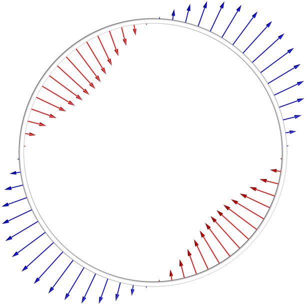


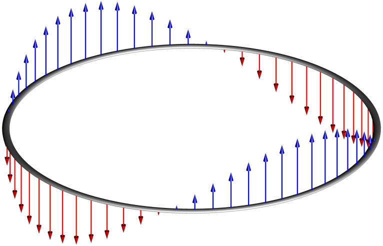
Regime of large curvature mismatch: $\zeta > \dfrac{a_2}{a_1}$.

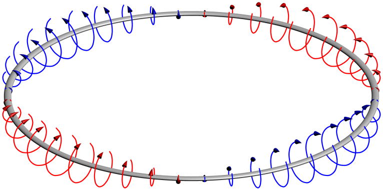


Basic perturbation modes

In-plane (left), out-of-plane (right), and twisting (bottom) perturbations.







Stability for small curvature mismatch

Dimensionless parameters:

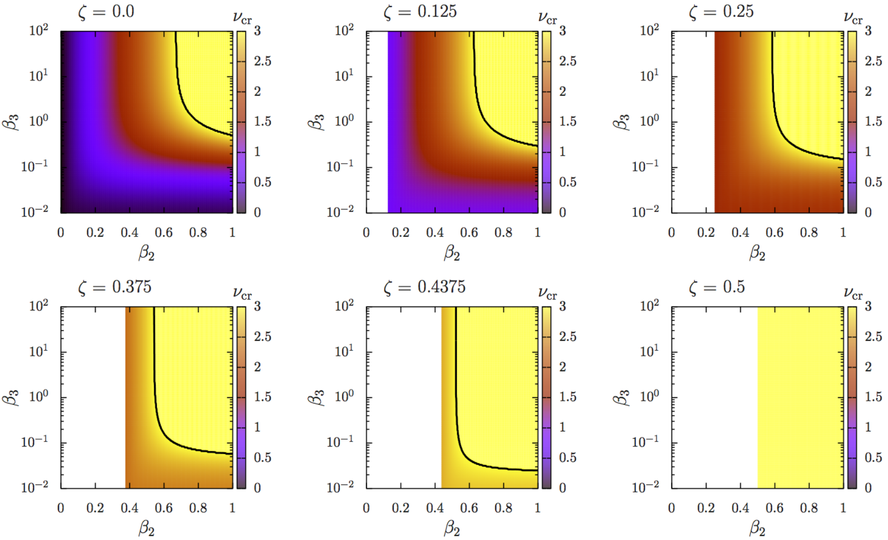
$$\nu := \frac{\sigma R^3}{a_1}\,, \qquad \beta_2 := \frac{a_2}{a_1}\,, \qquad \text{and} \qquad \beta_3 := \frac{a_3}{a_1}\,.$$

Necessary and sufficient stability conditions:

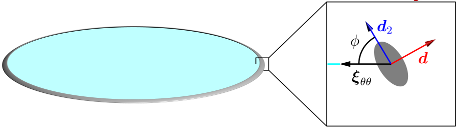
$$\begin{cases} \nu \leq 3\,, \\ \nu \leq 2\left(4\beta_2 - \zeta + \beta_3\right)\,, \\ \nu \leq \dfrac{6}{\beta_2 - \zeta + 4\beta_3}\left[\zeta(\beta_2 - \zeta + \beta_3) + 3\beta_2\beta_3\right]\,. \end{cases}$$

The circular cross-section case $\beta_2 = 1$ is always in this regime. But here the total actual twist is zero and there is no intrinsic twist.

Critical effective surface tension



Stability for large curvature mismatch



General case:

$$\nu^2\left[1+4\beta_3\frac{1-\beta_2}{(1-\beta_2)^2-(1-\zeta)^2}\right]$$
$$-\nu\left[9\beta_2+6\beta_3+12\beta_3(1+2\beta_2)\frac{1-\beta_2}{(1-\beta_2)^2-(1-\zeta)^2}\right]$$
$$+18\beta_2(\beta_2-\beta_3)+72\beta_2\beta_3\frac{1-\beta_2}{(1-\beta_2)^2-(1-\zeta)^2}\geq 0$$

In the limiting case $\zeta = 1$:

$$\nu \leq 3\beta_2 \qquad \Leftrightarrow \qquad \sigma R^3 \leq 3a_2$$

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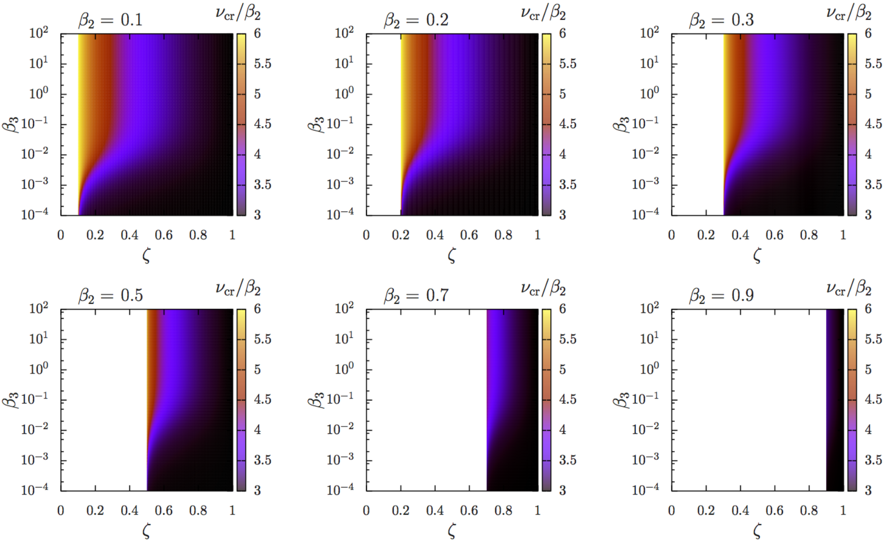
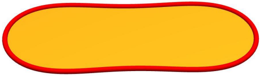


Work in progress

1. Consider different surface energies in connection with experiments
2. Retain the three-dimensional body of the rod, with a spanning film that touches the surface of the rod
3. Identify suitable numerical schemes to determine the nontrivial equilibrium shapes

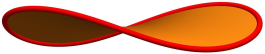
Some numerical results



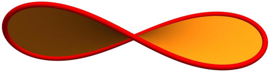
$\nu = 4.3$

$\nu = 4.415$

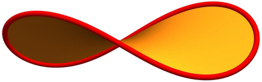
$\nu = 4.457$



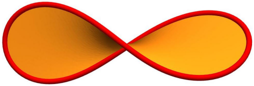
$\nu = 4.553$



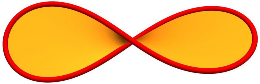
$\nu = 4.654$



$\nu = 4.711$



$\nu = 4.769$



$\nu = 4.885$

$\nu = 4.923$

Computational work by Abdul Majid