The shapes of a rod are traced in a Lie algebra

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GEMS, October 15, 2016

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Outline

1. Special Cosserat rods
   - What geometric structures describe their shape?
   - When are Lie algebraic objects essential?
2. Discrete rods
   - How can we discretize the rod shape?
   - When is such a discretization effective?
3. Framed curves
   - What is their relation with rods?
   - How can we represent their shape?

Shape: *A collection of properties invariant under rigid motions.*

A rigid-body motion . . .

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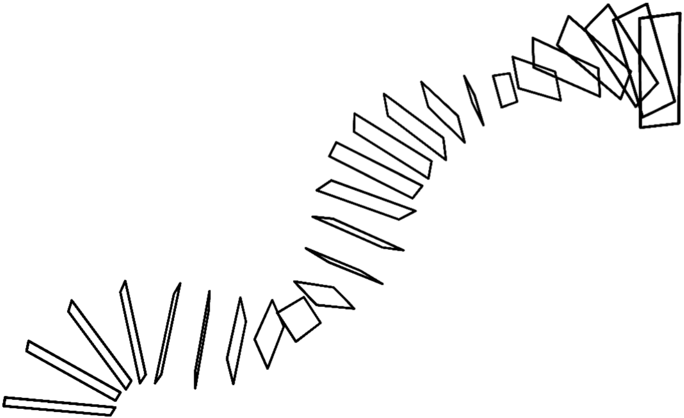
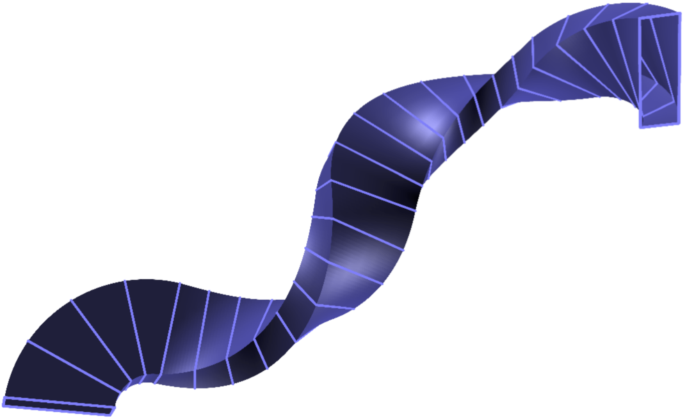


A rigid-body motion . . . or a rod?



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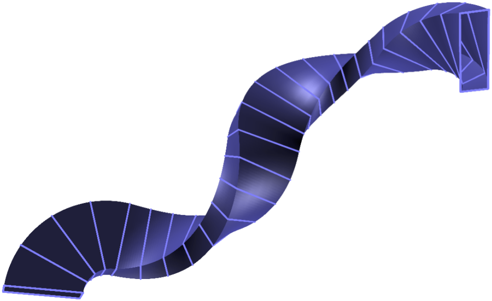

Shape: How it is traced, not where it goes.

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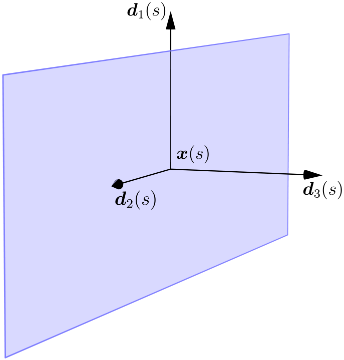
Also, although the motion of each cross-section of a rod is a rigid-body motion, a rod is deformable as a whole.

Turning the shape into a differential equation



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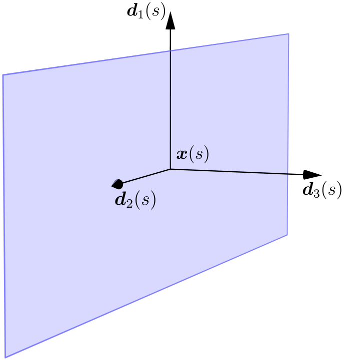

$$\begin{cases} \boldsymbol{x}'(s) = v_3(s)\boldsymbol{d}_3(s) + v_1(s)\boldsymbol{d}_1(s) + v_2(s)\boldsymbol{d}_2(s)\,, \\ \boldsymbol{d}_3'(s) = u_2(s)\boldsymbol{d}_1(s) - u_1(s)\boldsymbol{d}_2(s)\,, \\ \boldsymbol{d}_1'(s) = -u_2(s)\boldsymbol{d}_3(s) + u_3(s)\boldsymbol{d}_2(s)\,, \\ \boldsymbol{d}_2'(s) = u_1(s)\boldsymbol{d}_3(s) - u_3(s)\boldsymbol{d}_1(s)\,, \end{cases}$$

Turning the shape into a differential equation

If we now define the vector field $\mathcal{R} : [0, s_f] \to \mathbb{R}^{12}$ by $\mathcal{R} := (\boldsymbol{x}, \boldsymbol{d}_3, \boldsymbol{d}_1, \boldsymbol{d}_2)$ and the linear operator (O and I are 3×3 null and identity matrix)

$$\mathsf{L}(s) := \begin{pmatrix} \mathsf{O} & v_3(s)\mathsf{I} & v_1(s)\mathsf{I} & v_2(s)\mathsf{I} \\ \mathsf{O} & \mathsf{O} & u_2(s)\mathsf{I} & -u_1(s)\mathsf{I} \\ \mathsf{O} & -u_2(s)\mathsf{I} & \mathsf{O} & u_3(s)\mathsf{I} \\ \mathsf{O} & u_1(s)\mathsf{I} & -u_3(s)\mathsf{I} & \mathsf{O} \end{pmatrix},$$

it is possible to rewrite our equation as

$$\mathcal{R}' = \mathsf{L}\mathcal{R}\,.$$

Given the conditions $\mathcal{R}_0$ at $s = 0$ a unique solution exists and can be formally written as

$$\mathcal{R}(s) = \mathsf{U}(s; 0)\mathcal{R}_0\,,$$

where the operator $\mathsf{U}(s_1; s_0)$ represents the propagator of the solution from the point s_0 to s_1.

Rod: $\mathsf{U}\mathcal{R}_0$ is where it goes, L is how it is traced

The equation is encoded in L, which describes a possibly discontinuous path on the manifold of matrices generated by

$$V_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad V_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad V_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad U_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad U_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

The solution is encoded in $\mathcal{R}$, which represents a path in $\mathbb{R}^{12}$ starting at $\mathcal{R}_0$. The tracing of this path can be identified with the path described by the operators $\mathsf{U}(s;0)$, upon varying the parameter s, in their manifold.

Shape description and shape energy

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- The matrix manifold in which the operators $\mathsf{L}(s)$ live is a representation of the *special Euclidean algebra*.
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- The appearance of the filament in the ambient space is fully encoded in $\mathcal{R}$ (and in the shape of the cross-sections), and can be drawn by applying $\mathsf{U}(s;0)$ to the starting point $\mathcal{R}_0$.
- Any expression for the elastic energy of the filament that only depends on shape must depend on the components of L and not on $\mathcal{R}_0$ or any other derived quantity.

The stored elastic energy

Expressed in terms of Lie algebraic quantities:

$$\int_0^{s_f} \varphi\big(s, u_1(s), u_2(s), u_3(s), v_1(s), v_2(s), v_3(s)\big)\, ds.$$

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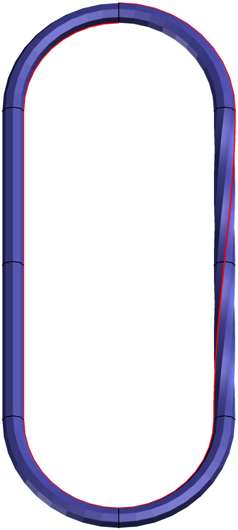
A special shape energy

For Kirchhoff rods, we assume unshearability and inextensibility:

$$\frac{1}{2}\int_0^{s_f} \big(a_1 u_1^2(s) + a_2 u_2^2(s) + a_3 u_3^2(s)\big)\, ds.$$

The constraints are exactly compatible with the discretization.

Shape relaxation of closed rods



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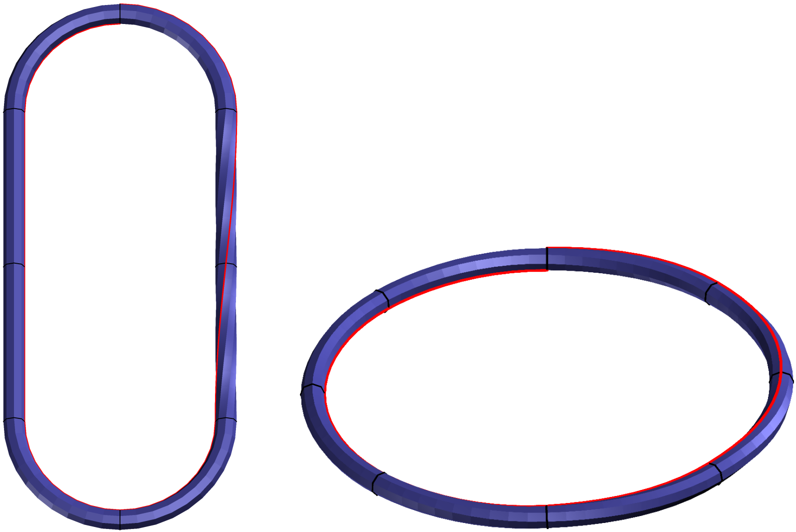
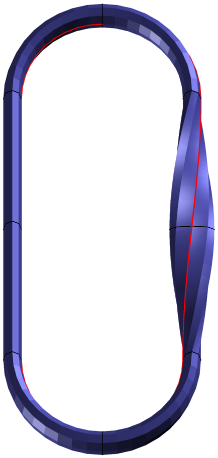


Shape relaxation with anisotropic bending stiffness



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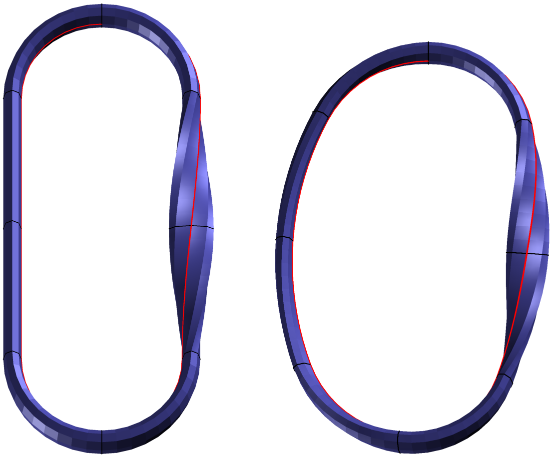

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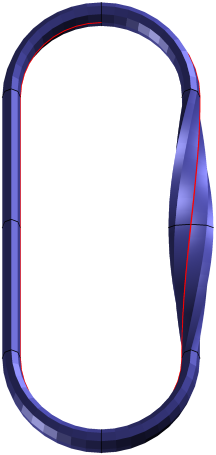
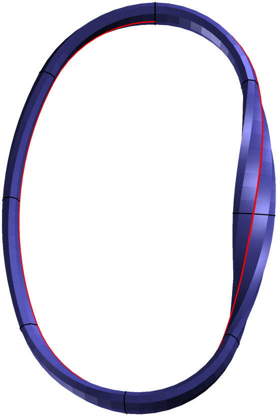
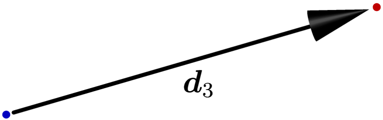




Degenerate cross-sections



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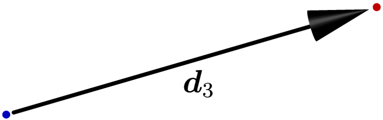

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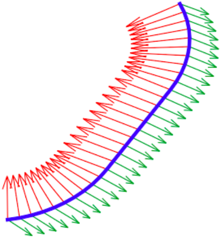
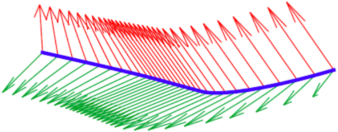
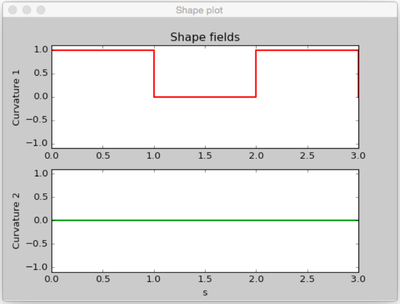
We obtain the curve and a *relatively parallel adapted frame* (Bishop).

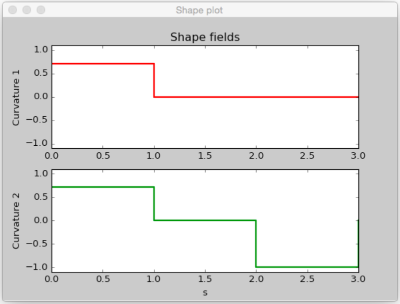
The relevant Lie algebra remains the same.

A simple comparison



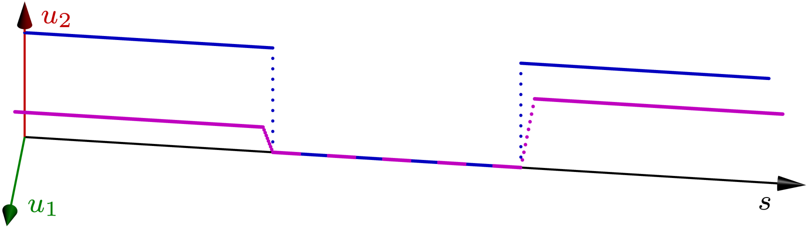






Geometric invariants

Two degrees of freedom: we can picture the shapes of framed curves by means of the *normal development*.



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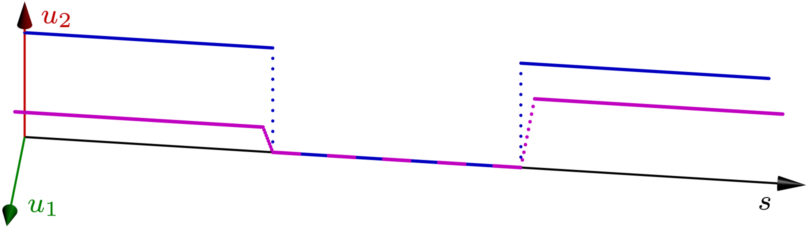

We introduce the Hasimoto transformation

$$\kappa(s)e^{i\theta(s)} = u_2(s) + iu_1(s)$$

and the geometric invariants are the *square-integrable* curvature κ and the *measure-valued* torsion $\tau = \theta'$.

Conclusions

1. Special Cosserat rods
    - The shape is a path in the special Euclidean algebra
    - The coordinate fields of this path determine the elastic energy
    - The shape energy should not depend on the torsion of the base curve
2. Discrete rods
    - Piecewise constant finite elements for the shape fields
    - Effective for shape relaxation with generic stiffness tensor
    - No interpolation needed to reconstruct the relevant information
3. Framed curves
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