Slender-body theory for viscous flow via dimensional reduction and hyperviscous regularization

Giulio G. Giusteri

Dipartimento di Matematica e Fisica “N. Tartaglia”
Università Cattolica del Sacro Cuore

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In collaboration with

Alfredo Marzocchi, Alessandro Musesti
Università Cattolica del Sacro Cuore (Brescia)

Eliot Fried
University of Washington (Seattle)
McGill University (Montréal)

Tae-Yeon Kim
McGill University (Montréal)

Creeping flow equation

In suspensions of microscopic solids of both technological and biological interest, flows are often characterized by very small Reynolds number, so that viscous effects are dominant.

Low-Reynolds-number linearization of the Navier–Stokes equation:

$$\operatorname{div} \mathbf{u} = 0\,,$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \mu \Delta \mathbf{u} + \rho \mathbf{b}\,,$$

with p the pressure field, $\mathbf{u}$ the velocity field, $\rho > 0$ the constant and homogeneous mass density, $\mu > 0$ the dynamic viscosity, and $\rho \mathbf{b}$ the volumetric force density.

Classical slender-body theory

The basic tool used to construct solutions to the **steady** Stokes problem is the Stokeslet, that is the Green's function for the Stokes operator in $\mathbb{R}^3$. Its classical expression is given by

$$p_s(\mathbf{x}) = \frac{\mathbf{h} \cdot \mathbf{x}}{4\pi|\mathbf{x}|^3} ,$$

$$\mathbf{s}(\mathbf{x}) = \mathsf{S}(\mathbf{x})\mathbf{h} ,$$

where $\mathbf{h} \in \mathbb{R}^3$ is any fixed vector, and S is the Oseen tensor, with components

$$S_{ij}(\mathbf{x}) := \frac{\delta_{ij}}{8\pi\mu|\mathbf{x}|} + \frac{x_i x_j}{8\pi\mu|\mathbf{x}|^3} ,$$

where δ_{ij} is Kronecker's symbol.

Classical slender-body theory

Solution by convolution:

$$\mathbf{u}(\mathbf{x}) := \rho \int_{\mathbb{R}^3} \mathsf{S}(\mathbf{x}-\mathbf{x}')\mathbf{b}(\mathbf{x}')\, d\mathcal{L}^3(\mathbf{x}')\,.$$

Classical ansatz for uniformly translating rigid rod:

$$\rho\mathbf{b}(\mathbf{x}) = \mathbf{f}(x_1)\chi_{[-a,a]}(x_1)\delta(x_2)\delta(x_3)\,.$$

But the rod is deformed during the motion . . .

Computational issues

Even though the force densities are concentrated on lines, the microscopic bodies are necessarily **three-dimensional**, because the Oseen tensor S is **divergent** at the origin.

Computational techniques has to face:

- moving boundaries;
- number of particles much greater than 1, in realistic situations;
- approximation of unsteady flows.

Moreover, non-linear inertial effects must be neglected.

Hyperviscosity and effective thickness

The hyperviscous Stokes equation, for a steady flow, is

$$\nabla p - \mu \Delta \mathbf{u} + \xi \Delta \Delta \mathbf{u} = \rho \mathbf{b} \,,$$

where the additional parameter $\xi > 0$ is called hyperviscosity.

We introduce an **effective thickness** $L > 0$, and set $\xi = \mu L^2$.

The dimensional reduction consists in the approximation of slender three-dimensional bodies with lower-dimensional entities, and L replaces the characteristic size of the body along the dimensions that are shrunken to zero in the slender-body limit.

Advantages

Analytical:

- Existence and uniqueness of regular solutions.
- The continuity of the velocity field permits to model adherence to lower-dimensional objects.
- The hyperviscous solution converges, in the limit $L \to 0$, to a suitable solution of the Navier-Stokes equation.

Computational:

- Simplification of the geometries.
- Moving boundaries replaced by a time-dependent constraint on function spaces.
- Reduction of instabilities related to singularities of the flow.

Resistance problems *vs* Mobility problems

In resistance problems, the velocities of the immersed objects are assigned and the goal is to determine the force required to sustain the motion.

In mobility problems, the forces acting on the system are prescribed and the objective is to determine the resulting velocity field. The solution can be found by convolution with the **regularized** Oseen tensor

$$Z_{ij}(\mathbf{x}) := \frac{\delta_{ij}}{8\pi\mu|\mathbf{x}|}\left[1 - 2e^{-\frac{|\mathbf{x}|}{L}} - \frac{2L}{|\mathbf{x}|}e^{-\frac{|\mathbf{x}|}{L}} - \frac{2L^2}{|\mathbf{x}|^2}\left(e^{-\frac{|\mathbf{x}|}{L}} - 1\right)\right] + \frac{x_i x_j}{8\pi\mu|\mathbf{x}|^3}\left[1 + 2e^{-\frac{|\mathbf{x}|}{L}} + \frac{6L}{|\mathbf{x}|}e^{-\frac{|\mathbf{x}|}{L}} + \frac{6L^2}{|\mathbf{x}|^2}\left(e^{-\frac{|\mathbf{x}|}{L}} - 1\right)\right].$$

But we must know the **reactive forces** within the body!

Flow past elastic bodies

The challenge: multiscale and multiphysics interaction.

- The fluid-structure interaction requires new analytical techniques, because it cannot be modeled using boundary conditions.
- With more than one immersed body, flows become unsteady.
- (In)compatibility between equations of different nature.
- Three-dimensional environment.

The goal: freely swimming filaments.

- Combination of resistance and mobility problems.
- Non-trivial elastic behavior of the slender bodies.

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Thank you