Paths in the special Euclidean algebra and rod shapes

Giulio G. Giusteri

Mathematical Soft Matter Unit (Prof. Fried)

NCTS, January 13, 2017

OIST

OKINAWA INSTITUTE OF SCIENCE AND TECHNOLOGY GRADUATE UNIVERSITY

Outline

1. Special Cosserat rods
   - What is the role of the special Euclidean algebra?
   - When are Lie algebraic objects essential?

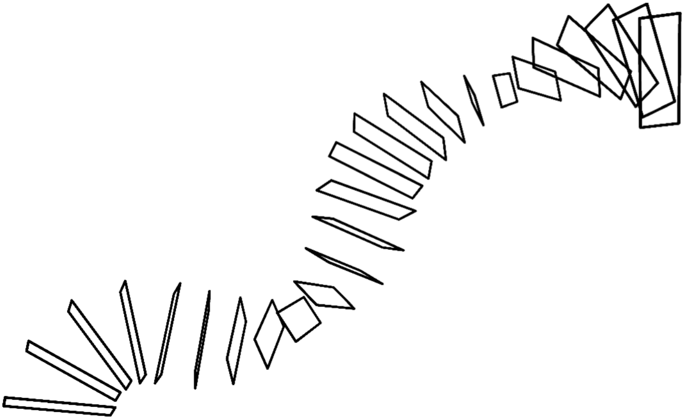
2. Framed curves
   - What is their relation with rods?
   - How can we represent their shape?

3. Discrete rods
   - How can we discretize the rod shape?
   - When is such a discretization effective?

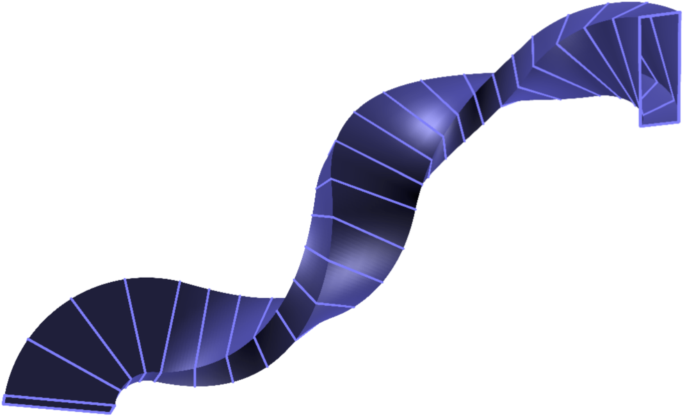
Shape: *A collection of properties invariant under rigid motions.*

A rigid-body motion . . .

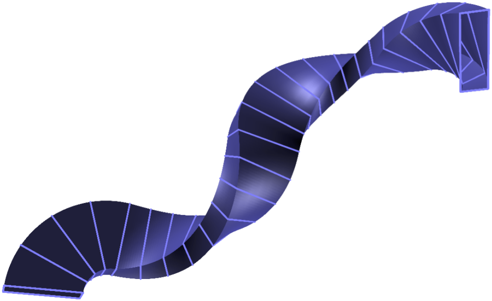
A rigid-body motion . . .



A rigid-body motion . . . or a rod?



A rigid-body motion . . . or a rod?



Shape: How it is traced, not where it goes.

A rigid-body motion . . . or a rod?

Common features:

- Rigid bodies are involved.
- Described by a family of special Euclidean transformations.

A rigid-body motion . . . or a rod?

Common features:

- Rigid bodies are involved.
- Described by a family of special Euclidean transformations.

Differences:

- The rigid-body motion is parametrized by time, the placement of cross-section by some s.
- When a single body moves, it can pass many times through the same region. Cross-sections cannot penetrate one another.

A rigid-body motion ... or a rod?

Common features:

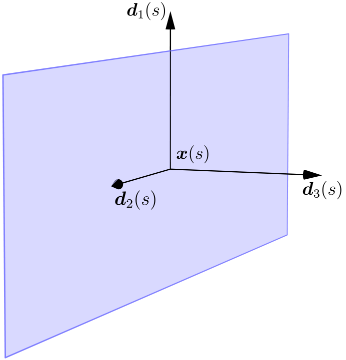
- Rigid bodies are involved.
- Described by a family of special Euclidean transformations.

Differences:

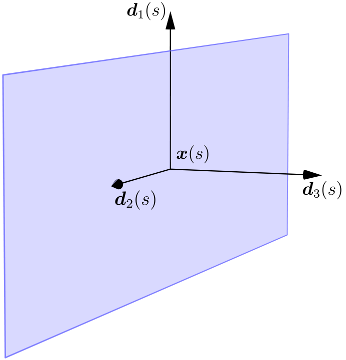
- The rigid-body motion is parametrized by time, the placement of cross-section by some s.
- When a single body moves, it can pass many times through the same region. Cross-sections cannot penetrate one another.

Also, although the motion of each cross-section of a rod is a rigid-body motion, a rod is deformable as a whole.

Turning the shape into a differential equation



Turning the shape into a differential equation



$$
\begin{cases}
\boldsymbol{x}'(s) = v_3(s)\boldsymbol{d}_3(s) + v_1(s)\boldsymbol{d}_1(s) + v_2(s)\boldsymbol{d}_2(s)\,, \\
\boldsymbol{d}_3'(s) = u_2(s)\boldsymbol{d}_1(s) - u_1(s)\boldsymbol{d}_2(s)\,, \\
\boldsymbol{d}_1'(s) = -u_2(s)\boldsymbol{d}_3(s) + u_3(s)\boldsymbol{d}_2(s)\,, \\
\boldsymbol{d}_2'(s) = u_1(s)\boldsymbol{d}_3(s) - u_3(s)\boldsymbol{d}_1(s)\,,
\end{cases}
$$

Turning the shape into a differential equation

If we now define the vector field $\mathcal{R} : [0, s_f] \to \mathbb{R}^{12}$ by $\mathcal{R} := (\boldsymbol{x}, \boldsymbol{d}_3, \boldsymbol{d}_1, \boldsymbol{d}_2)$ and the linear operator (O and I are 3×3 null and identity matrix)

$$\mathsf{L}(s) := \begin{pmatrix} \mathsf{O} & v_3(s)\mathsf{I} & v_1(s)\mathsf{I} & v_2(s)\mathsf{I} \\ \mathsf{O} & \mathsf{O} & u_2(s)\mathsf{I} & -u_1(s)\mathsf{I} \\ \mathsf{O} & -u_2(s)\mathsf{I} & \mathsf{O} & u_3(s)\mathsf{I} \\ \mathsf{O} & u_1(s)\mathsf{I} & -u_3(s)\mathsf{I} & \mathsf{O} \end{pmatrix} ,$$

it is possible to rewrite our equation as

$$\mathcal{R}' = \mathsf{L}\mathcal{R} .$$

Given the conditions $\mathcal{R}_0$ at $s = 0$ a unique solution exists and can be formally written as

$$\mathcal{R}(s) = \mathsf{U}(s; 0)\mathcal{R}_0 ,$$

where the operator $\mathsf{U}(s_1; s_0)$ represents the propagator of the solution from the point s_0 to s_1.

Rod: $\mathsf{U}\mathcal{R}_0$ is where it goes, L is how it is traced

The equation is encoded in L, which describes a possibly discontinuous path on the manifold of matrices generated by

$$V_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad V_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad V_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad U_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad U_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

The solution is encoded in $\mathcal{R}$, which represents a path in $\mathbb{R}^{12}$ starting at $\mathcal{R}_0$. The tracing of this path can be identified with the path described by the operators $\mathsf{U}(s;0)$, upon varying the parameter s, in their manifold.

Shape description and shape energy

- The matrix manifold in which the operators $\mathsf{L}(s)$ live is a representation of the *special Euclidean algebra*.

Shape description and shape energy

- The matrix manifold in which the operators $\mathsf{L}(s)$ live is a representation of the *special Euclidean algebra*.
- The shape of the filament is fully encoded in the path traced by L.

Shape description and shape energy

- The matrix manifold in which the operators $\mathsf{L}(s)$ live is a representation of the *special Euclidean algebra*.
- The shape of the filament is fully encoded in the path traced by L.
- The appearance of the filament in the ambient space is fully encoded in $\mathcal{R}$ (and in the shape of the cross-sections), and can be drawn by applying $\mathsf{U}(s;0)$ to the starting point $\mathcal{R}_0$.

Shape description and shape energy

- The matrix manifold in which the operators $\mathsf{L}(s)$ live is a representation of the *special Euclidean algebra*.
- The shape of the filament is fully encoded in the path traced by L.
- The appearance of the filament in the ambient space is fully encoded in $\mathcal{R}$ (and in the shape of the cross-sections), and can be drawn by applying $\mathsf{U}(s;0)$ to the starting point $\mathcal{R}_0$.
- Any expression for the elastic energy of the filament that only depends on shape must depend on the components of L and not on $\mathcal{R}_0$ or any other derived quantity.

The stored elastic energy

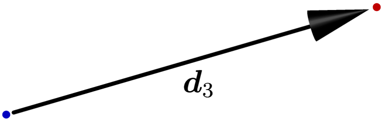
- Any expression for the elastic energy of the filament that only depends on shape must depend on the components of L and not on $\mathcal{R}_0$ or any other derived quantity.

Expressed in terms of Lie algebraic quantities:

$$\int_0^{s_f} \varphi\big(s, u_1(s), u_2(s), u_3(s), v_1(s), v_2(s), v_3(s)\big)\, ds.$$

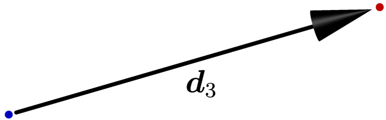
Quadratic case: L^2-norm $\rightarrow$ piecewise constant finite elements.

Degenerate cross-sections



Shearing and twisting loose their meaning: we set $v_1 = v_2 = 0$ and $u_3 = 0$.
Inextensibility can be imposed by setting $v_3 = 1$.

Degenerate cross-sections



Shearing and twisting loose their meaning: we set $v_1 = v_2 = 0$ and $u_3 = 0$. Inextensibility can be imposed by setting $v_3 = 1$.

$$\begin{cases} \boldsymbol{x}'(s) = \boldsymbol{d}_3(s)\,, \\ \boldsymbol{d}_3'(s) = u_2(s)\boldsymbol{d}_1(s) - u_1(s)\boldsymbol{d}_2(s)\,, \\ \boldsymbol{d}_1'(s) = -u_2(s)\boldsymbol{d}_3(s)\,, \\ \boldsymbol{d}_2'(s) = u_1(s)\boldsymbol{d}_3(s)\,, \end{cases}$$

We obtain the curve and a *relatively parallel adapted frame* (Bishop).

The relevant Lie algebra remains the same.

The inverse problem

$$\boldsymbol{t} = \boldsymbol{x}' \qquad \text{and} \qquad \boldsymbol{t}' = \boldsymbol{x}''\,.$$

If we now consider the integral equations

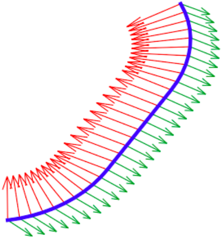
$$\boldsymbol{d}_1(s) = \boldsymbol{d}_1(0) - \int_0^s u_2(r)\boldsymbol{t}(r)\,dr\,,$$
$$\boldsymbol{d}_2(s) = \boldsymbol{d}_2(0) + \int_0^s u_1(r)\boldsymbol{t}(r)\,dr\,,$$

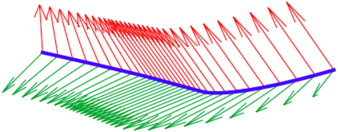
and take the scalar product with $\boldsymbol{t}' = u_2\boldsymbol{d}_1 - u_1\boldsymbol{d}_2$, we obtain

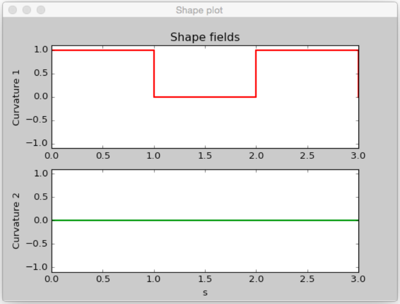
$$u_2(s) = \boldsymbol{d}_1(0)\cdot\boldsymbol{t}'(s) - \int_0^s u_2(r)\boldsymbol{t}(r)\cdot\boldsymbol{t}'(s)\,dr\,,$$
$$u_1(s) = \boldsymbol{d}_2(0)\cdot\boldsymbol{t}'(s) + \int_0^s u_1(r)\boldsymbol{t}(r)\cdot\boldsymbol{t}'(s)\,dr\,.$$

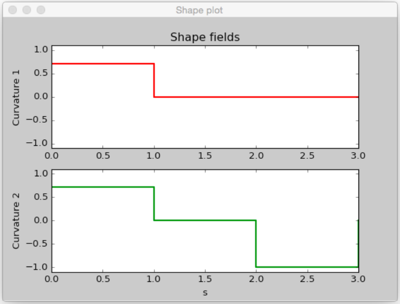
These are *Volterra equations of the second kind* and admit a unique solution on the interval $[0, s_f]$.

A simple comparison



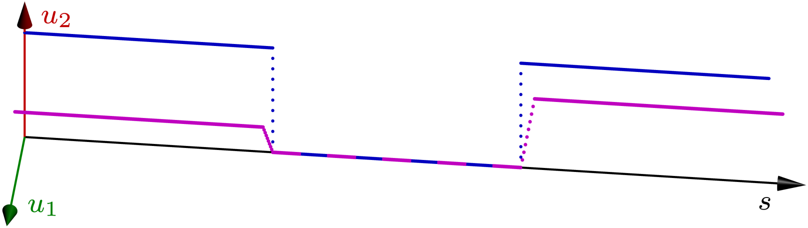






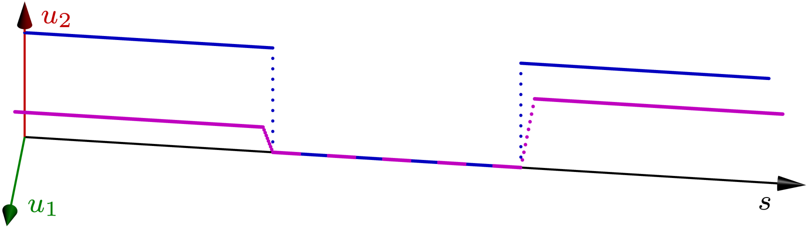
Geometric invariants

Two degrees of freedom: we can picture the shapes of framed curves by means of the *normal development*.



Geometric invariants

Two degrees of freedom: we can picture the shapes of framed curves by means of the *normal development*.



We introduce the Hasimoto transformation

$$\kappa(s)e^{i\theta(s)} = u_2(s) + iu_1(s)$$

and the geometric invariants are the *square-integrable* curvature κ and the *measure-valued* torsion $\tau = \theta'$.

Kirchhoff rods

For Kirchhoff rods, we assume unshearability and inextensibility:

$$v_1 = v_2 = 0 \qquad \text{and} \qquad v_3 = 1\,.$$

By substituting the identifications

$$\boldsymbol{d}_3 = \boldsymbol{t}\,, \qquad u_1 = -\kappa_2\,, \qquad u_2 = \kappa_1\,, \qquad \text{and} \qquad u_3 = \omega$$

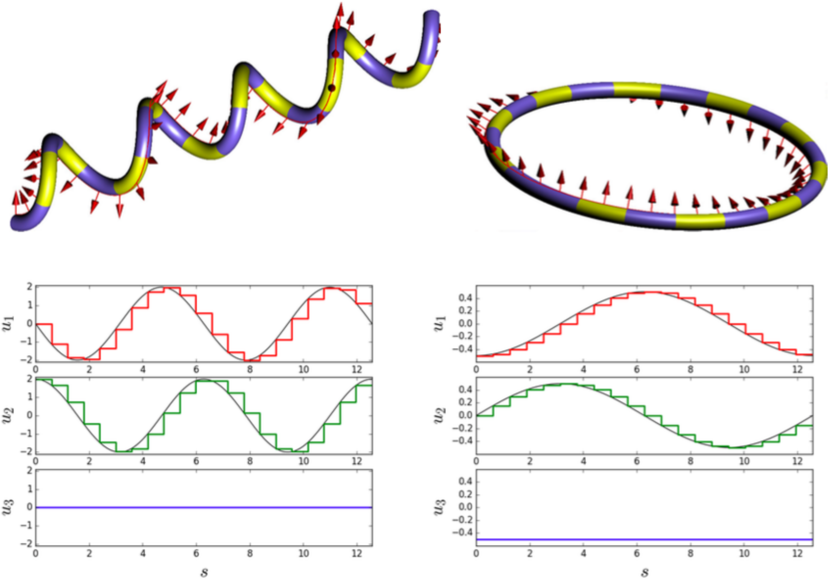
we find

$$\begin{cases} \boldsymbol{x}'(s) = \boldsymbol{t}(s)\,, \\ \boldsymbol{t}'(s) = \kappa_1(s)\boldsymbol{d}_1(s) + \kappa_2(s)\boldsymbol{d}_2(s)\,, \\ \boldsymbol{d}_1'(s) = -\kappa_1(s)\boldsymbol{t}(s) + \omega(s)\boldsymbol{d}_2(s)\,, \\ \boldsymbol{d}_2'(s) = -\kappa_2(s)\boldsymbol{t}(s) - \omega(s)\boldsymbol{d}_1(s)\,. \end{cases}$$

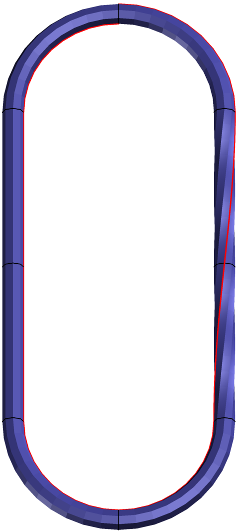
A quadratic shape energy is

$$\frac{1}{2}\int_0^{s_f} \left(a_1\kappa_2^2(s) + a_2\kappa_1^2(s) + a_3\omega^2(s)\right) ds.$$

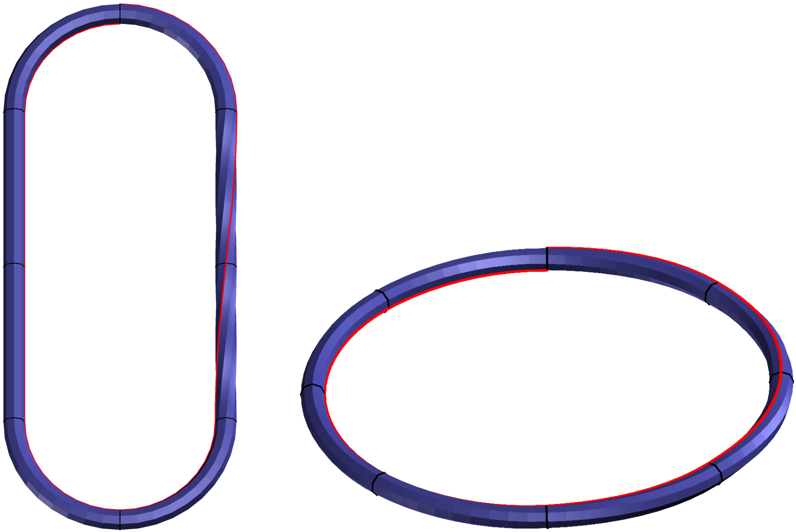
Twist is not Torsion



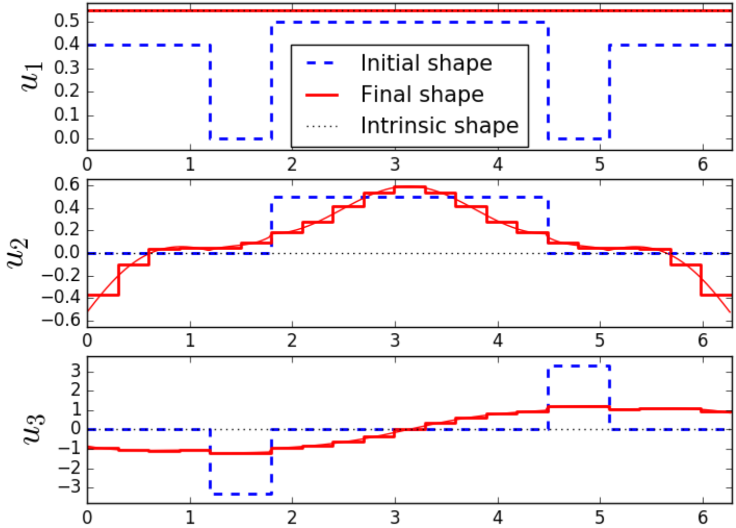
Shape relaxation of closed rods



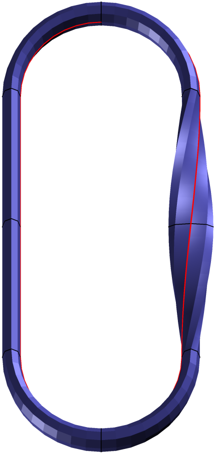
Shape relaxation of closed rods



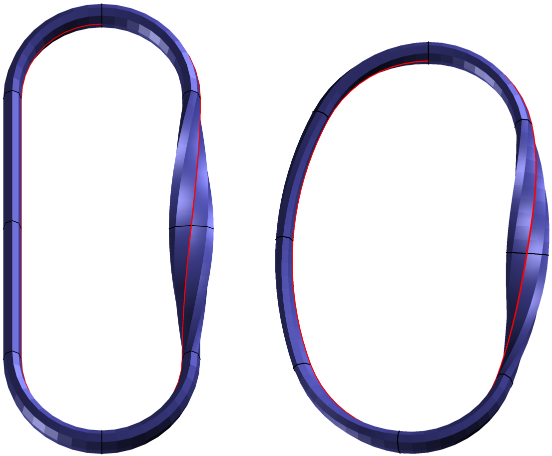
Shape relaxation with anisotropic bending stiffness



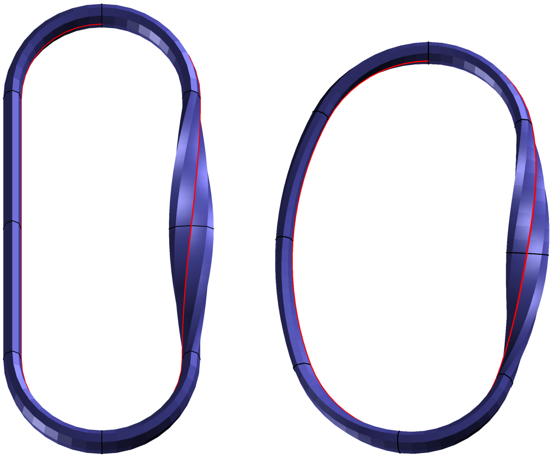
Shape relaxation with anisotropic bending stiffness



Shape relaxation with anisotropic bending stiffness



Shape relaxation with anisotropic bending stiffness



Conclusions

1. Special Cosserat rods
   - The shape is a path in the special Euclidean algebra
   - The coordinate fields of this path determine the elastic energy
   - The shape energy should not depend on the torsion of the base curve
2. Framed curves
   - Degenerate rods with point-like cross-sections
   - Relatively parallel adapted frames are the best choice
   - Shape described by an equivalence class of paths
   - The definition of geometric invariants does not require smoothness
3. Discrete rods
   - Piecewise constant finite elements for the shape fields
   - Effective for shape relaxation with generic stiffness tensor
   - No interpolation needed to reconstruct the relevant information

Conclusions

1. Special Cosserat rods
   - The shape is a path in the special Euclidean algebra
   - The coordinate fields of this path determine the elastic energy
   - The shape energy should not depend on the torsion of the base curve
2. Framed curves
   - Degenerate rods with point-like cross-sections
   - Relatively parallel adapted frames are the best choice
   - Shape described by an equivalence class of paths
   - The definition of geometric invariants does not require smoothness
3. Discrete rods
   - Piecewise constant finite elements for the shape fields
   - Effective for shape relaxation with generic stiffness tensor
   - No interpolation needed to reconstruct the relevant information

Thank you!