# Modeling the sedimentation of filaments in viscous fluids via dimensional reduction and hyperviscous regularization

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# Outline

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3. Fluid–structure interaction in presence of dimensional gap.
    - *Hydrodynamic force and torque on a filament.*
    - *Resistance tensors for low-Reynolds-number motion.*

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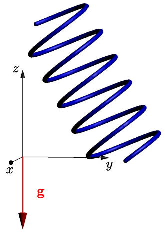
5. Summary and perspectives.

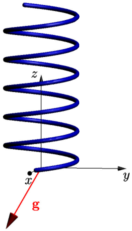
# Space and frames for the free fall

We consider a **rigid body** dropped from rest in an otherwise quiescent fluid filling all of space.
**Gravitational forces** give rise to a motion which is then influenced also by **hydrodynamic interactions**.
Such forces conspire to produce an asymptotic motion, which can be represented by a **steady** velocity field in a suitable reference frame.

**Inertial frame**




**Co-moving frame**




The gravitational acceleration vector **g** may not be constant in the co-moving frame during the transient motion, but it becomes asymptotically constant.

# Mathematical formulation

We introduce the **disturbance field u**: it is the difference between the actual flow and the flow which would take place in absence of the body, both observed in the co-moving frame.

$$\operatorname{div}\mathbf{u} = 0\,, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + Re\,\{[(\mathbf{u}-\mathbf{U})\cdot\nabla]\mathbf{u} + \boldsymbol{\omega}\times\mathbf{u}\} = \operatorname{div}\mathsf{T}(\mathbf{u},p) + \mathbf{g}\,, \tag{2}$$

$$\lim_{|\mathbf{x}|\to\infty}\mathbf{u}(\mathbf{x},t) = \mathbf{0}\,, \tag{3}$$

$$\mathbf{u}(\mathbf{x},t) = \mathbf{U}(\mathbf{x},t) = \boldsymbol{\xi}(t) + \boldsymbol{\omega}(t)\times\mathbf{x} \quad \text{on } \Sigma\times[0,+\infty)\,, \tag{4}$$

$$m\frac{d\boldsymbol{\xi}}{dt} + Re(m\boldsymbol{\omega}\times\boldsymbol{\xi}) = m_e\mathbf{g} + \mathbf{f}\,, \tag{5}$$

$$\mathsf{J}\frac{d\boldsymbol{\omega}}{dt} + Re[\boldsymbol{\omega}\times(\mathsf{J}\boldsymbol{\omega})] = -m_c(\mathbf{r}\times\mathbf{g}) + \mathbf{t}\,, \tag{6}$$

$$\frac{d\mathbf{g}}{dt} = Re(\mathbf{g}\times\boldsymbol{\omega})\,, \tag{7}$$

# Steady low-Reynolds-number flow

Every quantity is dimensionless and the Reynolds number is defined as $Re = \rho^2 g d^3/\mu^2$, thus measuring the relative importance of the gravitational and viscous forces. With this normalization $|\mathbf{g}| = 1$.

$$\operatorname{div} \mathbf{u} = 0\,, \tag{8}$$

$$\operatorname{div} \mathsf{T}(\mathbf{u}, p) + \mathbf{g} = 0\,, \tag{9}$$

$$\lim_{|\mathbf{x}|\to\infty} \mathbf{u}(\mathbf{x}) = \mathbf{0}\,, \tag{10}$$

$$\mathbf{u}(\mathbf{x}) = \boldsymbol{\xi} + \boldsymbol{\omega} \times \mathbf{x} \quad \text{on } \Sigma\,, \tag{11}$$

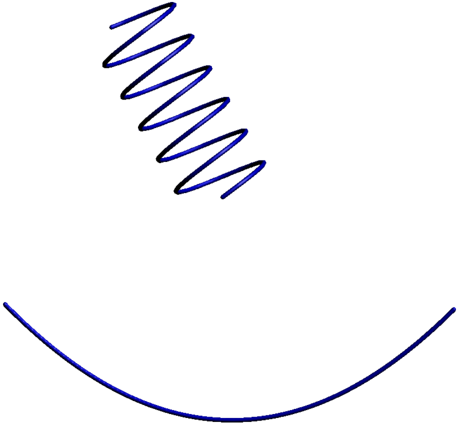
$$m_e \mathbf{g} = -\mathbf{f}\,, \tag{12}$$

$$m_c(\mathbf{r} \times \mathbf{g}) = \mathbf{t}\,, \tag{13}$$

$$\mathbf{g} \times \boldsymbol{\omega} = 0\,, \tag{14}$$

Note that only the last equation is non-linear.

# Slender bodies and tubular neighborhoods



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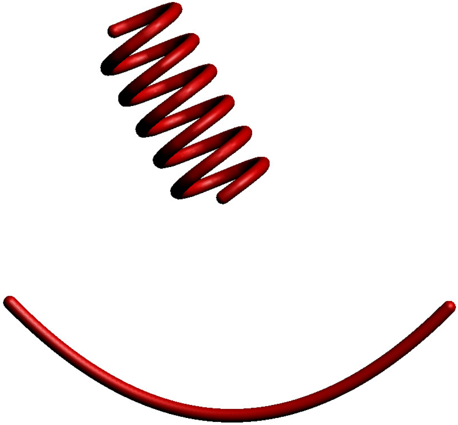

# Second-gradient dissipation functional

The classical Stokes equations for the steady low-Reynolds-number flow are related to a first-gradient dissipation functional:

$$\mathcal{D}_1 := \mu \int |\mathrm{Sym}\nabla \mathbf{v}|^2 \,.$$

We introduce a second-gradient dissipation functional:

$$\mathcal{D}_2 := \mu \int \left( |\mathrm{Sym}\nabla \mathbf{u}|^2 + \frac{L^2}{2} |\Delta \mathbf{u}|^2 \right) \,,$$

associated with the **effective stress tensor**

$$\mathsf{T}(\mathbf{u}, p) := -p\mathsf{I} + \mu \left( \nabla \mathbf{u} + \nabla \mathbf{u}^{\mathsf{T}} - L^2 \nabla \Delta \mathbf{u} \right) \,.$$

**Why?**

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**Why?**

Because 1- or 0-dimensional sets have vanishing $H^1$-capacity, that is, if we can only control the norm of the first gradient of the velocity, they are mechanically invisible, just as ghosts within the fluid.

# Hydrodynamic force and torque

Introducing the $r$-neighborhood of the filament $\Sigma$

$$V_r(\Sigma) := \left\{ \mathbf{x} \in \mathbb{R}^3 : d(x, \Sigma) \leq r \right\} ,$$

we can define the hydrodynamic force on $\Sigma$ as

$$\mathbf{f}(\mathbf{u}, p) := \lim_{r \to 0} \int_{\partial V_r(\Sigma)} \mathrm{T}(\mathbf{u}, p)\mathbf{n} \,,$$

and the hydrodynamic torque as

$$\mathbf{t}(\mathbf{u}, p) := \lim_{r \to 0} \int_{\partial V_r(\Sigma)} \mathbf{x} \times \mathrm{T}(\mathbf{u}, p)\mathbf{n} \,.$$

Those quantities are well defined, and would simply vanish for a Newtonian fluid.

# Auxiliary problems

Consider now the solutions $(\mathbf{h}^{(i)}, p^{(i)})$ and $(\mathbf{H}^{(i)}, P^{(i)})$ $(i = 1, 2, 3)$ of

$$\begin{cases} \operatorname{div} \mathbf{h}^{(i)} = 0 & \text{in } \mathbb{R}^3\,, \\ \operatorname{div} \mathbf{T}(\mathbf{h}^{(i)}, p^{(i)}) = 0 & \text{in } \mathbb{R}^3\,, \\ \mathbf{h}^{(i)} = \mathbf{e}_i & \text{on } \Sigma\,, \end{cases}$$

and

$$\begin{cases} \operatorname{div} \mathbf{H}^{(i)} = 0 & \text{in } \mathbb{R}^3\,, \\ \operatorname{div} \mathbf{T}(\mathbf{H}^{(i)}, P^{(i)}) = 0 & \text{in } \mathbb{R}^3\,, \\ \mathbf{H}^{(i)} = \mathbf{e}_i \times \mathbf{x} & \text{on } \Sigma\,. \end{cases}$$

By linearity, the combinations

$$\mathbf{u} = \sum_{i=1}^{3} [\xi_i \mathbf{h}^{(i)} + \omega_i \mathbf{H}^{(i)}] \quad , \qquad p = \sum_{i=1}^{3} [\xi_i p^{(i)} + \omega_i P^{(i)}] + \mathbf{g} \cdot \mathbf{x}\,, \qquad (15)$$

for suitable vectors $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$, solve the steady free fall problem.

# Resistance tensors

$$\mathsf{K}_{ji} := -\lim_{r\to 0}\int_{\partial V_r(\Sigma)} \mathsf{T}(\mathbf{h}^{(i)}, p^{(i)})\mathbf{n}\cdot\mathbf{e}_j\,,$$

$$\mathsf{S}_{ji} := -\lim_{r\to 0}\int_{\partial V_r(\Sigma)} \mathsf{T}(\mathbf{H}^{(j)}, P^{(j)})\mathbf{n}\cdot\mathbf{e}_i\,,$$

$$\mathsf{C}_{ji} := -\lim_{r\to 0}\int_{\partial V_r(\Sigma)} \mathbf{x}\times\mathsf{T}(\mathbf{h}^{(j)}, p^{(j)})\mathbf{n}\cdot\mathbf{e}_i\,,$$

$$\mathsf{B}_{ji} := -\lim_{r\to 0}\int_{\partial V_r(\Sigma)} \mathbf{x}\times\mathsf{T}(\mathbf{H}^{(i)}, P^{(i)})\mathbf{n}\cdot\mathbf{e}_j\,.$$

The matrices K, B, and

$$\mathsf{A} := \begin{pmatrix} \mathsf{K} & \mathsf{S} \\ \mathsf{C} & \mathsf{B} \end{pmatrix}$$

are **symmetric and positive definite**, thanks to a generalization of the Reciprocal Theorem.

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# Reduction to an algebraic problem

The geometric constraint $\mathbf{g} \times \boldsymbol{\omega} = 0$ implies that $\boldsymbol{\omega} = \lambda \mathbf{g}$ for some $\lambda \in \mathbb{R}$, and the rigid motion equations reduce to the following algebraic system in the unknowns $\boldsymbol{\xi}$, $\lambda$, and $\mathbf{g}$ (recall that $|\mathbf{g}| = 1$):

$$\begin{aligned} (-\mathbf{f} =) \qquad & \mathsf{K}\boldsymbol{\xi} + \lambda \mathsf{S}\mathbf{g} = m_e \mathbf{g}\,, \\ (-\mathbf{t} =) \qquad & \mathsf{C}\boldsymbol{\xi} + \lambda \mathsf{B}\mathbf{g} = -m_c(\mathbf{r} \times \mathbf{g})\,. \end{aligned} \tag{16}$$

This **non-linear** algebraic problem admits **at least a solution**, thanks to the properties of the resistance tensors.

# Symmetry under reflection

## Theorem

Assume that $\Sigma$ has a plane of material symmetry. Then there exists an orientation of the body, such that $\mathbf{g}$ lies in the same plane of symmetry, which gives rise to a purely translational solution.

## Corollary

If the body has two orthogonal planes of symmetry, then the free fall with $\mathbf{g}$ lying along the intersection of such planes gives rise to a purely translational motion.

# Symmetry under rotations

We say that a one-dimensional body $\Sigma$ is **helicoidally symmetric** if there exists a co-moving frame such that it is invariant under a discrete group of co-axial rotations of order strictly grater than 2.

## Theorem

If the body has helicoidal and fore-aft symmetry, then, for any given orientation, the body falls with a purely translational velocity given by

$$\boldsymbol{\xi} = m_e \mathsf{K}^{-1}\mathbf{g}\,.$$

# Summary and perspectives

1. If we want to use 1-dimensional filaments to model the effect of nanoparticles in viscous fluids we need a second-gradient dissipation.
2. Also the existence of solutions for the non-linear steady free fall has been obtained.
3. An important open issue is the numerical simulation of such problems with dimensional gap.
4. The free fall of elastic filaments is under investigation, but it involves a far-from-obvious extension of the mathematical techniques.

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**Thank you**