# Solution of the Kirchhoff–Plateau problem

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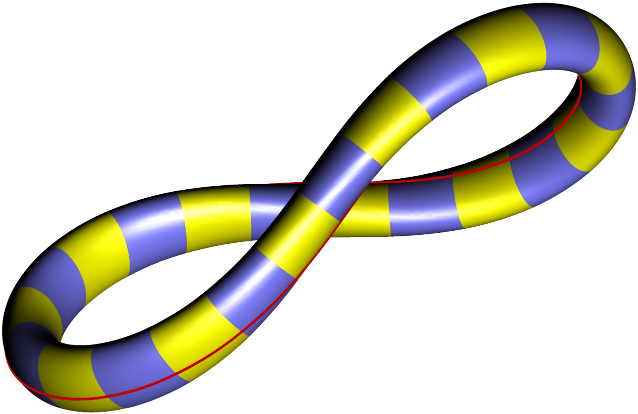
# Outline

1. Special Cosserat rods
2. Framed curves
3. The Kirchhoff–Plateau problem
4. Open issue: viscoelastic dynamics with topology changes

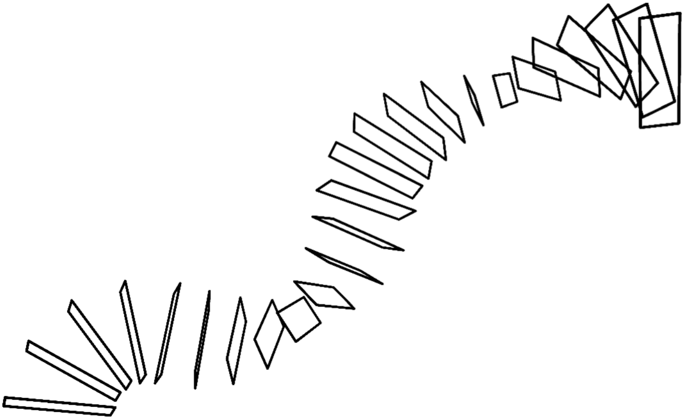
# References


- Schuricht, F.: *Global injectivity and topological constraints for spatial nonlinearly elastic rods*. J. Nonlinear Sci. **12**, 423–444 (2002)
- De Lellis, C., Ghiraldin, F., Maggi, F.: *A direct approach to Plateau's problem*. J. Eur. Math. Soc. (JEMS) **19**, 2219–2240 (2017)
- Giusteri, G. G., Lussardi, L., Fried, E.: *Solution of the Kirchhoff-Plateau problem*. J. Nonlinear Sci. **27**, 1043–1063 (2017)
- Giusteri, G. G., Fried, E.: *Importance and effectiveness of representing the shapes of Cosserat rods and framed curves as paths in the special Euclidean algebra*. J. Elast. **132**, 43–65 (2018)

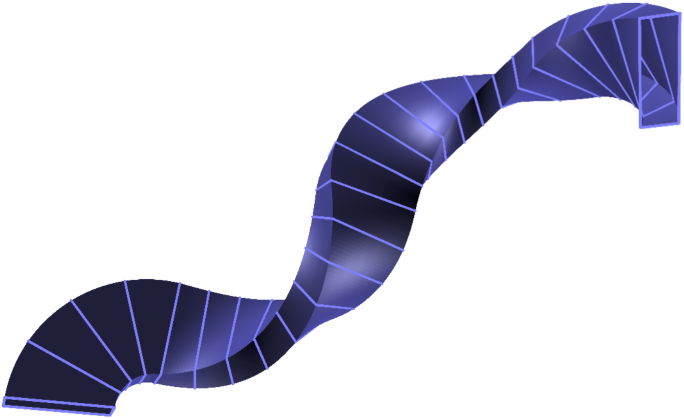
# Shape: properties invariant under rigid motions



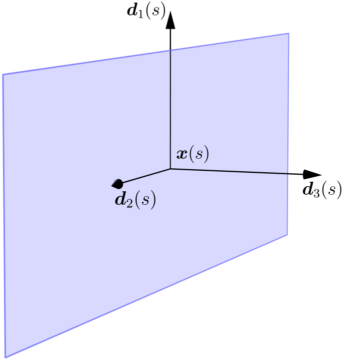
# A rigid-body motion . . .



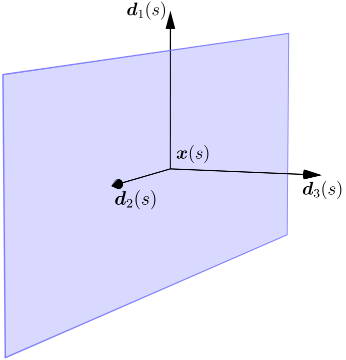
# A rigid-body motion . . . or a rod?



# Turning the shape into a differential equation



# Turning the shape into a differential equation




$$\begin{cases} \boldsymbol{x}'(s) = v_3(s)\boldsymbol{d}_3(s) + v_1(s)\boldsymbol{d}_1(s) + v_2(s)\boldsymbol{d}_2(s)\,, \\ \boldsymbol{d}_3'(s) = u_2(s)\boldsymbol{d}_1(s) - u_1(s)\boldsymbol{d}_2(s)\,, \\ \boldsymbol{d}_1'(s) = -u_2(s)\boldsymbol{d}_3(s) + u_3(s)\boldsymbol{d}_2(s)\,, \\ \boldsymbol{d}_2'(s) = u_1(s)\boldsymbol{d}_3(s) - u_3(s)\boldsymbol{d}_1(s)\,, \end{cases}$$

# Turning the shape into a differential equation

If we now define the field $\mathcal{R} : [0, s_f] \to \mathbb{R}^{4\times 3}$ by $\mathcal{R} := (\boldsymbol{x}, \boldsymbol{d}_3, \boldsymbol{d}_1, \boldsymbol{d}_2)$ and the linear operator

$$\mathsf{L}(s) := \begin{pmatrix} 0 & v_3(s) & v_1(s) & v_2(s) \\ 0 & 0 & u_2(s) & -u_1(s) \\ 0 & -u_2(s) & 0 & u_3(s) \\ 0 & u_1(s) & -u_3(s) & 0 \end{pmatrix},$$

it is possible to rewrite our equation as

$$\mathcal{R}' = \mathsf{L}\mathcal{R}\,.$$

Given the conditions $\mathcal{R}_0$ at $s = 0$ a unique solution exists and can be formally written as

$$\mathcal{R}(s) = \mathsf{U}(s; 0)\mathcal{R}_0\,,$$

where the operator $\mathsf{U}(s_1; s_0)$ represents the propagator of the solution from the point $s_0$ to $s_1$.

# Rod: $\mathrm{U}\mathcal{R}_0$ is where it goes, L is how it is traced

The equation is encoded in L, which describes a possibly discontinuous path on the manifold of matrices generated by

$$V_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad V_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad V_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad U_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad U_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

The solution is encoded in $\mathcal{R}$, which represents a path in $\mathbb{R}^{12}$ starting at $\mathcal{R}_0$. The tracing of this path can be identified with the path described by the operators $\mathrm{U}(s;0)$, upon varying the parameter $s$, in their manifold.

# Shape description and shape energy

- The matrix manifold in which the operators $\mathsf{L}(s)$ live is a representation of the *special Euclidean algebra*.
- The shape of the filament is fully encoded in the path traced by $\mathsf{L}$.
- The appearance of the filament in the ambient space is fully encoded in $\mathcal{R}$ (and in the shape of the cross-sections), and can be drawn by applying $\mathsf{U}(s;0)$ to the starting point $\mathcal{R}_0$.
- Any expression for the elastic energy of the filament that only depends on shape must depend on the components of $\mathsf{L}$ and not on $\mathcal{R}_0$ or any other derived quantity.

# The stored elastic energy

- Any expression for the elastic energy of the filament that only depends on shape must depend on the components of L and not on $\mathcal{R}_0$ or any other derived quantity.

Expressed in terms of Lie algebraic quantities:

$$\int_0^{s_f} \varphi\big(s, u_1(s), u_2(s), u_3(s), v_1(s), v_2(s), v_3(s)\big)\, ds.$$

Quadratic case: $L^2$-norm $\rightarrow$ linear elastic response with geometric nonlinearities.

# Kirchhoff rods

For Kirchhoff rods, we assume unshearability and inextensibility:

$$v_1 = v_2 = 0 \qquad \text{and} \qquad v_3 = 1\,.$$

By substituting the identifications

$$\boldsymbol{d}_3 = \boldsymbol{t}\,, \qquad u_1 = -\kappa_2\,, \qquad u_2 = \kappa_1\,, \qquad \text{and} \qquad u_3 = \omega$$

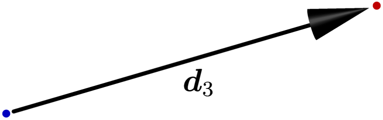
we find

$$\begin{cases} \boldsymbol{x}'(s) = \boldsymbol{t}(s)\,, \\ \boldsymbol{t}'(s) = \kappa_1(s)\boldsymbol{d}_1(s) + \kappa_2(s)\boldsymbol{d}_2(s)\,, \\ \boldsymbol{d}_1'(s) = -\kappa_1(s)\boldsymbol{t}(s) + \omega(s)\boldsymbol{d}_2(s)\,, \\ \boldsymbol{d}_2'(s) = -\kappa_2(s)\boldsymbol{t}(s) - \omega(s)\boldsymbol{d}_1(s)\,. \end{cases}$$

A widely-used quadratic shape energy is

$$\frac{1}{2}\int_0^{s_f} \left(a_1\kappa_2^2(s) + a_2\kappa_1^2(s) + a_3\omega^2(s)\right) ds.$$

# Degenerate cross-sections




Shearing and twisting loose their meaning: we set $v_1 = v_2 = 0$ and $u_3 = 0$. Inextensibility can be imposed by setting $v_3 = 1$.

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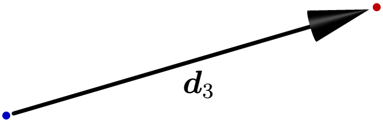


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We obtain the curve and a *relatively parallel adapted frame* (Bishop).

*The relevant Lie algebra remains the same.*

# The inverse problem

$$\boldsymbol{t} = \boldsymbol{x}' \qquad \text{and} \qquad \boldsymbol{t}' = \boldsymbol{x}''\,.$$

If we now consider the integral equations

$$\boldsymbol{d}_1(s) = \boldsymbol{d}_1(0) - \int_0^s u_2(r)\boldsymbol{t}(r)\,dr\,,$$
$$\boldsymbol{d}_2(s) = \boldsymbol{d}_2(0) + \int_0^s u_1(r)\boldsymbol{t}(r)\,dr\,,$$

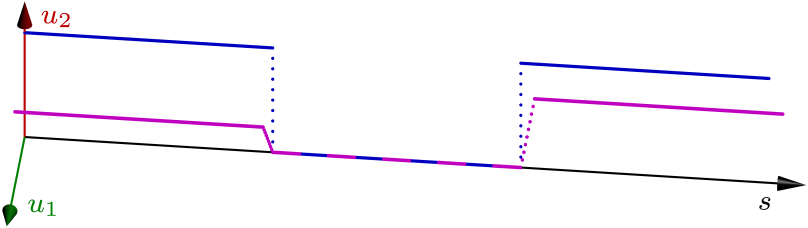
and take the scalar product with $\boldsymbol{t}' = u_2\boldsymbol{d}_1 - u_1\boldsymbol{d}_2$, we obtain

$$u_2(s) = \boldsymbol{d}_1(0)\cdot\boldsymbol{t}'(s) - \int_0^s u_2(r)\boldsymbol{t}(r)\cdot\boldsymbol{t}'(s)\,dr\,,$$
$$u_1(s) = \boldsymbol{d}_2(0)\cdot\boldsymbol{t}'(s) + \int_0^s u_1(r)\boldsymbol{t}(r)\cdot\boldsymbol{t}'(s)\,dr\,.$$

These are *Volterra equations of the second kind* and admit a unique solution on the interval $[0, s_f]$.

# Geometric invariants

Two degrees of freedom: we can picture the shapes of framed curves by means of the *normal development*.



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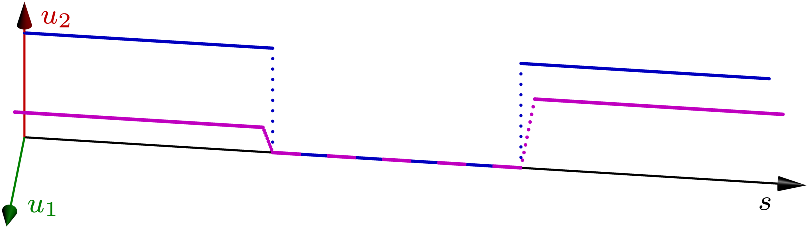
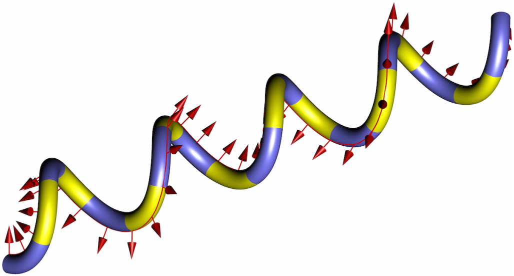


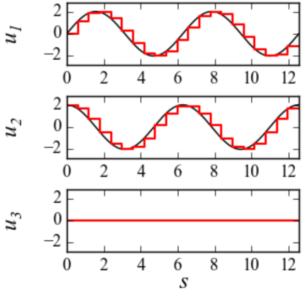

We introduce the Hasimoto transformation

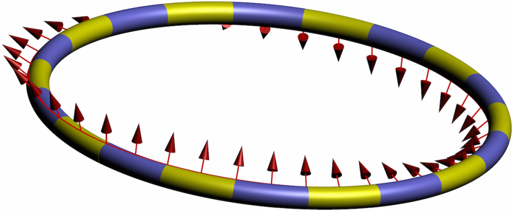
$$\kappa(s)e^{i\theta(s)} = u_2(s) + iu_1(s)$$

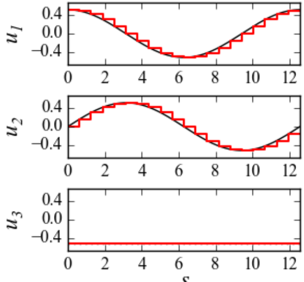
and the geometric invariants are the *square-integrable* curvature $\kappa$ and the *measure-valued* torsion $\tau = \theta'$.

# Twist is not Torsion







# Liquid films spanning rigid boundaries

We can make films without boundary: always spherical bubbles.

Or we can dip some frame in the liquid and build a film while extracting it.

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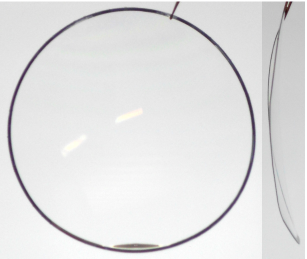
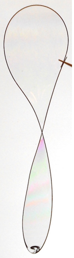
**What feature characterizes the shapes of the film?**
They describe a minimal surface for the prescribed boundary.

**How can we control those shapes?**
The only controllable entity is the shape of the rigid boundary.

# Flexible frames





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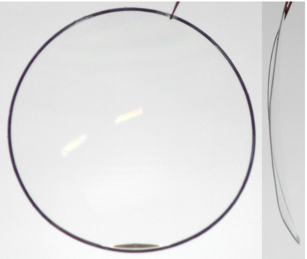
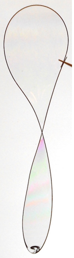
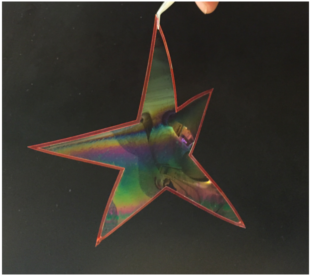
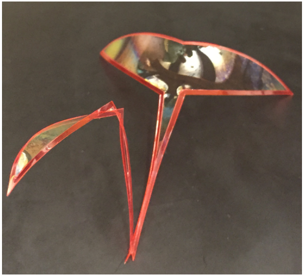
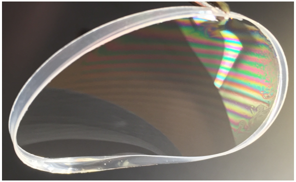
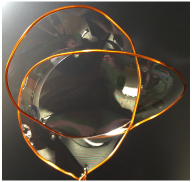


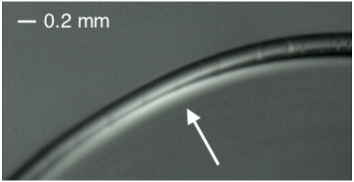
# Goals of the model

- Ensure that the midline of the loop is a closed space curve.
- Allow the midline of the loop to be twisted or knotted.





- Allow for self-contact of the loop without interpenetration
- Allow for loops with noncircular cross sections
- Distinguish the thicknesses of the film and the loop



# The Kirchhoff–Plateau problem

The Kirchhoff–Plateau problem concerns the **equilibrium shapes** of a system in which a flexible filament in the form of a closed loop is spanned by a soap film, with the filament being modeled as a **Kirchhoff rod** and the action of the spanning surface being solely due to **surface tension**.

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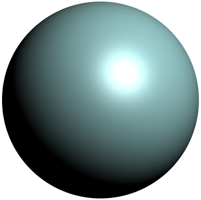
## The energetic approach

We define an energy and look for configurations that minimize it.



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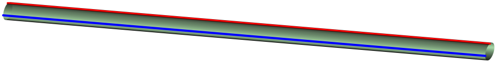

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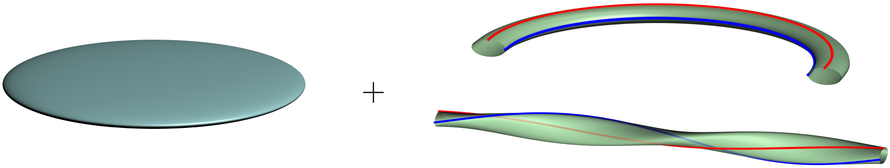
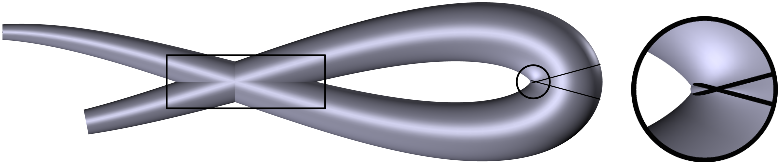
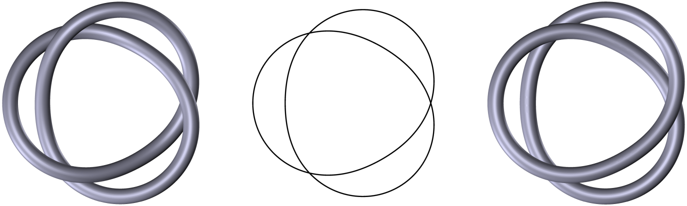


# Non-interpenetration of matter

This is an important constraint that has both global and local implications



With a finite rod thickness, it preserves topological features



# Energy of the liquid film

We define the homogeneous surface tension $\sigma$ of the liquid as the energy density per unit area of the liquid/air interface. We assume that the thickness of the film is negligible and represent it as a two-dimensional set $S \subset \mathbb{R}^3$, but we keep track of the fact that it consists of two interfaces:

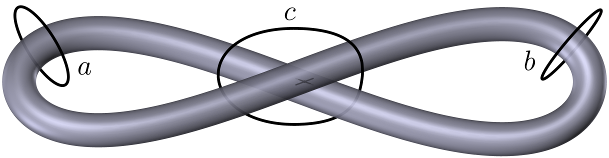
$$E_{\mathrm{film}}(S) := 2\sigma\mathcal{H}^2(S).$$

In the literature on minimal surfaces, the film energy was originally introduced via the mapping area integral and later defined through other notions from geometric measure theory. It was Reifenberg that first obtained important results considering directly the minimization of the Hausdorff measure. This purely spatial perspective has eventually proved to be most effective through the recent work of De Lellis, Ghiraldin & Maggi.

# The spanning condition

## Definition

Let $H$ be a closed set in $\mathbb{R}^3$. Given a collection $\mathcal{C}$ of loops (smooth embeddings of $S^1$ into $\mathbb{R}^3 \setminus H$) which is closed by homotopy, a relatively closed subset $K$ of $\mathbb{R}^3 \setminus H$ is a $\mathcal{C}$-spanning set of $H$ if $K \cap \gamma \neq \emptyset$ for every $\gamma$ in $\mathcal{C}$.




## Example

A spanning set relative to the homotopy class of the loop $a$ or $b$, will cover only the hole on the left or on the right, respectively.

If we consider the homotopy class of the loop $c$, both holes must be covered.

# Dimensional reduction via nonlocality

Let $\mathcal{D}_\Lambda$ denote the set of smooth embeddings $\gamma : S^1 \to \mathbb{R}^3 \setminus \Lambda$ that are not homotopically equivalent to a constant. We define the Kirchhoff–Plateau energy as

$$E_{\mathrm{loop}}(w) + \inf\left\{E_{\mathrm{film}}(S) : S \text{ is a } \mathcal{D}_{\Lambda[w]}\text{-spanning set of } \Lambda[w]\right\}.$$

This is justified by the following result.

## Theorem

Fix $w$ in $V$. If

$$\alpha := \inf\left\{E_{\mathrm{film}}(S) : S \text{ is a } \mathcal{D}_{\Lambda[w]}\text{-spanning set}\right\} < +\infty\,,$$

then there exists a relatively closed subset $S[w]$ of $\mathbb{R}^3 \setminus \Lambda$ that is a $\mathcal{D}_{\Lambda[w]}$-spanning with $E_{\mathrm{film}}(S[w]) = \alpha$. Furthermore, $S[w]$ enjoys the optimal soap film regularity identified by Almgren and Taylor.

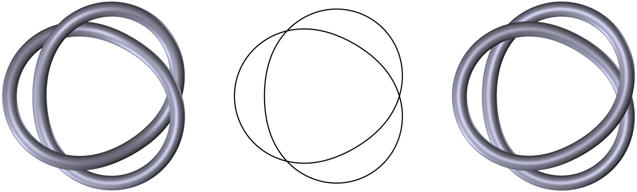
# Weak closure via physical constraints

## Lemma

Any subset $U$ of $V$ containing finite energy shape vectors $w$ with fixed clamping parameters, gluing condition, and knot type of the closed midline and such that the global injectivity condition holds is weakly closed in $V$.

## Remark

The global injectivity condition, together with a nonvanishing thickness, is crucial in obtaining this closure result. In particular, a finite cross-sectional thickness is essential to distinguish knot types in the presence of self-contact.



# Main existence result

## Theorem

Let the clamping parameters, the gluing condition, and the knot type of the closed midline be given. If there exists $\tilde{w}$ in $V$ with finite Kirchhoff–Plateau energy and that satisfies the physical constraints together with global injectivity, then there exists a solution $w$ belonging to $V$ for the Kirchhoff–Plateau problem. Furthermore, the spanning surface $S[w]$ associated with the energy minimizing configuration enjoys the optimal soap film regularity identified by Almgren and Taylor.

## Remark

Key point: weak lower semicontinuity of the nonlocal term

$$w \mapsto \inf\left\{E_{\text{film}}(S) : S \text{ is a } \mathcal{D}_{\Lambda[w]}\text{-spanning set of } \Lambda[w]\right\}.$$

# Sketch of the proof of the lower semicontinuity

- Fix a weakly convergent sequence $w_k \rightharpoonup w$ and let $S_k$ denote a $\mathcal{D}_{\Lambda[w_k]}$-spanning set with finite and minimal area.
- For any $k$ in $\mathbb{N}$, let $\mu_k := \mathcal{H}^2 \llcorner S_k$. Then, up to the extraction of a subsequence, we have $\mu_k \overset{*}{\rightharpoonup} \mu$ on $\mathbb{R}^3$ and we can set $S_0 := \mathrm{spt}(\mu) \setminus \Lambda[w]$.
- Since the convergence of $\{w_k\}$ in $L^p$ entails the uniform convergence of the midlines $\boldsymbol{x}[w_k]$ and the Hausdorff convergence of the bounding loops $\Lambda[w_k]$, we can deduce that

$$\mu \geq \mathcal{H}^2 \llcorner S_0 \text{ on subsets of } \mathbb{R}^3 \setminus \Lambda[w] \,.$$

- The same geometric convergence ensures that $S_0$ is a $\mathcal{D}_{\Lambda[w]}$-spanning set, namely a candidate minimal surface spanning $\Lambda[w]$, the which thing allows to conclude the proof.

# Features

- The description of the rod shape is based on the framework considered above
- The minimization is a two-step process in which we first prove that, for any acceptable rod configuration, there exists an energy-minimizing surface
- The surfaces are represented as supports of rectifiable Radon measures
- No notion of boundary is required for the considered surfaces
- The topology of the surface is not fixed a priori
- The energy-minimizing rod configuration can reach self-contact without self-overlap

# Viscoelastic dynamics with topology changes

- The pre-selection involved in the spanning condition seems to require some knowledge of "where the system will end up". This is not fully satisfactory if we wish to predict the behavior of the system via simulations.
- Dynamic simulations can overcome this issue as the final topology is determined by the evolution of a given (realizable) initial topology.
- Energy-minimizing numerical schemes may follow unphysical temporal paths to achieve their goal.
- The dynamics of a soap film spanning a flexible loop is characterized but not dominated by dissipation.
- Different time-scales can be identified in the evolving system and different asymptotic regimes can be studied.
- A surface discretization compatible with using supports of Radon measures is under development but highly nontrivial.

# Thanks for your attention!