Hyperviscous regularization of the Navier–Stokes equation and the motion of slender swimmers

Giulio G. Giusteri

Dipartimento di Matematica e Fisica “N. Tartaglia”
Università Cattolica del Sacro Cuore

International Conference on
New Trends in Fluid and Solid Models
Vietri sul Mare, April 4th–6th, 2013

Motivation

Composite systems where microscopic bodies move in a viscous fluid are ubiquitous in both biological and technological contexts, and the shape of such immersed bodies often displays some slenderness.

For both mechanical and computational reasons it is tempting to approximate such slender three-dimensional bodies with lower-dimensional objects, and that dimensional reduction requires a regularization of the Navier–Stokes equation.

In collaboration with
Alfredo Marzocchi, Alessandro Musesti
Università Cattolica del Sacro Cuore (Brescia)
Eliot Fried, Tae-Yeon Kim
McGill University (Montréal)

Classical slender-body theory

The basic tool used to construct solutions to the **steady Stokes problem** is the Stokeslet, that is the Green's function for the Stokes operator in $\mathbb{R}^3$.

Good approximation of relevant quantities are obtained by matching inner and outer asymptotic expansions at the boundary of **three-dimensional** bodies.

Computational techniques has to face:

- moving boundaries;
- number of particles much greater than 1, in realistic situations;
- approximation of unsteady flows.

Moreover, non-linear inertial effects must be neglected.

Navier–Stokes equation with hyperviscosity

The balance equations for mass and linear momentum become

$$\operatorname{div} \mathbf{u} = 0 \,,$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \mu \Delta \mathbf{u} + \xi \Delta\Delta \mathbf{u} = \rho \mathbf{b} \,,$$

with p pressure field, $\mathbf{u}$ velocity field, $\rho > 0$ constant and homogeneous mass density, $\mu > 0$ dynamic viscosity, $\rho\mathbf{b}$ volumetric force density, and the additional parameter $\xi > 0$ is the **hyperviscosity**.

We introduce an **effective thickness** $L > 0$, and set $\xi = \mu L^2$. In this way, we give a geometric meaning to the additional term: L replaces the characteristic size of the body along the dimensions that vanish in the slender-body limit.

Advantages

Analytical:

- Existence and uniqueness of regular solutions.
- The continuity of the velocity field permits to model adherence to lower-dimensional objects.
- The hyperviscous solution converges, in the limit $L \to 0$, to a suitable solution of the Navier-Stokes equation.

Computational:

- Simplification of the geometries.
- Moving boundaries replaced by a time-dependent constraint on function spaces.
- Reduction of instabilities related to singularities of the flow.

Force and torque on a slender body Σ

Hyperviscous stress tensor:

$$\mathsf{T}(\mathbf{u}, p) := -p\mathbf{I} + \mu\left(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathsf{T}} - L^2\nabla\Delta\mathbf{u}\right)$$

r-neighborhood of Σ:

$$V_r(\Sigma) := \left\{\mathbf{x} \in \mathbb{R}^3 : d(x, \Sigma) \leq r\right\}$$

Hydrodynamic force on Σ:

$$\mathbf{f}(\mathbf{u}, p) := \lim_{r\to 0}\int_{\partial V_r(\Sigma)} \mathsf{T}(\mathbf{u}, p)\mathbf{n}$$

Hydrodynamic torque on Σ:

$$\mathbf{t}(\mathbf{u}, p) := \lim_{r\to 0}\int_{\partial V_r(\Sigma)} \mathbf{x} \times \mathsf{T}(\mathbf{u}, p)\mathbf{n}$$

Free fall of a one-dimensional rigid body

The equation for the fluid should be stated in $\mathbb{R}^3 \setminus \Sigma$, and coupled to the equations for the rigid body motion. But we can also consider the following equation in **all of** $\mathbb{R}^3$:

$$\operatorname{div} \mathsf{T} = \rho(\mathbf{a} - \mathbf{g}) \,,$$

with ρ a **measure**, whose singular part is concentrated on Σ.

Such a singularity is necessary to give a non-zero weight to a body with vanishing volume. Due to the equation above, we expect also $\operatorname{div} \mathsf{T}$ to be a measure with singular part $(\operatorname{div} \mathsf{T})^{S}$ concentrated on Σ, and absolutely continuous part denoted by $(\operatorname{div} \mathsf{T})^{AC}$.

The hydrodynamic force is well-defined

Taking a ball of radius R large enough we have, for any $r > 0$ small enough,

$$\lim_{r\to 0}\int_{B_R\setminus V_r(\Sigma)} \operatorname{div} \mathsf{T} = \lim_{r\to 0}\int_{\mathbb{R}^3} (\operatorname{div} \mathsf{T})^{AC}\chi_{B_R\setminus V_r(\Sigma)}$$
$$= \int_{\mathbb{R}^3} (\operatorname{div} \mathsf{T})^{AC}\chi_{B_R\setminus\Sigma} = \int_{B_R\setminus\Sigma} \operatorname{div} \mathsf{T}\,.$$

Then

$$\mathbf{f} = \lim_{r\to 0}\int_{\partial V_r(\Sigma)} \mathsf{T}\mathbf{n} = \int_{\partial B_R} \mathsf{T}\mathbf{n} - \int_{B_R\setminus\Sigma} \operatorname{div} \mathsf{T}\,.$$

Remark: if $(\operatorname{div} \mathsf{T})^S = 0$, then $\operatorname{div} \mathsf{T} \in L^1(B_R)$ and $\mathbf{f} = 0$.

Ongoing work

- Existence and uniqueness of solution for the free fall of a one-dimensional rigid body in a hyperviscous fluid. *(done)*
- Analysis of the steady free fall of bodies with symmetries.
- Existence and uniqueness of solution for the free fall of a one-dimensional **deformable** structure in a hyperviscous fluid.
- Numerical scheme for the simulation of the coupled motion of one-dimensional rigid bodies in a hyperviscous fluid.

- Fried and Gurtin, Arch. Ration. Mech. Anal., 2006.
- Giusteri, Marzocchi, Musesti, Mech. Res. Commun., 2010.
- Giusteri, Marzocchi, Musesti, Acta Mech., 2011.
- Giusteri, ZAMP, 2012.
- Fried and Giusteri, *in preparation*.
- Giusteri, Marzocchi, Musesti, *in preparation*.

http://www.dmf.unicatt.it/~giusteri/

- Fried and Gurtin, Arch. Ration. Mech. Anal., 2006.
- Giusteri, Marzocchi, Musesti, Mech. Res. Commun., 2010.
- Giusteri, Marzocchi, Musesti, Acta Mech., 2011.
- Giusteri, ZAMP, 2012.
- Fried and Giusteri, *in preparation*.
- Giusteri, Marzocchi, Musesti, *in preparation*.

http://www.dmf.unicatt.it/~giusteri/

Thank you