Non-simple linear fluids surrounding 1D structures

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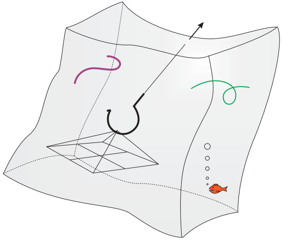
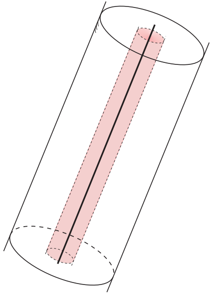
Berlin, STAMM 2010

1. The Model
   - Non-simple linear isotropic incompressible homogeneous viscous fluids
   - 1D structures in a 3D region
2. Results
   - Adherence of the fluid to 1D immersed rigid bodies
   - Well-posedness of the Cauchy problem
3. Puzzling Questions
   - Do edges and lines exist?
   - Do rigid bodies exist?





a joint work with Alessandro Musesti and Alfredo Marzocchi

Virtual Works framework

- Choose the *space manifold*: $\Omega \times (0, T)$ in Euclidean 4D space-time.
- Choose the *virtual fields*: Eulerian incompressible velocity field.
- Consider the *virtual work* on the virtual fields, represented as

$$\langle\, W \,,\, \mathbf{v} \,\rangle = \sum_{j=0}^{k} \int_0^T \int_\Omega F_j \cdot \nabla^{(j)} \mathbf{v} \, d\mathbf{x} \, dt$$

for internal and external interactions.

- Choose *constitutive laws* for the internal and external densities F_j as functions of the relevant real fields $\mathbf{u}$.
- Require the balance of internal and external virtual work

$$\langle\, W_{\text{int}} \,,\, \mathbf{v} \,\rangle = \langle\, W_{\text{ext}} \,,\, \mathbf{v} \,\rangle$$

for every virtual field (Principle of Virtual Works).

Non-simple: of second gradient

Internal virtual work of second gradient satisfying Galilean invariance:

$$\langle\, W_{\text{int}}\,,\, \mathbf{v}\,\rangle = \int_0^T \left(\int_\Omega \mathrm{T}\cdot\nabla\mathbf{v} + \int_\Omega \mathbf{G}\cdot\nabla\nabla\mathbf{v} \right)$$

External virtual work of second gradient in an alternative representation:

$$\langle\, W_{\text{ext}}\,,\, \mathbf{v}\,\rangle = \int_0^T \left(\int_\Omega (\mathbf{b}-\mathbf{b}_{\text{in}})\cdot\mathbf{v} + \int_S \mathbf{t}\cdot\mathbf{v} + \int_S \mathbf{m}\cdot\frac{\partial\mathbf{v}}{\partial n} + \int_E \mathbf{f}_c\cdot\mathbf{v} \right)$$

The inertial body forces will take the usual nonlinear form:

$$\mathbf{b}_{\text{in[ertial]}} = \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}$$

Linear, isotropic, incompressible, homogeneous

...the simplest ...

$$\operatorname{div} \mathbf{u} = 0 \qquad \text{and} \qquad \rho(\mathbf{x}, t) = 1$$

...still one has the following constitutive prescriptions:

$$\begin{aligned} \mathsf{T}_{ij} &= \mu(\mathbf{u}_{i,j} + \mathbf{u}_{j,i}) - p\,\delta_{ij} \\ \mathbf{G}_{ijk} &= \eta_1 \mathbf{u}_{i,jk} + \eta_2(\mathbf{u}_{j,ki} + \mathbf{u}_{k,ij} - \mathbf{u}_{i,ss}\delta_{jk}) \\ &\quad + \eta_3(\mathbf{u}_{j,ss}\delta_{ki} + \mathbf{u}_{k,ss}\delta_{ij} - 4\mathbf{u}_{i,ss}\delta_{jk}) - \mathbf{p}_k\delta_{ij} \end{aligned}$$

In intrinsic notation $\mathsf{T} = 2\mu\,\mathrm{Sym}\,\nabla\mathbf{u} - p\,\mathsf{I}$ and

$$\begin{aligned} \mathbf{G} &= (\eta_1 - \eta_2)\nabla\nabla\mathbf{u} + 3\eta_2\,\mathrm{Sym}\,\nabla\nabla\mathbf{u} \\ &\quad - (\eta_2 + 5\eta_3)\Delta\mathbf{u} \otimes \mathsf{I} + 3\eta_3\,\mathrm{Sym}\,(\Delta\mathbf{u} \otimes \mathsf{I}) - \mathsf{I} \otimes \mathbf{p}\,. \end{aligned}$$

Thermodynamical constraints

The instant dissipation must be nonnegative for any flow.

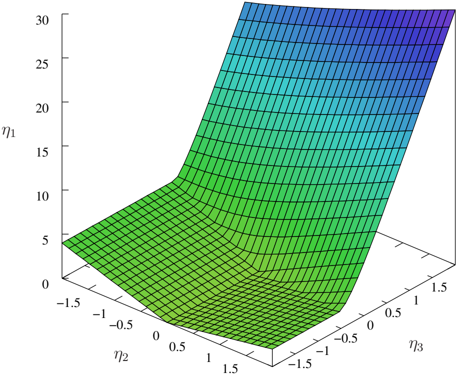
$$\Updownarrow$$

$$\mu \geq 0$$

$$\eta_1 + 2\eta_2 \geq 0$$

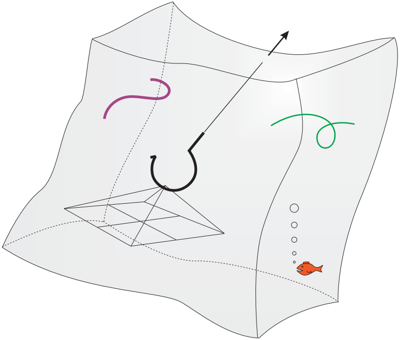
$$\eta_1 - \eta_2 \geq 0$$

$$\eta_1 - \eta_2 - 6\eta_3 - 2\sqrt{\eta_2^2 + 4\eta_2\eta_3 + 9\eta_3^2} \geq 0$$



For the Cauchy problem to be well-posed strict inequalities are needed.

1D rigid bodies in a 3D region

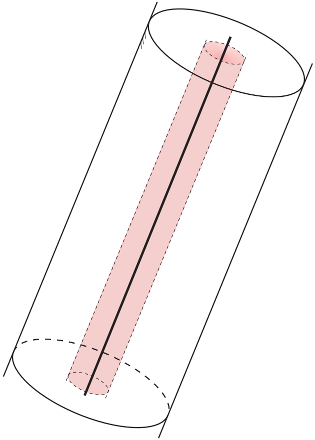


Situations:

1. A pressure gradient drives the flow producing in- and out-flows through some parts of the boundary.
2. A closed container where thin objects swim.

We choose to model the thin moving objects as rigid bodies with finite 1D Hausdorff measure, so that their motion plainly satisfies the incompressibility condition (though unfortunately they cannot swim).

Adherence in a cylindrical cavity



Fluid dragged along the axial direction by the inner cylinder, adherence at R_1 and R_2.

Newtonian liquid:

$$u(r) = -\log(r/R_2)/\log(R_2/R_1)$$

No continuous extension for $R_1 \to 0$.

Second-gradient linear liquid:

$$u(r) = c_1 + c_2 I_0(r/L) + c_3[\log(r/L) + K_0(r/L)]$$

Continuous extension for $R_1 \to 0$ since

$$\lim_{r\to 0}[\log(r/L) + K_0(r/L)] < +\infty\,.$$

Adherence interaction for the present model

The adherence is encoded in the functional setting.

- H: completion with respect to the L^2 norm of divergence-free smooth vector fields.
- H^2_d: completion with respect to the H^2 norm of divergence-free smooth vector fields.
- $\Lambda(t)$: position of the 1D rigid body at time t.
- A crucial role is played by the functional spaces

$$X_t := \{\, \mathbf{v} \in H^2(\Omega;\mathbb{R}^3) : \mathbf{v} = 0 \text{ on } \partial\Omega \cup \Lambda(t) \,\} \cap H^2_d$$

whose elments are continuous functions.

The Cauchy problem

The nonzero velocity on $\Lambda(t)$ is recovered by constructing a suitable divergence-free interpolator $\mathbf{w}$. It is not a trivial step, but it can be done.

Theorem

For every initial datum $\mathbf{u}_0 \in H$ there exists a unique

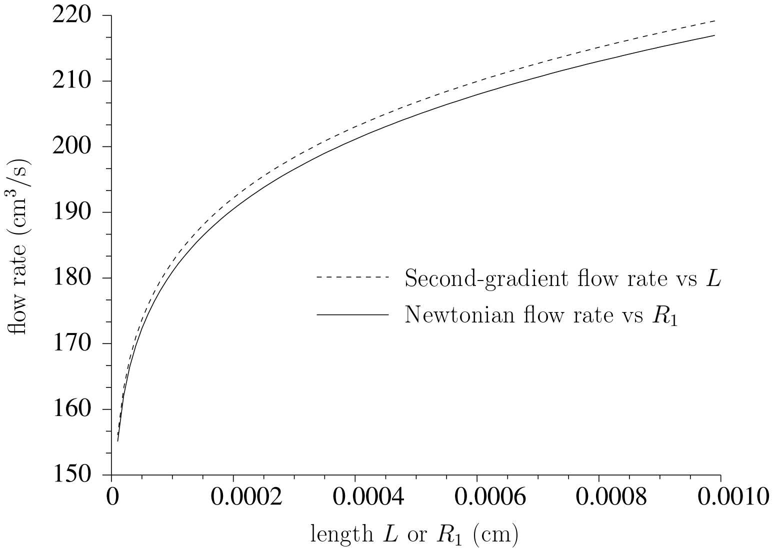
$$\mathbf{u} \in L^2([0,T];X_t) \cap C^0([0,T];H) \cap H^1([0,T];X_t')$$

such that the field $\mathbf{u} + \mathbf{w}$ satisfies the balance

$$\langle\, W_{\mathrm{int}} \,,\, \mathbf{v} \,\rangle = \langle\, W_{\mathrm{ext}} \,,\, \mathbf{v} \,\rangle$$

for every virtual velocity field $\mathbf{v}$.

A charming plot



To-do list

- Clearly identify the modeling meaning of the parameters η_1, η_2, η_3 and $L = \sqrt{(\eta_1 - \eta_2 - 4\eta_3)/\mu}$.
- Coupling between a more general dynamics for the 1D structures and the fluid dynamics.
- To provide an efficient scheme for numerical simulations.

- Degiovanni, Marzocchi, Musesti, Ann. Mat. Pura Appl., 2006.
- Fried, Gurtin, Arch. Rat. Mech. Anal., 2006.
- Podio-Guidugli, Vianello, Cont. Mech. Thermodyn., 2010.
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Thank you