Tempered holomorphic solutions of $\mathcal{D}$-modules on a complex curve

Giovanni Morando

April 10, 2006

Let X be a complex *curve*, $\mathcal{O}_X$ the sheaf of holomorphic functions on X.

Definition

Let U be a relatively compact open subset of X contained in some chart of X.

$$\mathcal{O}^t_{X_{sa}}(U) := \left\{ f \in \mathcal{O}_X(U);\ \exists\ M, C > 0\ s.t.\ |f(z)| \leq \frac{C}{\mathrm{dist}(z, \partial U)^M} \right\}$$

Remark: the gluing property for *arbitrary* open sets fails, i.e. for U_j an open set and $f_j \in \mathcal{O}^t(U_j)$ $(j = 1, 2)$

$$f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2} \not\Rightarrow \exists f \in \mathcal{O}^t(U_1 \cup U_2) \text{ s.t. } f|_{U_j} = f_j$$

in particular $U \longmapsto \mathcal{O}^t_{X_{sa}}(U)$ does not define a sheaf on X,

The family of *subanalytic sets* in $\mathbb{R}^2$ is the smallest family closed under *finite* $\cup, \cap$ and complements containing sets locally defined by

$$f(x, y) > 0$$

for f an analytic function.

The gluing property on finite relatively compact open subanalytic sets holds. i.e. if U_j is rel comp open subanalytic and $f_j \in \mathcal{O}^t(U_j)$ $(j = 1, 2)$

$$f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2} \Rightarrow \exists f \in \mathcal{O}^t(U_1 \cup U_2) \text{ s.t. } f|_{U_j} = f_j \qquad (1)$$

Define the subanalitic site X_{sa} underlying X by

1. open sets are relatively compact subanalytic open subsets of X, denoted $\mathrm{Op}^c_{X_{sa}}$,
2. coverings are finite coverings.

Then the gluing property for tempered holomorphic functions on relatively compact subanalytic open sets implies that

$$\mathcal{O}^t \text{ is a sheaf on } X_{sa}.$$

There are some particular ODE of degree 1 which play an important role in the classification of ODE.

Let $p \in z^{-1}\mathbb{C}[z^{-1}]$, $p \neq 0$.

Let P_p be the differential operator of degree 1 defined on $\mathbb{C}$ having $\exp(p(z))$ as a holomorphic solution outside the origin.

P_p is an irregular differential operator.

For any p, q the sheaves of holomorphic solutions of P_p and P_q are isomorphic.

Theorem

The sheaves of tempered holomorphic solutions of P_p and P_q are isomorphic if and only if $\exists \lambda > 0$ such that $p = \lambda q$.

Let $U \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}}$. P_p has tempered holomorphic solutions on U if and only if $\exp(p) \in \mathcal{O}^t(U)$.

Equivalent statement

$$\left\{U \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}};\ \exp(p) \in \mathcal{O}^t(U)\right\} = \left\{U \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}};\ \exp(q) \in \mathcal{O}^t(U)\right\}$$

if and only if

$$\exists \lambda > 0 \text{ such that } p = \lambda q.$$

Lemma

Let $U \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}}$. Then $\exp(p(z)) \in \mathcal{O}^t(U)$ if and only if there exists $A > 0$ such that

$$U \quad \subset \quad U_{p,A} \quad := \quad \left\{z \in \mathbb{C};\ \mathrm{Re}\,(p(z)) < A\right\}.$$

$\Uparrow$: straightforward

$$p = \lambda q,\ \lambda > 0 \quad \Rightarrow \quad U_{p,A} = U_{q,\frac{A}{\lambda}}$$

$\Rightarrow$ P_p and P_q have the same sheaves of tempered holomorphic solutions

It remains to prove
⇓: if $p \neq \lambda q$ then

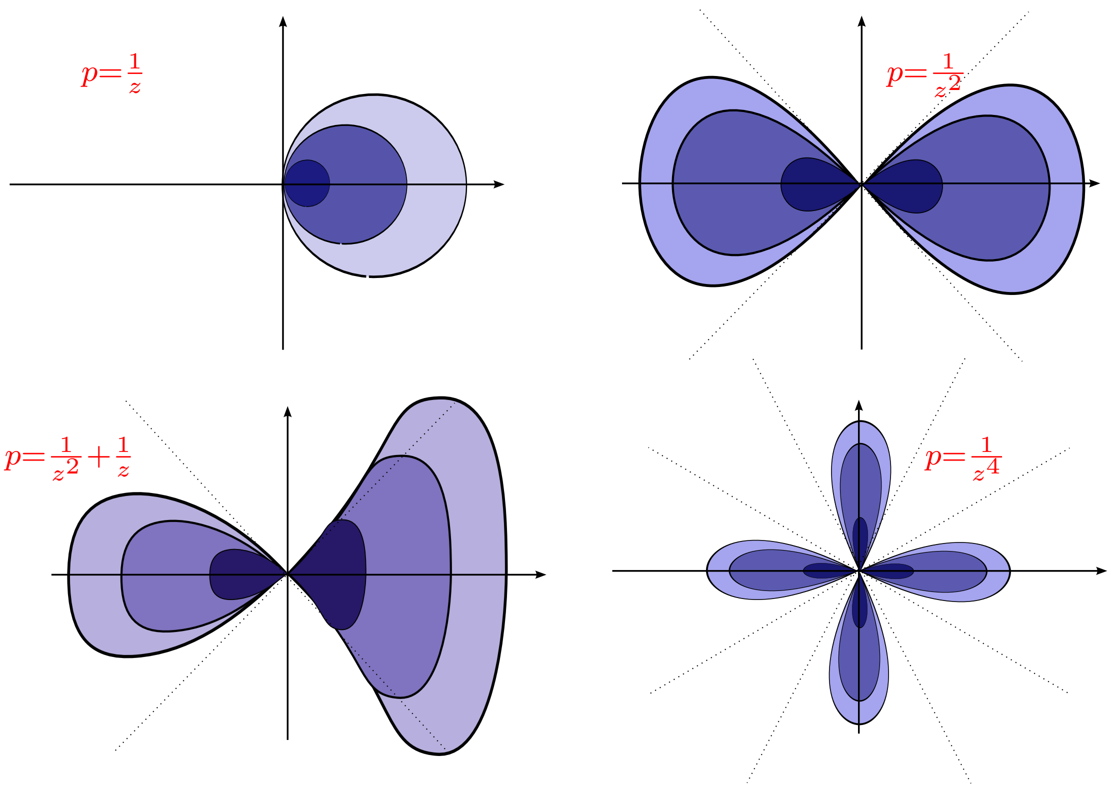
$$\exists\, U_1 \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}};\ \exp(p) \in \mathcal{O}^t(U_1),\ \exp(q) \notin \mathcal{O}^t(U_1)\ .$$

$$\exists\, U_2 \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}};\ \exp(p) \notin \mathcal{O}^t(U_2),\ \exp(q) \in \mathcal{O}^t(U_2)\ .$$

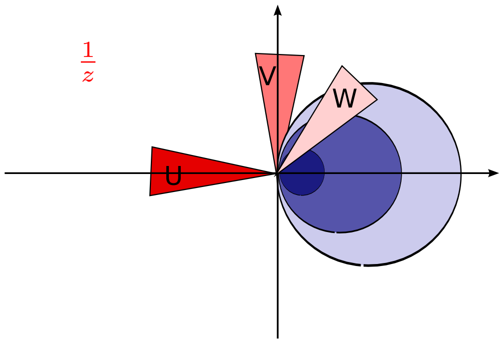
Which is equivalent to
Claim

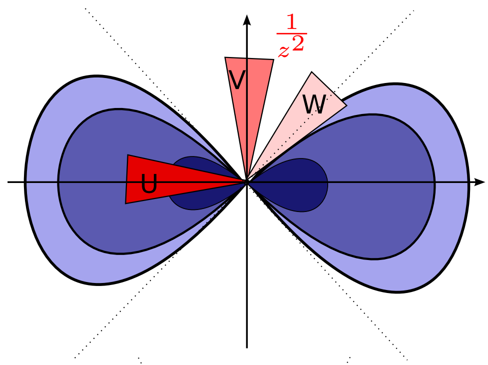
$$p \neq \lambda q \quad \Rightarrow \quad \exists U \quad \begin{array}{lll} \subset & U_{p,A} & \forall A > \overline{A} \\ \not\subset & U_{q,A} & \forall A \end{array}$$

The sets complementary to $U_{p,A}$

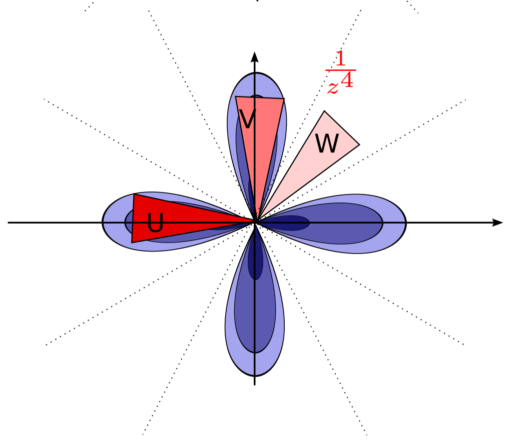


The sets complementary to $U_{p,A}$



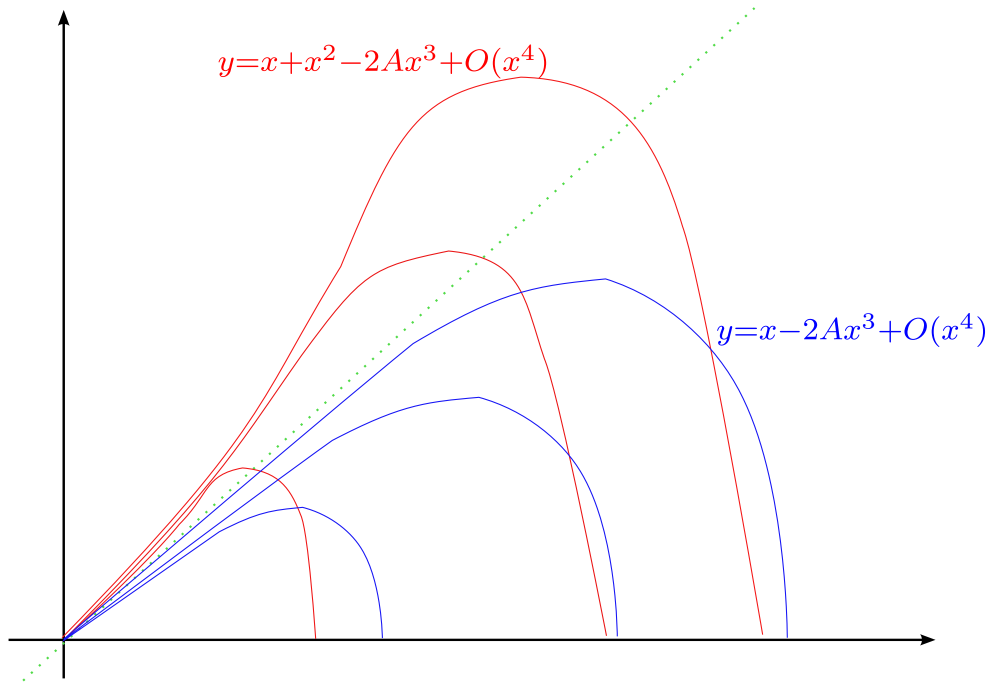


Is $\exp(p)$ tempered on	U	V	W ?
$p=\frac{1}{z}$	YES	NO	NO
$p=\frac{1}{z^2}$	NO	YES	NO
$p=\frac{1}{z^4}$	NO	NO	YES



If p and q have different degrees
then the claim is true

The boundaries of $U_{\frac{1}{z^2},A}$ and $U_{\frac{1}{z^2}+\frac{1}{z},A}$



Let $X \subset \mathbb{C}$ be an open neighborhood of 0. Consider the differential operator

$$P\left(z, \frac{d}{dz}\right) = a_n(z)\frac{d^n}{dz^n} + \ldots + a_1(z)\frac{d}{dz} + a_0(z) \ ,$$

where $n \in \mathbb{N}$ and $a_j \in \mathcal{O}(X)$ $(j = 0, \ldots, n)$.

Let $0 \in X$ be the only possible singular point for P.

Theorem

Let $U \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}}$, $g \in \mathcal{O}^t(U)$. There exist an open subanalytic covering $\{U_j\}$ of U, $v_j \in \mathcal{O}^t(U_j)$ such that

$$Pv_j = g|_{U_j} \ .$$

We prove the Theorem in three steps.

1. First we add the hypothesis that U is an open sector of sufficiently small amplitude.
2. Second we add the hypothesis that U is biholomorphic to an open sector.
3. Third we prove the generic case $U \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}}$.

Theorem (Hukuhara-Turrittin)

There exist $l \in \mathbb{Z}_{>0}$, $p_j \in z^{-1/l}\mathbb{C}[z^{-1/l}]$ and for any $\vartheta \in S^1$ there exist an open sector S containing ϑ, $f_j \in \mathcal{O}(\overline{S} \setminus \{0\})$, $C, M > 0$ such that ($j = 1, \dots, n$)

1. $C|z|^M \leq |f_j(z)| \leq C^{-1}|z|^{-M} \qquad \forall z \in S$
2. $P\ f_j(z)\exp(p_j(z)) = 0$

For sake of simplicity, we will suppose that

$$P = z^N \frac{d}{dz} + h(z) \ ,$$

for $h \in \mathcal{O}(X)$ and $N \in \mathbb{N}$.

By Hukuhara-Turrittin's Theorem, the holomorphic solutions of P on a sufficiently small sector S are multiples of $f(z)\exp(p(z))$, for $p \in z^{-1}\mathbb{C}[z^{-1}]$ and f invertible such that f and $\frac{1}{f}$ are tempered on S.

Given $g \in \mathcal{O}^t(U)$, $Pv = g$ has holomorphic solution

$$I(g)(z) = f(z)\exp(p(z)) \int_\Gamma \exp(-p(\zeta)) \frac{g(\zeta)}{\zeta^N f(\zeta)} d\zeta \in \mathcal{O}(U) \ .$$

Is $I(g)$ tempered on U?

Theorem (Honda-Malgrange-Hukuhara ...)

If S is an open sector of amplitude sufficiently small then $I(g) \in \mathcal{O}^t(S)$.

Technique: find good path of integration. They depends highly on p.

Remark: the nice boundary of a sector S allows easy estimates of the distance from the boundary. It is hard to generalize the procedure to an arbitrary open subanalytic set.

The Theorem is true supposing additionally that

1. U is an open sector of sufficiently small amplitude. ✓
2. U is biholomorphic to an open sector.
3. $U \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}}$.

Theorem (1)

Let $U \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}}$, $\varphi \in \mathcal{O}(\overline{U})$, injective on $\overline{U}$, $h \in \mathcal{O}(\varphi(U))$. Then $h \in \mathcal{O}^t(\varphi(U))$ if and only if $h \circ \varphi \in \mathcal{O}^t(U)$.

Idea:

1. consider S an open sector of sufficiently small amplitude and $\varphi : S \longrightarrow T$ a biholomorphism injective up to the boundary of S.
2. Consider $g \in \mathcal{O}^t(T)$ and $I(g) \in \mathcal{O}(T)$, the holomorphic solution of $Pv = g$.
3. Consider $I(g) \circ \varphi \in \mathcal{O}(S)$ and check that it is tempered on S with the case of small sectors.
4. Use Theorem (1) to obtain that $I(g)$ is tempered on T.

1. U is an open sector of sufficiently small amplitude. ✓
2. U is biholomorphic to an open sector. ✓
3. $U \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}}$.

Theorem

Let $U \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}}$. Locally

$$U = \bigcup_{j=1}^{d} \left(\varphi_{j,1}(S_{j,1}) \cap \varphi_{j,2}(S_{j,2})\right) .$$

Where $\varphi_{j,k}$ is a biholomorphism on $\overline{S}_{j,k}$, for $S_{j,k}$ an open sector.

Lemma

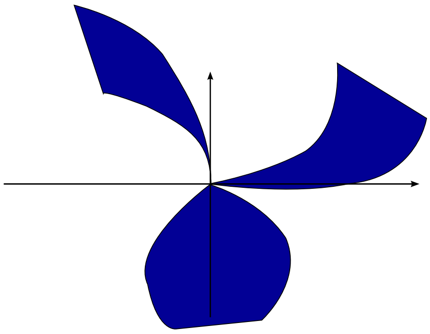
Let $U, V \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}}$. The following sequence is exact

$$0 \longrightarrow \mathcal{O}^t(U \cup V) \longrightarrow \mathcal{O}^t(U) \oplus \mathcal{O}^t(V) \longrightarrow \mathcal{O}^t(U \cap V) \longrightarrow 0 .$$

✓the Theorem is proved.

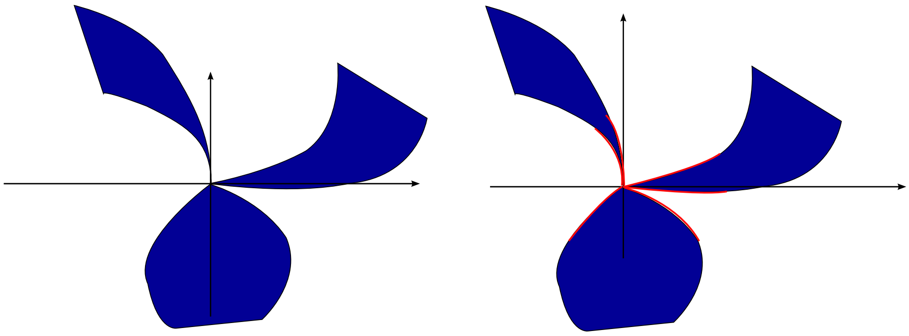
Proof of $U = \bigcup_{j=1}^{d} \left(\varphi_{j,1}(S_{j,1}) \cap \varphi_{j,2}(S_{j,2})\right)$.

Using ccd or triangulation theorem



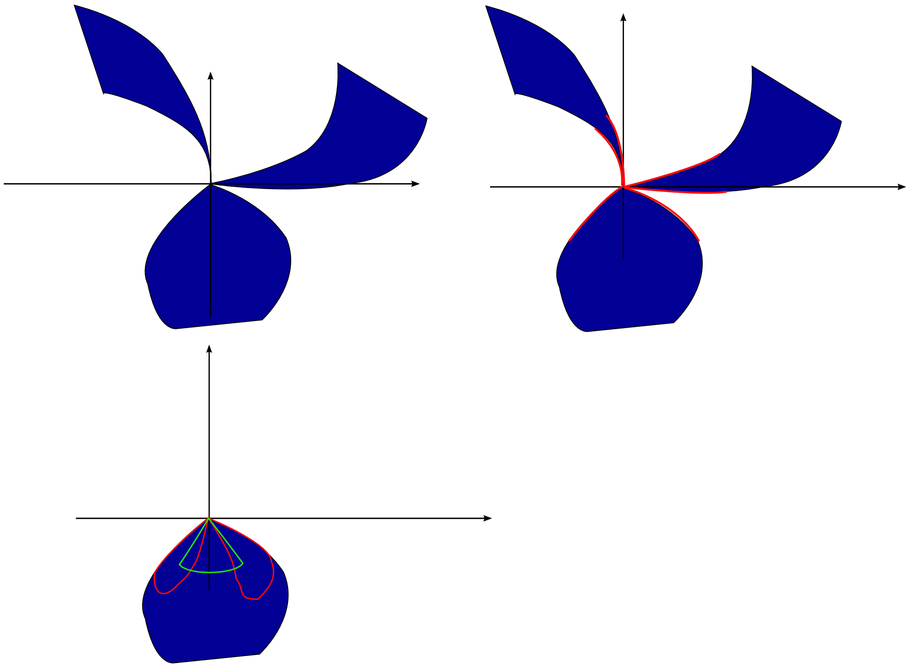
Proof of $U = \bigcup_{j=1}^{d} \left(\varphi_{j,1}(S_{j,1}) \cap \varphi_{j,2}(S_{j,2})\right)$.

Locally the boundaries are semi-analytic arcs.



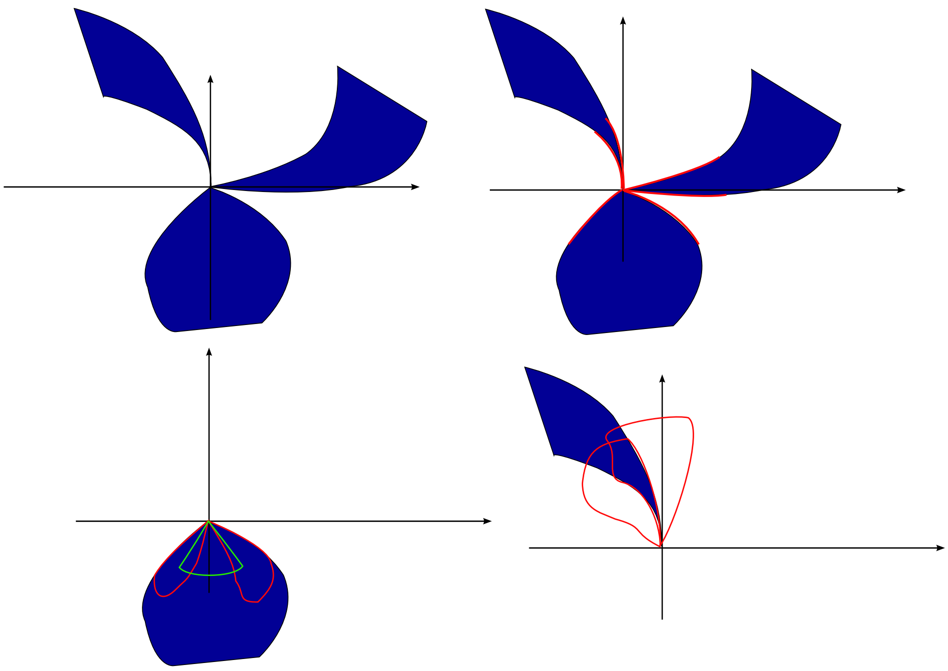
Proof of $U = \bigcup_{j=1}^{d} \left(\varphi_{j,1}(S_{j,1}) \cap \varphi_{j,2}(S_{j,2})\right)$.

Arcs with different tangents lead unions.



Proof of $U = \bigcup_{j=1}^{d} \left(\varphi_{j,1}(S_{j,1}) \cap \varphi_{j,2}(S_{j,2})\right)$.

Arcs with the same tangents need intersection.



Let X be a complex analytic manifold, $\mathcal{M} \in D^b_{rh}(\mathcal{D}_X)$.

$$\begin{aligned} \mathcal{S}ol\mathcal{M} &:= R\varrho_* \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \in D^b(X_{sa}) \\ \mathcal{S}ol^t\mathcal{M} &:= \mathrm{R}\mathcal{H}om_{\varrho_!\mathcal{D}_X}(\varrho_!\mathcal{M}, \mathcal{O}^t_{X_{sa}}) \in D^b(X_{sa}) \ . \end{aligned}$$

Theorem (Kashiwara 1984)

The natural morphism in $D^b(X_{sa})$

$$\mathcal{S}ol^t\mathcal{M} \longrightarrow \mathcal{S}ol\mathcal{M}$$

is an isomorphism.

Let X be a complex curve, $\mathcal{M}$ a holonomic $\mathcal{D}_X$-module.

Theorem

Then

$$H^1\left(\mathcal{S}ol^t(\mathcal{M})\right) \longrightarrow H^1\left(\mathcal{S}ol(\mathcal{M})\right)$$

is an isomorphism.

Proposition

Locally, there exist $\mathcal{M}_{reg}$ a regular holonomic $\mathcal{D}_X$-module, a differential operator P and a distinguished triangle

$$\mathcal{M}_{reg} \longrightarrow \mathcal{M} \longrightarrow \frac{\mathcal{D}_X}{\mathcal{D}_X \cdot P} \xrightarrow{+1} \ .$$

$$\mathcal{S}ol^t\left(\frac{\mathcal{D}_X}{\mathcal{D}_X \cdot P}\right): \qquad 0 \longrightarrow \mathcal{O}^t_{X_{sa}} \xrightarrow{P} \mathcal{O}^t_{X_{sa}} \longrightarrow 0$$

$$\mathcal{S}ol\left(\frac{\mathcal{D}_X}{\mathcal{D}_X \cdot P}\right): \qquad 0 \longrightarrow \mathcal{O}_X \xrightarrow{P} \mathcal{O}_X \longrightarrow 0$$

Proposition

The natural morphism of sheaves on X_{sa}

$$\frac{\mathcal{O}^t_{X_{sa}}}{P\mathcal{O}^t_{X_{sa}}} \longrightarrow \varrho_* \frac{\mathcal{O}_X}{P\mathcal{O}_X},$$

is an isomorphism.

Lemma ($0 \notin U$)

For any $g \in \mathcal{O}^t(U)$, there exist a subanalytic open covering $\{U_j\}_{j\in J}$ of U and $v_j \in \mathcal{O}^t(U_j)$ such that $Pv_j = g|_{U_j}$.

Lemma (U disc centered at 0)

The natural morphism

$$\frac{\mathcal{O}^t_{X_{sa}}(U)}{P\mathcal{O}^t_{X_{sa}}(U)} \xrightarrow{\varphi_t} \frac{\mathcal{O}_X(U)}{P\mathcal{O}_X(U)}$$

is an isomorphism.

Proposition

The natural morphism of sheaves on X_{sa}

$$\frac{\mathcal{O}^t_{X_{sa}}}{P\mathcal{O}^t_{X_{sa}}} \longrightarrow \varrho_* \frac{\mathcal{O}_X}{P\mathcal{O}_X} \ ,$$

is an isomorphism.

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