Tempered holomorphic solutions of $\mathcal{D}$-modules on complex curves

Giovanni Morando

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Let X be a complex analytic *curve*. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X.

Let X_{sa} be the subanalytic site induced by X.

The sheaf of tempered holomorphic functions on X_{sa}, $\mathcal{O}^t_{X_{sa}}$, is defined as the solution complex of the Cauchy-Riemann system in the subanalytic sheaf of tempered distributions.

If U is a relatively compact subanalytic open subset of $\mathbb{C}$, then

$$\mathcal{O}^t_{X_{sa}}(U) = \Big\{ f \in \mathcal{O}(U);\ \text{there exist } C, N > 0 \text{ such that } |f(z)| \leq \frac{C}{\operatorname{dist}(z, \partial U)^N} \Big\} .$$

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Given a holonomic $\mathcal{D}_X$-module, $\mathcal{M}$, it makes sense to consider the complex of subanalytic sheaves of tempered holomorphic solutions of $\mathcal{M}$,

$$R\mathcal{H}om_{\varrho_!\mathcal{D}_X}\left(\varrho_!\mathcal{M}, \mathcal{O}^t_{X_{sa}}\right) .$$

For $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$ denote by $\mathcal{L}^\varphi$ the meromorphic connection represented by

$$\left(\mathbb{C}(\{z\}), \frac{d}{dz} - \frac{d\varphi}{dz}\right) .$$

For $\alpha \in \mathbb{C}$ and $r \in \mathbb{Z}_{\geq 0}$, denote by $\mathcal{R}_{\alpha,r}$ the meromorphic connection represented by

$$\left(\mathbb{C}(\{z\})^r, \frac{d}{dz} - \begin{pmatrix} \alpha & 1 & & 0 \\ & \alpha & \ddots & \\ & & \ddots & 1 \\ 0 & & & \alpha \end{pmatrix}\right) .$$

A *good model* is a meromorphic connection of the form

$$\bigoplus_{j=1}^{p} \mathcal{L}^{\varphi_j} \otimes \mathcal{R}_{\alpha_j,r_j} .$$

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Definitions
Main Theorem

For $E \bigoplus_{j=1}^{p} \mathcal{L}^{\varphi_j} \otimes \mathcal{R}_{\alpha_j,r_j}$ a good model, set

$$\kappa(E) := \max_{j=1,\dots,p} \{\deg \varphi_j\} .$$

$\kappa(E)$ is the *Katz invariant* of E.

Denote by GM_k the abelian category of good models with Katz invariant strictly smaller than k.

Let us consider good models as $\mathcal{D}_{\mathbb{C}}$-modules. For $k \in \mathbb{Z}_{>0}$, set

$$\begin{aligned} \mathcal{S}^t(E) &:= \mathcal{H}om_{\varrho_! \mathcal{D}_X}\left(\varrho_! E, \mathcal{O}^t_{X_{sa}}\right) , \\ \mathcal{S}^t_k(E) &:= \mathcal{S}^t(E \otimes \mathcal{L}^{1/z^k}) \end{aligned}$$

Theorem

1. *For* $j = 1, 2$, *let* $\varphi_j \in z^{-1}\mathbb{C}[z^{-1}]$, $\alpha_j \in \mathbb{C}$, $r_j \in \mathbb{Z}_{\geq 0}$. *Then*

$$Hom_{\mathbb{C}_{sa}}\left(\mathcal{S}^t(\mathcal{L}^{\varphi_1} \otimes \mathcal{R}_{\alpha_1,r_1}), \mathcal{S}^t(\mathcal{L}^{\varphi_2} \otimes \mathcal{R}_{\alpha_2,r_2})\right) =$$

$$= \begin{cases} 0 \text{ if for any } \lambda > 0 \ \varphi_1 \neq \lambda\varphi_2 \\ Hom(\mathcal{R}_{\alpha_1,r_1}, \mathcal{R}_{\alpha_2,r_2}) \text{ otherwise} \end{cases}$$

2. *The functor*

$$\mathcal{S}^t_k : \mathsf{GM}_k \longrightarrow \mathrm{Mod}(\mathbb{C}_{\mathbb{C}_{sa}})$$

is fully faithful.

Let us recall the definition of the sheaf $\mathcal{A}^{\leq 0}$ on $S^1 \times \mathbb{R}_{\geq 0}$.
On $S^1 \times \mathbb{R}_{>0}$, $\mathcal{A}^{\leq 0}$ is the sheaf of holomorphic functions.
For $\vartheta \in S^1$,

$$\mathcal{A}^{\leq 0}_{(\vartheta,0)} := \Big\{ f \in \mathcal{O}(S);\ S \text{ an open sector containing } \vartheta \\ \text{there exist } C, N > 0 \text{ such that } |f(z)| \leq \frac{C}{|z|^N} \Big\} .$$

Example: let $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$,

$$\varphi(z) = \frac{\varrho e^{i\tau}}{z^n} + \sum_{j=1}^{n-1} \frac{a_j}{z^j} \qquad (\varrho > 0, \tau \in \mathbb{R}) .$$

Then

$$\exp(\varphi) \in \mathcal{A}^{\leq 0}_{(\vartheta,0)} \iff \cos(\tau - n\vartheta) < 0 .$$

Lemma
Let $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$. Then

$$\exp(\varphi) \in \mathcal{O}^t(U) \iff \exists A > 0 \text{ such that } \forall z \in U,\ \mathrm{Re}\,\varphi(z) < A .$$

In particular $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$ determines an increasing family of open sets, for $A > 0$,

$$U_{\varphi,A} := \{z \in \mathbb{C}^\times;\ \mathrm{Re}\varphi(z) < A\},$$

satisfying

$$\exp(\varphi) \in \mathcal{O}^t(U) \iff \exists A > 0 \text{ such that } U \subset U_{\varphi,A} .$$

Lemma

$$\mathcal{S}^t(\mathcal{L}^\varphi) \simeq \varinjlim_{A>0} \mathbb{C}_{U_{\varphi,A}} .$$

Definition

Let $\tau \in \mathbb{R}$. A relatively compact subanalytic open set $U \subset \mathbb{C}$ is said τ-concentrated if U is connected, $0 \in \partial U$ and given an open sector S containing the direction τ, the germ of U is contained in S.

Proposition

Let $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$ of degree n. There exist $\tau \in \mathbb{R}$ and $U_0, \dots, U_{2n-1}$ relatively compact subanalytic open sets such that

1. *U_j is $\left(\tau + j\frac{\pi}{n}\right)$-concentrated ($j = 0, \dots, 2n-1$),*
2. *for any $j = 0, \dots, 2n-1$, $\exp(\varphi), \exp(-\varphi) \in \mathcal{O}^t(U_j)$.*

Remark

1. The sets τ-concentrated are not open in the topology of $S^1 \times \mathbb{R}_{\geq 0}$.
2. $\nexists \vartheta \in S^1$ such that $\exp(\varphi), \exp(-\varphi) \in \mathcal{A}^{\leq 0}_{(\vartheta,0)}$.

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