Tempered solutions of $\mathcal{D}$-modules on complex curves

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Tempered holomorphic solutions

Let X be a complex analytic *curve*. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X.

Let X_{sa} be the subanalytic site induced by X. i.e. the open sets of X_{sa} are relatively compact open subanalytic subsets of X and coverings are locally finite ones.

The sheaf of tempered holomorphic functions on X_{sa}, $\mathcal{O}^t_{X_{sa}}$, is defined as the solution complex of the Cauchy-Riemann system with values in the subanalytic sheaf of tempered distributions.

If U is a relatively compact subanalytic open subset of $\mathbb{C}$, then

$$\mathcal{O}^t_{X_{sa}}(U) = \left\{ f \in \mathcal{O}_X(U);\ \text{there exist } C, N > 0 \text{ such that } |f(z)| \leq \frac{C}{\mathrm{dist}(z, \partial U)^N} \right\}.$$

Given a holonomic $\mathcal{D}_X$-module, $\mathcal{M}$, it makes sense to consider the complex of subanalytic sheaves of tempered holomorphic solutions of $\mathcal{M}$,

$$\mathcal{S}ol^t\mathcal{M} \ := \ R\mathcal{H}om_{\varrho_!\mathcal{D}_X}\left(\varrho_!\mathcal{M}, \mathcal{O}^t_{X_{sa}}\right) \ .$$

Recall

$$\mathcal{S}ol\mathcal{M} \ := \ R\varrho_*\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}\left(\mathcal{M}, \mathcal{O}_X\right) \in D^b\left(X_{sa}\right) \ .$$

Outline
First case: small sectors
Second case: sets biholomorphic to sectors
Third case: $U \in \mathrm{Op}^c_{\mathbb{R}^2}$
$\mathbb{R}$-constructibility for tempered solutions

Let X be a complex analytic manifold.

Theorem (M. Kashiwara 1979)

Let $\mathcal{M} \in D^b_{rh}(\mathcal{D}_X)$. The natural morphism in $D^b(X_{sa})$

$$\mathcal{S}ol^t\mathcal{M} \longrightarrow \mathcal{S}ol\mathcal{M}$$

is an isomorphism.

Let $D^b_h(\mathcal{D}_X)$ be the bounded derived category of complexes of $\mathcal{D}_X$-modules with holonomic cohomology.

Let $D^b_{\mathbb{R}-c}(\mathbb{C}_X)$ be the bounded derived category of complexes of sheaves with $\mathbb{R}$-constructible cohomology. Let $D^b(\mathbb{C}_{X_{sa}})$ be the bounded derived category of subanalytic sheaves.

Definition

An element $F \in D^b(\mathbb{C}_{X_{sa}})$ is said $\mathbb{R}$-constructible on X_{sa} if, for any $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$,

$$\varrho^{-1}\mathrm{R}\mathcal{H}om_{\mathbb{C}_{X_{sa}}}(\varrho_* G, F) \ \in \ D^b_{\mathbb{R}-c}(\mathbb{C}_X) \ .$$

Corollary

Let $\mathcal{M} \in D^b_{rh}(\mathcal{D}_X)$. Then $\mathcal{S}ol^t\mathcal{M}$ is $\mathbb{R}$-constructible as a subanalytic sheaf.

Tempered solutions of meromorphic connections

For $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$, set

$$\mathcal{L}^{\varphi} := \mathcal{D}_{\mathbb{C}} \exp(\varphi) \ .$$

For $\alpha \in \mathbb{C}$ and $r \in \mathbb{Z}_{\geq 0}$, set

$$\mathcal{R}_{\alpha,r} := \frac{\mathcal{D}_{\mathbb{C}}}{\mathcal{D}_{\mathbb{C}} \cdot \left(z\frac{d}{dz} - \alpha\right)^r} \ .$$

A *good model* is a $\mathcal{D}_{\mathbb{C}}$ module of the form

$$\bigoplus_{j=1}^{p} \mathcal{L}^{\varphi_j} \otimes \mathcal{R}_{\alpha_j,r_j} \ .$$

Outline
First case: small sectors
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Third case: $U \in \mathrm{Op}^c_{\mathbb{R}^2}$
$\mathbb{R}$-constructibility for α-tempered solutions

Theorem (Levelt-Turrittin)

Let E be a meromorphic connection. There exist $l \in \mathbb{Z}_{>0}$ and a good model $\bigoplus_{j=1}^{p} \mathcal{L}^{\varphi_j} \otimes \mathcal{R}_{\alpha_j,r_j}$ such that

$$\pi^*(E) \otimes \mathbb{C}((z)) \simeq \bigoplus_{j=1}^{p} \mathcal{L}^{\varphi_j} \otimes \mathcal{R}_{\alpha_j,r_j} \otimes \mathbb{C}((z)) \ ,$$

for $\pi(z) = z^l$.

Set

$$\Phi := \bigcup_{l \in \mathbb{Z}_{>0}} z^{-1/l}\mathbb{C}[z^{-1/l}] \ .$$

Let γ be the *formal monodromy.*
i.e. given $\varphi \in \Phi$, $(\gamma\varphi)(z) = \varphi(e^{2i\pi}z)$.

Definition

Define the category Gr_1 *as follows. The objects of* Gr_1 *are triples* $(V, \{V_\varphi\}_{\varphi\in\Phi}, \gamma_V)$ *where*

1. V *is a finite dimensional* $\mathbb{C}$*-vector space,*
2. $\{V_\varphi\}_{\varphi\in\Phi}$, $V_\varphi \subset V$ *satisfy* $V = \oplus V_\varphi$,
3. $\gamma_V \in \mathrm{Aut}_{\mathbb{C}}(V)$ *such that* $\gamma_V(V_\varphi) = V_{\gamma\varphi}$.

Theorem (Levelt-Turrittin)

The category of formal meromorphic connections is equivalent to Gr_1.

For E a meromorphic connection, let $\kappa(E)$ be the *Katz invariant* of E.

For $E \simeq \bigoplus_{j=1}^{p} \mathcal{L}^{\varphi_j} \otimes \mathcal{R}_{\alpha_j,r_j}$ a good model,

$$\kappa(E) = \max_{j=1,\dots,p} \{\deg \varphi_j\} \; .$$

Denote by $\widehat{\mathsf{MC}}_k$ the abelian category of formal meromorphic connections with Katz invariant strictly smaller than k.
For $k \in \mathbb{Z}_{>0}$, set

$$\begin{aligned}\mathcal{S}^t(E) &:= \mathcal{H}om_{\varrho_! \mathcal{D}_X}\left(\varrho_! E, \mathcal{O}^t_{X_{sa}}\right) \;, \\ \mathcal{S}^t_k(E) &:= \mathcal{S}^t(E \otimes \mathcal{L}^{1/z^k}) \;.\end{aligned}$$

Theorem

1. *For $j = 1, 2$, let $\varphi_j \in z^{-1}\mathbb{C}[z^{-1}]$, $\alpha_j \in \mathbb{C}$, $r_j \in \mathbb{Z}_{\geq 0}$. Then*

$$\mathrm{Hom}_{\mathbb{C}_{sa}}\left(\mathcal{S}^t(\mathcal{L}^{\varphi_1} \otimes \mathcal{R}_{\alpha_1,r_1}), \mathcal{S}^t(\mathcal{L}^{\varphi_2} \otimes \mathcal{R}_{\alpha_2,r_2})\right) =$$

$$= \begin{cases} 0 \ \textit{if for any } \lambda > 0 \ \varphi_1 \neq \lambda\varphi_2 \\ \mathrm{Hom}(\mathcal{R}_{\alpha_1,r_1}, \mathcal{R}_{\alpha_2,r_2}) \ \textit{otherwise} \ . \end{cases}$$

2. *The functor $\mathcal{S}^t_k : \widehat{\mathsf{MC}_k} \longrightarrow \mathrm{Mod}(\mathbb{C}_{\mathbb{C}_{sa}})$ is fully faithful.*

Theorem (Hukuhara-Turrittin)

On sufficiently small open sectors S, the holomorphic solutions of E are $\mathbb{C}$-linear combinations of

$$h_{j,k}(z)\exp(\varphi_j) \qquad (j = 1, \ldots, p;\ k = 1, \ldots, r_j) \ ,$$

for $\varphi_j \in \Phi$ and $h_{j,k}$ satisfying $h_{j,k}, h^{-1}_{j,k} \in \mathcal{O}^t(S)$.

For $\varphi \in \Phi$, set

$$\mathrm{Fr}(\varphi) := \{d \in \mathbb{R};\ \exp(\varphi(re^{id})) \text{ is maximally decreasing}\} \ .$$

Definition

Define the category Gr_2 *as follows. The objects of* Gr_2 *are tuples* $(V, \{V_\varphi\}_{\varphi\in\Phi}, \gamma_V, \{St_{V,d}\}_{d\in\mathbb{R}})$ *where*

1. $(V, \{V_\varphi\}_{\varphi\in\Phi}, \gamma_V)$ *is in* Gr_1,
2. *for any* $d \in \mathbb{R}$, $St_{V,d} \in \mathrm{Aut}(V)$ *is equal to*

$$\mathrm{id} + \sum_{\varphi_1,\varphi_2,d\in \mathrm{Fr}(\varphi_2-\varphi_1)} N_{\varphi_2,\varphi_1}$$

   for $N_{\varphi_2,\varphi_1} : V \xrightarrow{pr_2} V_{\varphi_2} \longrightarrow V_{\varphi_1} \longrightarrow V$.
3. $\gamma_V^{-1} St_{V,d}\gamma_V = St_{V,d+2\pi}$.

Theorem (van der Put)

The category of meromorphic connections is equivalent to Gr_2.

Let E be a meromorphic connection.

Let $(V, \{V_\varphi\}_{\varphi\in\Phi}, \gamma_V, \{St_{V,d}\}_{d\in\mathbb{R}})$ be the associated object of Gr_2.

If $V_\varphi \neq 0$, then φ is said a *determinant polynomial* of E.

Topological monodromy, formal monodromy and Stokes matrices are related as follows, $\gamma_M = \gamma_V \cdot St_{V,d_1} \cdot \ldots \cdot St_{V,d_m}$.

Theorem

Let E_1, E_2 *be meromorphic connections with Katz invariant strictly smaller than* k. *The following two conditions are equivalent.*

1. $\mathcal{S}^t_k(E_1) \simeq \mathcal{S}^t_k(E_2)$.
2. E_1 *and* E_2 *have the same determinant polynomials and conjugate topological monodromies.*

Existence Theorem for tempered solutions and $\mathbb{R}$-constructibility on X_{sa}

Using subanalytic geometry and classical existence theorems for functional spaces with growth conditions on sufficiently small open sectors, we proved the following results.

Proposition

Let $X \subset \mathbb{C}$ be a open neighbourhood of 0, P a linear ordinary differential operator on X. Let U be a relatively compact subanalytic open subset of X.

There exists an open neighbourhood W of 0 and an open covering $\{U_j\}_{j\in J}$ of $U \cap W$ such that, for any $j \in J$,

$$\mathcal{O}^t(U_j) \xrightarrow{P(\cdot)} \mathcal{O}^t(U_j) \ ,$$

is an epimorphism.

We prove the Theorem in three steps.

1. First we prove the result in the case where U is an open sector of sufficiently small amplitude.
2. Second we prove the result in the case where U is biholomorphic to an open sector.
3. Third we prove the generic case $U \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}}$.

For sake of simplicity, we will suppose that

$$P = z^N \frac{d}{dz} + h(z) \ ,$$

for $h \in \mathcal{O}(X)$ and $N \in \mathbb{N}$.

By Hukuhara-Turrittin's Theorem, the holomorphic solutions of P on a sufficiently small sector S are multiples of $h(z)\exp(\varphi(z))$, for $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$ and h invertible such that h and $\frac{1}{h}$ are tempered on S.

Given $g \in \mathcal{O}^t(U)$, $Pv = g$ has holomorphic solution

$$I(g)(z) = f(z)\exp(\varphi(z))\int_\Gamma \exp(-\varphi(\zeta))\frac{g(\zeta)}{\zeta^N f(\zeta)}d\zeta \ \in \ \mathcal{O}(U) \ .$$

Is $I(g)$ tempered on U?

Theorem (Honda-Malgrange-Hukuhara ...)

If S is an open sector of amplitude sufficiently small then $I(g) \in \mathcal{O}^t(S)$.

Technique: find good path of integration. They depends highly on φ.

Remark: the nice boundary of a sector S allows easy estimates of the distance from the boundary. It is hard to generalize the procedure to an arbitrary open subanalytic set.

The Theorem is true supposing additionally that

1. U is an open sector of sufficiently small amplitude. ✓
2. U is biholomorphic to an open sector.
3. $U \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}}$.

Theorem (1)

Let $U \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}}$, $f \in \mathcal{O}(\overline{U})$, injective on $\overline{U}$, $h \in \mathcal{O}(\varphi(U))$. Then $h \in \mathcal{O}^t(f(U))$ if and only if $h \circ f \in \mathcal{O}^t(U)$.

Idea:

1. consider S an open sector of sufficiently small amplitude and $f : S \longrightarrow T$ a biholomorphism injective up to the boundary of S.
2. Consider $g \in \mathcal{O}^t(T)$ and $I(g) \in \mathcal{O}(T)$, the holomorphic solution of $Pv = g$.
3. Consider $I(g) \circ f \in \mathcal{O}(S)$ and check that it is tempered on S with the case of small sectors.
4. Use Theorem (1) to obtain that $I(g)$ is tempered on T.

1. U is an open sector of sufficiently small amplitude. ✓
2. U is biholomorphic to an open sector. ✓
3. $U \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}}$.

Theorem

Let $U \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}}$. *Locally*

$$U = \bigcup_{j=1}^{d} \left(f_{j,1}(S_{j,1}) \cap f_{j,2}(S_{j,2}) \right) .$$

Where $f_{j,k}$ *is a biholomorphism on* $\overline{S}_{j,k}$, *for* $S_{j,k}$ *an open sector.*

Lemma

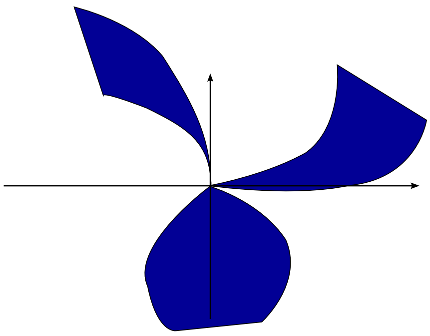
Let $U, V \in \mathrm{Op}^c_{\mathbb{R}^2_{sa}}$. The following sequence is exact

$$0 \longrightarrow \mathcal{O}^t(U \cup V) \longrightarrow \mathcal{O}^t(U) \oplus \mathcal{O}^t(V) \longrightarrow \mathcal{O}^t(U \cap V) \longrightarrow 0 .$$

✓the Theorem is proved.

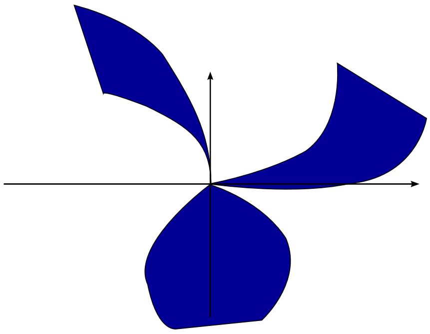
Proof of $U = \bigcup_{j=1}^{d} \left(f_{j,1}(S_{j,1}) \cap f_{j,2}(S_{j,2}) \right)$.

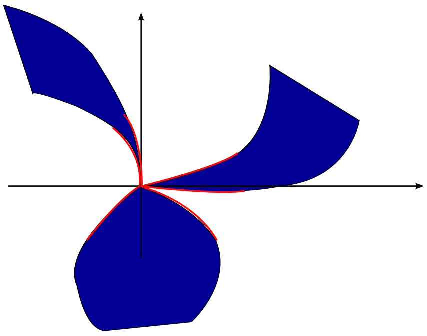
Using ccd or triangulation theorem



Proof of $U = \bigcup_{j=1}^{d} \left(f_{j,1}(S_{j,1}) \cap f_{j,2}(S_{j,2}) \right)$.

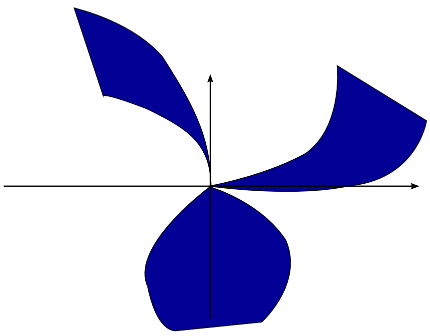
Locally the boundaries are semi-analytic arcs.

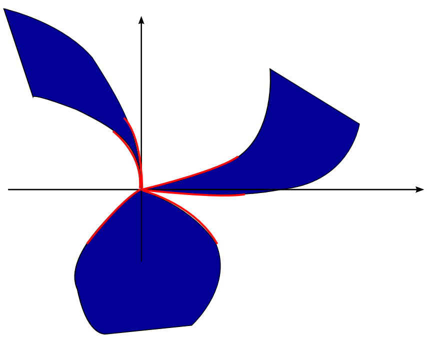


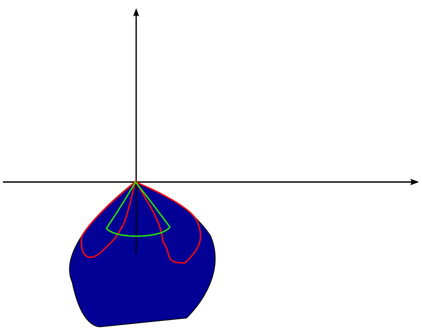


Proof of $U = \bigcup_{j=1}^{d} \left(f_{j,1}(S_{j,1}) \cap f_{j,2}(S_{j,2}) \right)$.

Arcs with different tangents lead unions.

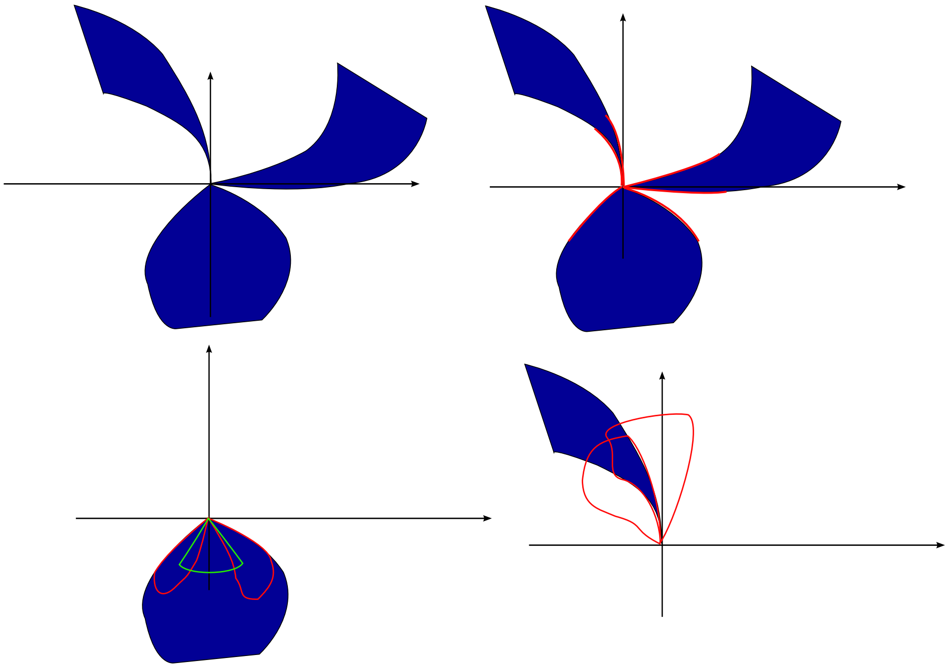






Proof of $U = \bigcup_{j=1}^{d} \left(f_{j,1}(S_{j,1}) \cap f_{j,2}(S_{j,2})\right)$.

Arcs with the same tangents need intersection.



Theorem

Let X be a complex curve, $\mathcal{M}$ a holonomic $\mathcal{D}_X$-module. Then

$$H^1\left(\mathcal{S}ol^t(\mathcal{M})\right) \longrightarrow H^1\left(\mathcal{S}ol(\mathcal{M})\right)$$

is an isomorphism.

Theorem

Let $\mathcal{M} \in D^b_h(\mathcal{D}_X)$. Then $\mathrm{R}\mathcal{H}om_{\varrho_!\mathcal{D}_X}\left(\varrho_!\mathcal{M}, \mathcal{O}^t_{X_{sa}}\right)$ is $\mathbb{R}$-constructible on X_{sa}.

Peculiarities of X_{sa} and $\mathcal{O}^t_{X_{sa}}$

Outline
First case: small sectors
Second case: sets biholomorphic to sectors
Third case: $U \in \mathrm{Op}^c_{\mathbb{R}^2}$
$\mathbb{R}$-constructibility for tempered solutions

Let us recall the definition of the sheaf $\mathcal{A}^{\leq 0}$ on $S^1 \times \mathbb{R}_{\geq 0}$.
On $S^1 \times \mathbb{R}_{>0}$, $\mathcal{A}^{\leq 0}$ is the sheaf of holomorphic functions.
For $\vartheta \in S^1$,

$$\mathcal{A}^{\leq 0}_{(\vartheta,0)} := \Big\{ f \in \mathcal{O}(S);\ S \text{ an open sector containing } \vartheta \\ \text{there exist } C, N > 0 \text{ such that } |f(z)| \leq \frac{C}{|z|^N} \Big\} .$$

Example: let $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$,

$$\varphi(z) = \frac{\varrho e^{i\tau}}{z^n} + \sum_{j=1}^{n-1} \frac{a_j}{z^j} \qquad (\varrho > 0, \tau \in \mathbb{R}) .$$

Then

$$\exp(\varphi) \in \mathcal{A}^{\leq 0}_{(\vartheta,0)} \quad \Longleftrightarrow \quad \cos(\tau - n\vartheta) < 0 .$$

Lemma

Let $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$. Then

$$\exp(\varphi) \in \mathcal{O}^t(U) \quad \Longleftrightarrow \quad \exists A > 0 \text{ such that } \forall z \in U,\ \mathrm{Re}\,\varphi(z) < A\,.$$

In particular $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$ determines an increasing family of open sets, for $A > 0$,

$$U_{\varphi,A} := \{z \in \mathbb{C}^\times;\ \mathrm{Re}\varphi(z) < A\},$$

satisfying

$$\exp(\varphi) \in \mathcal{O}^t(U) \quad \Longleftrightarrow \quad \exists A > 0 \text{ such that } U \subset U_{\varphi,A}\ .$$

Lemma

$$\mathcal{S}^t(\mathcal{L}^\varphi) \simeq \varinjlim_{A>0} \mathbb{C}_{U_{\varphi,A}}\ .$$

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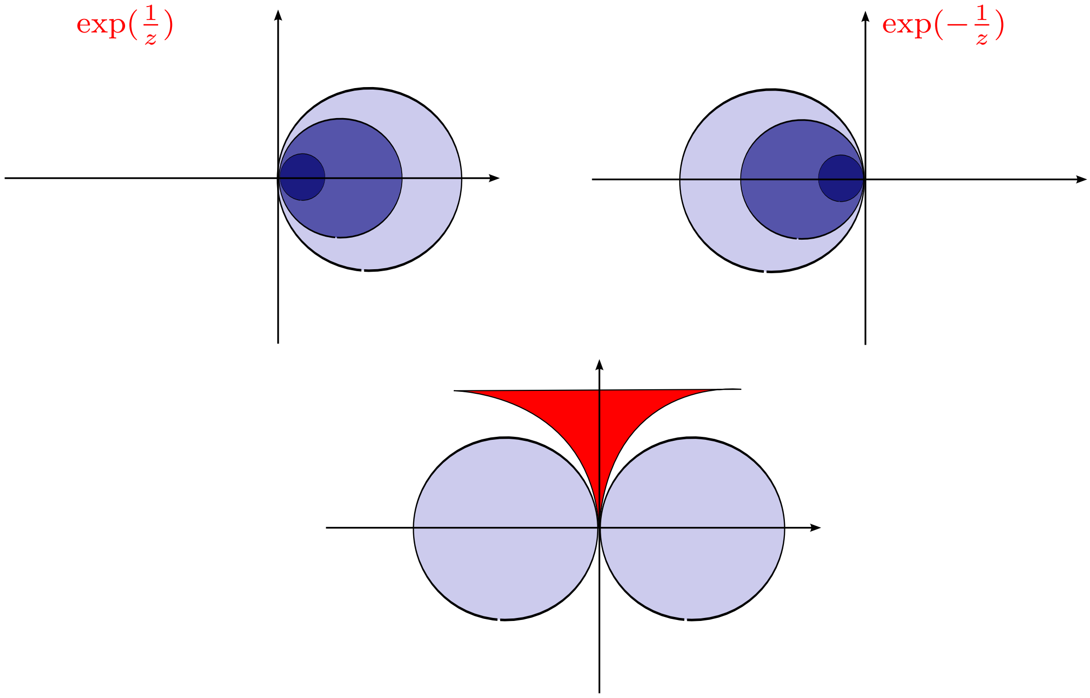

Definition

Let $\tau \in \mathbb{R}$. A relatively compact subanalytic open set $U \subset \mathbb{C}$ is said τ-concentrated if U is connected, $0 \in \partial U$ and given an open sector S containing the direction τ, the germ of U is contained in S.

Proposition

Let $\varphi \in z^{-1}\mathbb{C}[z^{-1}]$ of degree n. There exist $\tau \in \mathbb{R}$ and $U_0, \dots, U_{2n-1}$ relatively compact subanalytic open sets such that

1. *U_j is $\left(\tau + j\frac{\pi}{n}\right)$-concentrated ($j = 0, \dots, 2n-1$),*
2. *for any $j = 0, \dots, 2n-1$, $\exp(\varphi), \exp(-\varphi) \in \mathcal{O}^t(U_j)$.*

Remark

1. The τ-concentrated sets are not open in the topology of $S^1 \times \mathbb{R}_{\geq 0}$.
2. $\nexists \vartheta \in S^1$ such that $\exp(\varphi), \exp(-\varphi) \in \mathcal{A}^{\leq 0}_{(\vartheta,0)}$.

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